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Original Research Paper

Generalized Ternary Hom-Derivations and Jensen ρ-Functional Equation: Solving and Stability

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Abstract. In this research, first we present the concept of the new generalized Jensen ρ-functional equation and then, by utilizing ternary homomorphisms and derivations, we define the new generalized ternary hom-derivations linked to this equation within ternary Banach algebras. We demonstrate that the generalized Jensen ρ-functional equation belongs to the category of additive functions. Furthermore, by employing the fixed point theory, we establish the stability of both the generalized Jensen ρ-functional equation and the associated generalized ternary hom-derivations, using control functions inspired by Găvruta and Rassias. Finally, we investigate the Jordan property as it pertains to generalized ternary hom-derivations linked to this equation within ternary Banach algebras, alongside the generalized ternary (Jordan) hom-derivations can be stable.

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1 Introduction

This statement examines the historical importance of utilizing 3-ary operations with cubic matrices pioneered by Cayley in the 19th century. Cayley's contributions marked a significant milestone in the exploration of n-ary algebraic structures across various fields. N-ary algebras, particularly focusing on ternary structures, have showcased their versatility in a wide range of applications including data processing, physics, and Nambu mechanics for a comprehensive understanding of this matter, it is advisable to refer to sources for information [15, 23]. In 2008, the work of Bagger and Lambert [2] involved a detailed investigation into gauge symmetry and the proposition of a supersymmetric theory for multiple M2-branes. This study introduced a novel algebra incorporating ternary operations called Bagger-Lambert algebras. This advancement further broadened the utilization of ternary algebraic structures in contemporary theoretical physics for more information see [31, 33].

In the year 1940, the notion of the stability problem was brought to light, questioning when a function that nearly satisfies a functional equation ($\mathbb{FE}$) must be close to an exact solution of that equation. This pioneering discussion on the stability of $\mathbb{FE}$s was initiated by Ulam [36]. The subsequent year, in 1941, Hyers [11] offered a positive resolution to Ulam's problem, particularly for additive mappings in Banach spaces. This marked a significant step forward in understanding the stability of $\mathbb{FE}$s. Building upon Hyers' work, Rassias [30], in 1978, expanded the scope by formulating a generalized Hyers-Ulam (HU) stability and introduced a novel stability concept utilizing a control function

$$\varepsilon(\|u_1\|^i + \|u_2\|^i), \qquad \varepsilon > 0,\ i < 1, \tag{1}$$

specifically for additive mappings. Shortly thereafter, in 1994, Găvruta [8] further refined this model by supplanting Rassias's control function with $\varphi(u_1, u_2)$ and demonstrated its stability. Continuing this progress, in 2004, Cădariu and Radu [4] adopted a fixed-point approach to corroborate the stability of the Cauchy additive $\mathbb{FE}$, thereby contributing to a deeper understanding and broader application of these stability concepts in $\mathbb{FE}$s. For those interested in diverse $\mathbb{FE}$s across various spaces, further reading and research can be found in [16, 17, 18, 24, 29, 35]. Additional studies have delved deeper into the Hyers-Ulam-Rassias (HUR) stability, extending this line of inquiry to Dolinar's results concerning isometries, Fibonacci numbers, linear operators and optimization theory [6, 9, 13, 14, 34].

In 1993, an important focus was also directed towards the stability of linear differential equations by Obloza [25]. This pivotal work laid the groundwork for the concept of stability in fractional differential equations, attracting substantial attention from the research community. Researchers have since extended the classical theories of fractional differential equations to encapsulate the principles of HU stability, with notable explorations in works referenced by [5, 7, 10, 32]. Moreover, the overarching theory of fractional calculus has itself seen significant expansions and developments, ensuring stability within the realm of fractional differential equations [3, 19, 20, 22, 37, 38]. One of the applications of HU stability is modeling, such as a fractional optimal control model for COVID-19 and diabetes co-dynamics, demonstrating the positivity and boundedness of solutions using Laplace transform techniques [26].

In the year 2008, an innovative approach to ternary algebras (TAs) was presented by the authors in [1], with a specific emphasis on ternary Banach algebras (TBAs) and their significance in the realms of physics and quantum mechanics. The framework of a TA $\mathfrak{A}$ is delineated through a ternary product $(u_1, u_2, u_3) \to [u_1, u_2, u_3]$, mapping from $\mathfrak{A}^3$ to $\mathfrak{A}$. This algebra, regarded as a complex vector space, exhibits a product that is $\mathbb{C}$-linear in the outer variables, conjugate linear in the middle variable, and, crucially, associative by nature. It adheres to specific criteria, such as $\|[u_1, u_2, u_3]\| \leq \|u_1\| \cdot \|u_2\| \cdot \|u_3\|$ and the condition that $\|[u, u, u]\| = \|u\|^3$. Encompassing the properties of a Banach space, a TA $\mathfrak{A}$ is recognized within the discipline as a TBAs. Most recently, the groundbreaking work by Park *et al.* [28] unveiled the concept of hom-derivations within BAs. After that, in a significant development, Jahedi and Keshavarz [12] delved into the realm of ternary hom-derivations for additive and quadratic mappings in 2022.

In the following, we expand upon this exploration, a comprehensive and refined generalization of the Jensen ρ-functional (where $\rho \neq 0, \pm 1$ denotes a complex number) and ternary hom-derivations between TBAs is introduced in the subsequent discussions.

The continuation of the work is as follows: In the Section 2, we define the concept of the new generalized Jensen ρ-$\mathbb{FE}$ where $\rho \neq 0, \pm 1$ denotes a complex number and by utilizing ternary homomorphisms and derivations, we define the new generalized ternary hom-derivations linked to this equation within TBAs. In the Section 3, we solve the new generalized Jensen ρ-$\mathbb{FE}$, demonstrating that any function satisfying this equation belongs to a specific category known as additive mappings. Additionally, By employing the method of fixed points, we delve into ascertaining the HU stability properties associated with the given equation. Furthermore, we extend our analysis to cover generalized ternary hom-derivations within the framework of TBAs. In our investigation, we utilize control functions defined by Găvruta and Rassias, which are specific types of control functions used for stability.

2 Preliminaries

First, utilizing the principles outlined by additive and Jensen mappings, we will introduce the definition of the generalized Jensen ρ-$\mathbb{FE}$.

Definition 2.1. Consider a mapping $f : \mathfrak{A} \to \mathfrak{A}$ which satisfies the following relation:

$$\begin{aligned} f\left(\tfrac{a+b}{2} + c\right) + f\left(\tfrac{a+c}{2} + b\right) + f\left(\tfrac{b+c}{2} + a\right) \\ - 2f(a) - 2f(b) - 2f(c) \\ = \rho\left(f(a+b+c) - f(a) - f(b) - f(c)\right), \end{aligned} \tag{2}$$

where $\rho \neq 0, \pm 1$ denotes a complex number. The aforementioned relation is referred to as the definition of the generalized Jensen ρ-$\mathbb{FE}$.

Within the context of this elucidation, the framework considers $\mathfrak{A}$ as a pivotal TBA. Within this algebraic framework, a mapping $h : \mathfrak{A} \to \mathfrak{A}$ called a ternary homomorphism if the mapping h map linearity over the complex numbers $\mathbb{C}$ and the property,

$$h([a, b, c]) = [h(a), h(b), h(c)]. \tag{3}$$

Furthermore, consider a linear map denoted by $d : \mathfrak{A} \to \mathfrak{A}$ which takes elements from $\mathfrak{A}$ to itself. Such a map is termed a ternary derivation if it satisfies the property

$$d([a, b, c]) = [d(a), b, c] + [a, d(b), c] + [a, b, d(c)]. \tag{4}$$

Definition 2.2. Let the mapping $h : \mathfrak{A} \to \mathfrak{A}$ be established as a ternary homomorphism as previously defined. We then introduce the concept of a linear map $D : \mathfrak{A} \to \mathfrak{A}$, which is termed a generalized ternary hom-derivation. This designation requires the existence of a derivation $d : \mathfrak{A} \to \mathfrak{A}$ that conforms to the aforementioned derivational property. For D to be a generalized ternary hom-derivation, it must satisfy the following condition:

$$D([a, b, c]) = [D(a), h(b), h(c)] + [h(a), d(b), h(c)] + [h(a), h(b), d(c)]. \tag{5}$$

Theorem 2.3 ([21]). *Consider (A, d) to be a complete generalized metric space and let $F : A \to A$ be a strictly contractive mapping with a Lipschitz constant β such that $\beta < 1$. Then, for any given element $u \in A$, one of the following two scenarios must occur:*

I) *For every nonnegative integer i, the distance $d(F^i u, F^{i+1} u)$ is infinite.*

II) *There exists a positive integer i_0 such that the following conditions hold true:*

   a) *For all integers $i \geq i_0$, the distance $d(F^i u, F^{i+1} u)$ remains finite;*
   b) *The sequence $\{F^i u\}$ converges to a fixed point v^* of F;*
   c) *The point v^* is the unique fixed point of F within the set*

$$B = \Big\{ v \in A \, : \, d(F^{i_0} u, v) < \infty \Big\}; \tag{6}$$

   d) *For each $v \in B$, the inequality $d(v, v^*) \leq \frac{1}{1-\beta} d(v, Fv)$ is satisfied.*

3 Main Results

In the stability section, we consider a TBA denoted by $\mathfrak{A}$. Here, λ is situated within the confines of the unit circle $\mathbb{T}^1$, defined as the set of complex numbers ζ such that $|\zeta| = 1$. Finally, the number ρ embodies a non-negative real value distinct from $0, 1$, and -1.
To establish the main theorems, we need to introduce and utilize the following proportions. Initially, in the upcoming proportion, we will demonstrate the property of additivity for the mapping f.

Proposition 3.1. *If a mapping $f : \mathfrak{A} \to \mathfrak{A}$ satisfies Eq. (2), then the mapping f is additive.*

Proof. Assume that the function f mapping from $\mathfrak{A}$ to $\mathfrak{A}$ satisfies Eq. (2). If we substitute $a = b = c = 0$ into Eq. (2), we obtain $f(0) = 0$.
Now, by setting $b = c = 0$ in Eq. (2), we get:

$$\tfrac{1}{2} f(a) = f\left(\tfrac{a}{2}\right). \tag{7}$$

Again, putting $c = 0$ and replacing a by $-b$ in Eq. (2) and using (7), we get:

$$-f(a) = f(-a). \tag{8}$$

In following replacing c by $-b$ and using (7) and (8), we obtain

$$f\left(\tfrac{a-b}{2}\right) + f\left(\tfrac{a+b}{2}\right) - f(a) = 0 \tag{9}$$

finally, putting $a = a + b$ and $b = a - b$ in (9), we have

$$f(a + b) = f(a) + f(b). \tag{10}$$

Hence, f is additive. □

In the proportion that follows next, we demonstrate that the newly introduced generalized function f is indeed a $\mathbb{C}-$linear mapping under the specified conditions.

Proposition 3.2. *If a mapping $f : \mathfrak{A} \to \mathfrak{A}$ fulfills Eq. (11) and $t \in \mathbb{T}^1$, then the mapping f is $\mathbb{C}-$linear, with Eq. (11) defined as:*

$$\begin{aligned} f\left(\tfrac{ta+tb}{2} + tc\right) + f\left(\tfrac{ta+tc}{2} + tb\right) + f\left(\tfrac{tb+tc}{2} + a\right) \\ - 2tf(a) - 2tf(b) - 2tf(c) \\ = \rho\left(f(t(a + b + c)) - tf(a) - tf(b) - tf(c)\right). \end{aligned} \tag{11}$$

Proof. For proof, we use proportion 3.1 to demonstrate that the function f is additive. We achieve this demonstration by substituting $c = -b$ into Eq. (11), resulting in the equation

$$f\left(\tfrac{ta-tb}{2}\right) + f\left(\tfrac{ta+tb}{2}\right) - f(ta) - 2tf(a) = 0. \tag{12}$$

Subsequently, by substituting $b = a$ into Eq. (12), we find that $f(ta) = tf(a)$. By employing analogous reasoning to the demonstration in [27, Theorem 2.1], we can infer that the function f is $\mathbb{C}-$linear. □

We ascertain the stability of the generalized Jensen ρ-$\mathbb{FE}$ under the HU framework is investigated using the fixed point method. Găvruta's control function, as a comprehensive control function within the HU stability context, is employed for this purpose. The integration of fixed point theory and control functions offers a solid foundation for comprehending and demonstrating the stability characteristics of $\mathbb{FE}$s in TBAs.

Before prove main theorem in section, suppose that ψ be mapping from $\mathfrak{A}^3$ into $[0,\infty)$, such that

$$\psi(a,b,c) \leq \tfrac{L}{2}\psi(2a,2b,2c), \tag{13}$$

holds for some constant $0 < L < 1$. If we assign $a = b = c = 0$, then $\psi(0,0,0) = 0$. By considering Eq. (13), we deduce that

$$\lim_{i\to\infty} 2^i\psi\left(\tfrac{a}{2^i}, \tfrac{b}{2^i}, \tfrac{c}{2^i}\right) = 0. \tag{14}$$

Theorem 3.3. *For a given mapping $f : \mathfrak{A} \to \mathfrak{A}$, let there be functions $\psi : \mathfrak{A}^3 \to [0,\infty)$ satisfying (13) and*

$$\begin{aligned}\Big\| f\left(\tfrac{ta+tb}{2} + tc\right) + f\left(\tfrac{ta+tc}{2} + tb\right) + f\left(\tfrac{tb+tc}{2} + ta\right) - 2tf(a)\\ - 2tf(b) - 2tf(c) - \rho\big(f(t(a+b+c))\\ - tf(a) - tf(b) - tf(c)\big)\Big\|\\ \leq \psi(a,b,c), \qquad \forall\, a,b,c \in \mathfrak{A}.\end{aligned} \tag{15}$$

Then, there exists a unique $\mathbb{C}-$linear mapping $T : \mathfrak{A} \to \mathfrak{A}$ satisfying

$$\|f(a) - T(a)\| \leq \tfrac{L}{1-L}\psi(a,0,0), \qquad \forall\, a \in \mathfrak{A}. \tag{16}$$

Proof. When substituting t with 1, and b and c with 0 in Eq. (15), the resulting expression is obtained as:

$$\left\|2f\left(\frac{a}{2}\right)-f(a)\right\| \leq \psi(a,0,0), \qquad \forall\, a \in \mathfrak{A}. \tag{17}$$

Let's now consider the set Ψ of mappings $g:\mathfrak{A}\to\mathfrak{A}$ where $g(0)=0$. Let the function d be defined on this set Ψ in the following manner: For any two functions g and h belonging to Ψ, we define $d(g,h)$ as the infimum of the set of positive values k such that the inequality $\|g(a)-h(a)\| \leq k\psi(a,0,0)$ holds for all elements a in $\mathfrak{A}$. It is evident that the pair (Ψ,d) forms a generalized metric space, following the definitions outlined above. Now, we consider the linear mappings $\Lambda:\Psi\to\Psi$ such that

$$\Lambda g(a) = 2g\left(\frac{a}{2}\right), \qquad \forall\, a \in \mathfrak{A}. \tag{18}$$

Suppose we are provided with g and h from Ψ such that $d(g,h)=\varepsilon$, then,

$$\|g(a)-h(a)\| \leq \varepsilon\psi(a,0,0), \qquad \forall\, a \in \mathfrak{A}. \tag{19}$$

Since

$$\begin{aligned}\|\Lambda g(a)-\Lambda h(a)\| &= \left\|2g\left(\frac{a}{2}\right)-2h\left(\frac{a}{2}\right)\right\| \\ &\leq 2\varepsilon\psi\left(\frac{a}{2},0,0\right) \leq 2\varepsilon\frac{L}{2}\psi(a,0,0) \\ &= L\varepsilon\psi(a,0,0).\end{aligned} \tag{20}$$

So, we have $d(\Lambda g,\Lambda h)\leq L\varepsilon$. This expression implies that

$$d(\Lambda g,\Lambda h) \leq Ld(g,h), \qquad \forall\, g,h \in \Psi. \tag{21}$$

Following the relation presented in Eq. (13), we can observe that

$$\left\|f(a)-\frac{1}{2}f(2a)\right\| \leq \frac{1}{2}\psi(2a,0,0) \leq L\psi(a,0,0). \tag{22}$$

We have $d(f,\Lambda f)\leq L$. Based on Theorem 2.3, it is established that a function $T:\mathfrak{A}\to\mathfrak{A}$ exists satisfying the subsequent conditions.

1) T serves as a fixed point for Λ, ensuring the following property:

$$T(a) = 2T\left(\frac{a}{2}\right), \qquad \forall\, a \in \mathfrak{A}. \tag{23}$$

The function T represents a distinct fixed point for Λ. This circumstance suggests the existence of a unique function that satisfies (23), where there exists a $k \in (0, \infty)$ that meets the condition.

$$\|f(a) - T(a)\| \leq k\psi(a, 0, 0), \qquad \forall a \in \mathfrak{A}. \tag{24}$$

2) The statement $d(\Lambda^i f, T) \to 0$ as $i \to \infty$ suggests that the following equality holds:

$$\lim_{i\to\infty} 2^i f\left(\frac{a}{2^i}\right) = T(a), \qquad \forall a \in \mathfrak{A}.$$

3) The inequality $d(f, T) \leq \frac{1}{1-L} d(f, \Lambda f)$ implies the following result:

$$\|f(a) - T(a)\| \leq \frac{L}{1-L}\psi(a, 0, 0), \qquad \forall\, a \in \mathfrak{A}.$$

Subsequent to the equations denoted by (14) and (15), it can be inferred that

$$\begin{aligned}
&\Big\| T\left(\tfrac{a+b}{2} + c\right) + T\left(\tfrac{a+c}{2} + b\right) + T\left(\tfrac{b+c}{2} + a\right) - 2T(a) - 2T(b) \\
&\qquad - 2T(c) - \rho\big(T(a+b+c) - T(a) - T(b) - T(c)\big)\Big\| \\
&= \lim_{i\to\infty} 2^i \Big\| f\left(\tfrac{a+b}{2^{i+1}} + \tfrac{c}{2^i}\right) + f\left(\tfrac{a+c}{2^{i+1}} + \tfrac{b}{2^i}\right) \\
&\qquad + f\left(\tfrac{b+c}{2^{i+1}} + \tfrac{a}{2^i}\right) - 2f\left(\tfrac{a}{2^i}\right) - 2f\left(\tfrac{b}{2^i}\right) - 2f\left(\tfrac{c}{2^i}\right) \\
&\qquad - \rho\left(f\left(\tfrac{a+b+c}{2^i}\right) - f\left(\tfrac{a}{2^i}\right) - f\left(\tfrac{b}{2^i}\right) - f\left(\tfrac{c}{2^i}\right)\right)\Big\| \\
&\leq \lim_{i\to\infty} 2^i \psi\left(\tfrac{a}{2^i}, \tfrac{b}{2^i}, \tfrac{c}{2^i}\right) = 0.
\end{aligned} \tag{25}$$

So

$$\begin{aligned}
T\left(\tfrac{a+b}{2} + c\right) + T\left(\tfrac{a+c}{2} + b\right) + T\left(\tfrac{b+c}{2} + a\right) \\
- 2T(a) - 2T(b) - 2T(c) \\
= \rho\left(T(a+b+c) - T(a) - T(b) - T(c)\right).
\end{aligned} \tag{26}$$

Based on the proportion 3.2, it can be inferred that the mapping T exhibits $\mathbb{C}-$linearity. $\square$

In the corollary that is to follow, our exploration will be centered around the investigation of HU stability through the application of Rassias' control function on $\mathfrak{A}$.

Corollary 3.4. *Suppose $r < 1$, $s \in \mathbb{R}^+ \cup \{0\}$, and let $f : \mathfrak{A} \to \mathfrak{A}$ be a mapping with the conditions $f(0) = 0$ and*

$$\begin{aligned}\Big\| f\left(\tfrac{ta+tb}{2} + tc\right) + f\left(\tfrac{ta+tc}{2} + tb\right) - f\left(\tfrac{tb+tc}{2} + ta\right) \\ - 2tf(a) - 2tf(b) - 2tf(c) \\ - \rho\big(f\left(t(a+b+c)\right) - tf(a) - tf(b) - tf(c)\big)\Big\| \\ \leq s(\|a\|^r + \|b\|^r + \|c\|^r), \qquad \forall\, a, b, c \in \mathfrak{A}. \qquad (27)\end{aligned}$$

Then, there exists a unique additive mapping $T : \mathfrak{A} \to \mathfrak{A}$ such that

$$\|f(a) - T(a)\| \leq \tfrac{2s}{2-2^r}\|a\|^r, \qquad \forall\, a \in \mathfrak{A}. \qquad (28)$$

Proof. The proof technique can be obtained by referring to Theorem 3.5 with the establishment of $\psi(a, b, c) := s(\|a\|^r + \|b\|^r + \|c\|^r)$ and L being assigned as 2^{1-r} for all elements a, b, and c in $\mathfrak{A}$. By following this approach, the proof methodology can be constructed and verified within the defined framework. □

The following theorem offers a thorough and detailed explanation of how ternary hom-derivations on $\mathfrak{A}$ can be stable. For this work, we the utilization of Găvruta's control function approach within the context of the fixed point theorem for HU stability on $\mathfrak{A}$.

Theorem 3.5. *Consider a function $\psi : \mathfrak{A}^3 \to [0, \infty)$, where there exists a value $L < 1$ such that*

$$\psi(a, b, c) \leq \tfrac{L}{2^3}\psi(2a, 2b, 2c), \qquad (29)$$

where f, g, and $\delta : \mathfrak{A} \to \mathfrak{A}$ are mappings that satisfy (15), and

$$\|g[a, b, c] - [g(a), g(b), g(c)]\| \leq \psi(a, b, c), \qquad (30)$$

$$\|d([a, b, c]) - [d(a), b, c] - [a, d(b), c] - [a, b, d(c)]\| \leq \psi(a, b, c), \qquad (31)$$

$$\begin{aligned}\|f([a, b, c]) - [f(a), g(b), g(c)] - [g(a), d(b), h(c)] \\ -[h(a), h(b), d(c)]\| \leq \psi(a, b, c), \qquad (32)\end{aligned}$$

for each $a, b, c \in \mathfrak{A}$. Then there are a unique ternary homomorphism H and ternary derivation d on $\mathfrak{A}$ such that $D : \mathfrak{A} \to \mathfrak{A}$ is a generalized ternary hom-derivations and for all $a \in \mathfrak{A}$,

$$\|g(a) - H(a)\| \leq \tfrac{L}{1-L}\psi(a, 0, 0), \tag{33}$$

$$\|d(a) - \delta(a)\| \leq \tfrac{L}{1-L}\psi(a, 0, 0), \tag{34}$$

$$\|f(a) - D(a)\| \leq \tfrac{L}{1-L}\psi(a, 0, 0). \tag{35}$$

Proof. Firstly, we take $a = b = c = 0$, then $\psi(0, 0, 0) = 0$ in (29). It follows from (29) that

$$\lim_{i\to\infty} 2^{3i}\psi\left(\tfrac{a}{2^i}, \tfrac{b}{2^i}, \tfrac{c}{2^i}\right) = 0. \tag{36}$$

By applying the identical procedure as delineated in the proof of Theorem 3.3, it is possible for us to define and establish the mappings H, δ, and D, which are all mappings from $\mathfrak{A}$ to $\mathfrak{A}$.

$$\begin{aligned} H(a) &:= \lim_{i\to\infty} 2^i h\left(\tfrac{a}{2^i}\right), \\ \delta(a) &:= \lim_{i\to\infty} 2^i d\left(\tfrac{a}{2^i}\right), \\ D(a) &: = \lim_{i\to\infty} 2^i f\left(\tfrac{a}{2^i}\right) \end{aligned} \tag{37}$$

for all $a \in \mathfrak{A}$. It follows from (30), (31), (32) and (37), we have

$$\begin{aligned} &\|H([a, b, c]) - [H(a), H(b), H(c)]\| \\ &\quad = \lim_{i\to\infty} 2^{3i}\Big\| h\left(\left[\tfrac{a}{2^i}, \tfrac{b}{2^i}, \tfrac{c}{2^i}\right]\right) \\ &\qquad - \left[h\left(\tfrac{a}{2^i}\right), h\left(\tfrac{b}{2^i}\right), h\left(\tfrac{c}{2^i}\right)\right] \Big\| \\ &\quad \leq \lim_{i\to\infty} 2^{3i}\psi\left(\tfrac{a}{2^i}, \tfrac{b}{2^i}, \tfrac{c}{2^i}\right) = 0, \end{aligned} \tag{38}$$

$$\begin{aligned} &\|\delta([a, b, c]) - [\delta(a), b, c] - [a, \delta(b), c] - [a, b, \delta(c)]\| \\ &\quad = \lim_{i\to\infty} 2^{3i}\Big\| d\left(\left[\tfrac{a}{2^i}, \tfrac{b}{2^i}, \tfrac{c}{2^i}\right]\right) - \left[d\left(\tfrac{a}{2^i}\right), \tfrac{b}{2^i}, \tfrac{c}{2^i}\right] \\ &\qquad - \left[\tfrac{a}{2^i}, d\left(\tfrac{b}{2^i}\right), \tfrac{c}{2^i}\right] - \left[\tfrac{a}{2^i}, \tfrac{b}{2^i}, d\left(\tfrac{c}{2^i}\right)\right] \Big\| \\ &\quad \leq \lim_{i\to\infty} 2^{3i}\psi\left(\tfrac{a}{2^i}, \tfrac{b}{2^i}, \tfrac{c}{2^i}\right) = 0 \end{aligned} \tag{39}$$

and

$$\begin{aligned}
\|D([a,b,c]) - [D(a),H(b),H(c)] - [H(a),\delta(b),H(c)] \\
- [H(a),H(b),\delta(c)]\| \\
= \lim_{i\to\infty} 2^{3i}\Big\| f\left(\left[\tfrac{a}{2^i},\tfrac{b}{2^i},\tfrac{c}{2^i}\right]\right) - \left[f\left(\tfrac{a}{2^i}\right), h\left(\tfrac{b}{2^i}\right), h\left(\tfrac{c}{2^i}\right)\right] \\
- \left[h\left(\tfrac{a}{2^i}\right), \delta\left(\tfrac{b}{2^i}\right), h\left(\tfrac{c}{2^i}\right)\right] \\
- \left[h\left(\tfrac{a}{2^i}\right), h\left(\tfrac{b}{2^i}\right), \delta\left(\tfrac{c}{2^i}\right)\right]\Big\| \\
\le \lim_{i\to\infty} 2^{3i}\psi\left(\tfrac{a}{2^i},\tfrac{b}{2^i},\tfrac{c}{2^i}\right) = 0. \qquad (40)
\end{aligned}$$

It follows that

$$\begin{aligned}
H([a,b,c]) &= [H(a),H(b),H(c)], \\
\delta([a,b,c]) &= [\delta(a),b,c] + [a,\delta(b),c] + [a,b,\delta(c)], \\
D([a,b,c]) &= [D(a),H(b),H(c)] + [H(a),\delta(b),H(c)] \\
&\quad + [H(a),H(b),\delta(c)], \qquad \forall\, a,b,c \in \mathfrak{A}. \qquad (41)
\end{aligned}$$

Hence the mappings $H,\delta : \mathfrak{A} \to \mathfrak{A}$ and $D : \mathfrak{A} \to \mathfrak{A}$ are ternary homomorphism, ternary derivation and generalized ternary hom-derivations, respectively. $\square$

In The statement of Theorem 3.5, if we designate $L = 2^{1-r}$ for $0 < r < 1$, and define

$$\psi(a,b,c) := s\left(\|a\|^r + \|b\|^r + \|c\|^r\right), \qquad (42)$$

in the context of $a,b,c \in \mathfrak{A}$ and $s \in \mathbb{R}^+ \cup \{0\}$, we reach the resulting conclusion following Rassias' theorem regarding ternary hom-derivations.

Corollary 3.6. *Let $\delta(a,b,c) = s(\|a\|^r+\|b\|^r+\|c\|^r)$ where $r < 1$ and $s \in \mathbb{R}_+$. Suppose that the functions $f,h,d : \mathfrak{A} \to \mathfrak{A}$ satisfy Eqs. (15), (30), (31) and (32). Then there are a unique ternary homomorphism H and ternary derivation d on $\mathfrak{A}$ such that $D : \mathfrak{A} \to \mathfrak{A}$ is a generalized ternary hom-derivations and*

$$\begin{cases}
\|h(a) - H(a)\| \le \frac{2s}{2-2^r}\|a\|^r, \\
\|d(a) - \delta(a)\| \le \frac{2s}{2-2^r}\|a\|^r, \\
\|f(a) - D(a)\| \le \frac{2s}{2-2^r}\|a\|^r.
\end{cases} \qquad (43)$$

In the following, considering the importance, the concept of ternary Jordan derivations has applications in operator theory, non-associative algebras, and quantum physics. For instance, a derivation D on ternary system T can analyze quantum operations' interaction with ternary structures. We can investigate the Jordan property of generalized ternary (Jordan) hom-derivations on TBAs. Next, we can investigate the stability of generalized ternary (Jordan) hom-derivations by control functions of Găvruta and Rassias through the theorem of Margolis and Diaz. Before commencing the stability theorem, we define the concept of generalized ternary (Jordan) hom-derivations.

Definition 3.7. Assume the mapping $h : \mathfrak{A} \to \mathfrak{A}$ is a ternary homomorphism. We then introduce the concept of a linear map $D : \mathfrak{A} \to \mathfrak{A}$, which is termed a generalized ternary (Jordan) hom-derivation. This designation requires the existence of a derivation $d : \mathfrak{A} \to \mathfrak{A}$ that conforms to the aforementioned derivational property. For D to be a generalized ternary (Jordan) hom-derivation, it must satisfy the following condition,

$$\begin{aligned} D([a,a,a]) = \big[D(a), h(a), h(a)\big] + \big[h(a), d(a), h(a)\big] \\ + \big[h(a), h(a), d(a)\big], \qquad \forall\, a \in \mathfrak{A}. \end{aligned} \tag{44}$$

Remark 3.8. Consider a function ψ satisfies (29) and the mappings f, g, δ satisfying (15) and

$$\big\|g[a,b,c] - [g(a), g(b), g(c)]\big\| \le \psi(a,b,c),$$

$$\begin{aligned} &\big\|d([a,a,a]) - [d(a),a,a] - [a,d(a),a] - [a,a,d(a)]\big\| \\ &\qquad \le \psi(a,a,a), \end{aligned} \tag{45}$$

$$\begin{aligned} &\big\|f([a,a,a]) - [f(a), g(a), g(a)] - [g(a), d(a), h(a)] \\ &\qquad - [h(a), h(a), d(a)]\big\| \le \psi(a,a,a). \end{aligned} \tag{46}$$

Then, the stability conditions analogous to those in Theorem 3.5, hold for generalized ternary (Jordan) hom-derivations on $\mathfrak{A}$.

Remark 3.9. Let $\delta(a,a,a) = 3s(\|a\|^r)$ where $r < 1$ and $s \in \mathbb{R}_+$. suppose that the functions $f, h, d : \mathfrak{A} \to \mathfrak{A}$ satisfy Eqs. (15), (30), (45), and (46). Then, the stability conditions analogous to those in Corollary 3.6, hold for generalized ternary (Jordan) hom-derivations on $\mathfrak{A}$.

4 Conclusions

Acknowledging the significance attributed to TBAs, generalized ternary derivations, and their significant applications in the realm of mathematical physics, our initiative primarily focused on the introduction of the pioneering concept of generalized ternary hom-derivations within the framework of TBAs. Through the theorem of Margolis and Diaz, we have demonstrated the stability associated with generalized ternary hom-derivations and the generalized Jensen ρ-$\mathbb{FE}$ (2) within the context of stability within TBAs. Finally, we explored the Jordan property to generalized ternary (Jordan) hom-derivations on $\mathfrak{A}$ and the stability of generalized ternary (Jordan) hom-derivations on $\mathfrak{A}$ by control functions of Găvruta and Rassias.

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