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Mond-Weir Duality Results for Nondifferentiable Mathematical Programming with Vanishing Constraints

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Abstract. In this paper, we introduce a Mond-Weir type dual problem for the optimization problems with vanishing constraints (MPVC) defined by nondifferentiable locally Lipschitz functions. Then, we present the weak, the strong, the converse, the restricted converse, and the strict converse duality results for this new dual problem. This article can be considered as an extension of Mishra *et al.* (Ann. Oper. Res. 243(1):249–272, 2016), and a supplement of Gobadzadeh *et al.* (J. Math. Ext 9(7):1–17, 2022).

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1 Introduction

A difficult class of optimization problems, named mathematical programming with vanishing constraints (MPVC, in brief), and its applications in topological optimization have been introduced by Kanzow and his co-authors in [1, 7]. Some Karush-Kuhn-Tucker (KKT for short)

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type optimality conditions, named stationarity conditions, are presented under the Abadie and Guignard type constraints qualifications (CQ, in short) by several authors; see [1, 5, 8, 9] in differentiable case and [13, 14, 16, 17, 18] in nondifferentiable case.

Following [13], In this paper, we consider the following MPVC:

$$\min f(x) \quad \text{subject to } x \in \mathcal{M}, \tag{1}$$

where the feasible set $\mathcal{M}$ is defined as

$$\mathcal{M} := \big\{x \in \mathbb{R}^n \mid H_i(x) \geq 0,\ H_i(x)G_i(x) \leq 0, \quad i \in I\big\}, \tag{2}$$

in which the index set I is finite, and the functions f, H_i, and G_i are locally Lipschitz from $\mathbb{R}^n$ to $\mathbb{R}$ for all $i \in I := \{1, \ldots, m\}$. It should be noted that the general form of an MPVC, introduced in [1], includes inequality constraints $g_j(x) \leq 0$ as $j \in J$ and equality constraints $h_t(x) = 0$ as $t \in T$ for some finite index sets J and T. Since adding these constraints to problem (1) does not increase the technical problems of the issue and just prolongs the formulas, we ignore them and just deal with problem (1).

Since it is difficult to introduce new dualities for MPVCs, few articles have been written in this field. Recently, [6, 10, 15] introduced some Wolf and Mond-Weir types dual problems for MPVCs and presented some weak, strong, converse, restricted converse, and strict converse duality results for these dual problems. We should be noted that the mentioned papers consider the duality for MPVCs with continuously differentiable (i.e., smooth) functions. Very recently, the Wolf type duality presented for MPVCs with nonsmooth functions in [4], and there are no articles that study Mond-Weir type duality for nonsmooth MPVCs. In this paper, we will full this gap.

The structure of subsequent sections of this paper is as follows: In Sec. 2, we define required definitions and preliminary results which are requested in sequel. The main results, which include the introduction of a Mond-Weir type dual problem for nonsmooth MPVC (1) and the statement of weak, strong, converse, restricted converse, and strict converse duality results, are presented in Section 3.

2 Notations and Preliminaries

In this section we present some definitions and auxiliary results from [3] that will be needed in what follows.

The function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is called locally Lipschitz if for each $x^* \in \mathbb{R}^n$ there exist a neighbourhood U of x^* and a positive constant L_U such that

$$|\varphi(x) - \varphi(y)| \leq L_U \|x - y\|, \qquad \forall x, y \in U.$$

For a given locally Lipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $x^* \in \mathbb{R}^n$, the Clarke directional derivative of φ at x^* in the direction $\nu \in \mathbb{R}^n$ and the Clarke subdifferential of φ at x^* are respectively defined by

$$\varphi^0(x^*; \nu) := \limsup_{x \to x^*,\ t \downarrow 0} \frac{\varphi(x + t\nu) - \varphi(x)}{t},$$
$$\partial_c \varphi(x^*) := \big\{\xi \in \mathbb{R}^n \mid \langle \xi, \nu \rangle \leq \varphi^0(x^*; \nu) \quad \text{for all } \nu \in \mathbb{R}^n \big\},$$

where $\langle a, b \rangle$ exhibits the standard inner product of $a, b \in \mathbb{R}^n$. The zero vector of $\mathbb{R}^n$ is denoted by 0_n.

It is known that when φ is continuously differentiable (smooth) at x^*, then $\partial_c \varphi(x^*) = \{\nabla \varphi(x^*)\}$, where $\nabla \varphi(x^*)$ denotes the standard gradient of φ at x^*. We know from [3] that if φ and ψ are locally Lipschitz functions form $\mathbb{R}^n$ to $\mathbb{R}$, one has

$$\partial_c(\mu\varphi + \eta\psi)(x^*) \subseteq \mu \partial_c \varphi(x^*) + \eta \partial_c \psi(x^*), \quad \forall \mu, \eta \in \mathbb{R}.$$

The following concepts plays a very important role in this paper.

Definition 2.1. The locally Lipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be

(i): ∂_c-quasiconvex at $x^* \in \mathbb{R}^n$, if for all $x \in \mathbb{R}^n$, one has

$$\varphi(x) \leq \varphi(x^*) \implies \langle \xi, x - x^* \rangle \leq 0, \quad \forall \xi \in \partial_c \varphi(x^*).$$

(ii): ∂_c-pseudlinear at $x^* \in \mathbb{R}^n$, if φ and $-\varphi$ are ∂_c-quasiconvex at $x^* \in \mathbb{R}^n$. In other word, φ is ∂_c-pseudlinear at x^* if for all $x \in \mathbb{R}^n$, one has

$$\varphi(x) = \varphi(x^*) \implies \langle \xi, x - x^* \rangle = 0, \quad \forall \xi \in \partial_c \varphi(x^*).$$

(iii): ∂_c-pseudoconvex at $x^* \in \mathbb{R}^n$, if for all $x \in \mathbb{R}^n$, one has

$$\varphi(x) < \varphi(x^*) \implies \langle \xi, x - x^* \rangle < 0, \quad \forall \xi \in \partial_c \varphi(x^*).$$

(iv): ∂_c-strictly pseudoconvex at $x^* \in \mathbb{R}^n$, if for all $x \in \mathbb{R}^n$ with $x \neq x^*$, one has

$$\varphi(x) \leq \varphi(x^*) \implies \langle \xi, x - x^* \rangle < 0, \quad \forall \xi \in \partial_c \varphi(x^*).$$

To see details, examples, properties, and characterizations of the above concepts, we can refer to book by Bagirov *et. al.* [2] and the papers [12, 11].

3 Main Results

As the beginning of this section, we assign some symbols for the whole of this article.

Suppose that the feasible set $\mathcal{M}$, defined in (2), is nonempty. For each $x \in \mathcal{M}$, the index set I can be partitioned as

$$I = I_{+0}(x) \cup I_{+-}(x) \cup I_{0+}(x) \cup I_{0+}(x) \cup I_{0-}(x),$$

in which

$$\begin{aligned} I_{+0}(x) &:= \{i \in I \mid H_i(x) > 0,\ G_i(x) = 0\}, \\ I_{+-}(x) &:= \{i \in I \mid H_i(x) > 0,\ G_i(x) < 0\}, \\ I_{0+}(x) &:= \{i \in I \mid H_i(x) = 0,\ G_i(x) > 0\}, \\ I_{00}(x) &:= \{i \in I \mid H_i(x) = 0,\ G_i(x) = 0\}, \\ I_{0-}(x) &:= \{i \in I \mid H_i(x) = 0,\ G_i(x) < 0\}. \end{aligned}$$

For simplicity, put

$$\begin{aligned} I_{+}(x) &:= I_{+0}(x) \cup I_{+-}(x) = \{i \in I \mid H_i(x) > 0\}, \\ I_{0}(x) &:= I_{0+}(x) \cup I_{00}(x) \cup I_{0-}(x) = \{i \in I \mid H_i(x) = 0\}. \end{aligned}$$

Let the feasible point $x \in \mathcal{M}$ be given. We define the following Mond-Weir type dual problem for MPVC (1):

$$MWD(x): \qquad \max f(u),$$

$$s.t. \left| \begin{array}{ll} 0_n \in \partial_c f(u) + \sum_{i \in I} \big(-\mu_i \partial_c H_i(u) + \eta_i \partial_c G_i(u) \big), & \\ -\mu_i H_i(u) \geq 0, & i \in I, \\ \eta_i G_i(u) \geq 0, & i \in I, \\ \mu_i \geq 0, & i \in I_+(x), \\ \eta_i \leq 0, & i \in I_{0+}(x), \\ \eta_i \geq 0, & i \in I_{0-}(x) \cup I_{+-}(x). \end{array} \right. \tag{3}$$

The feasible sets of $MWD(x)$ is denoted by

$$\mathcal{M}(x) := \big\{ (u, \mu, \eta) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \mid (u, \mu, \eta) \text{ fulfills } (3) \big\}.$$

Following [4], for each $(u, \mu, \eta) \in \mathcal{M}(x)$ we define the following index sets:

$$\begin{aligned} I_+^+(x) &:= \{ i \in I_+(x) \mid \mu_i > 0 \}, \\ I_0^+(x) &:= \{ i \in I_0(x) \mid \mu_i > 0 \}, \\ I_0^-(x) &:= \{ i \in I_0(x) \mid \mu_i < 0 \}, \\ I_{0+}^-(x) &:= \{ i \in I_{0+}(x) \mid \eta_i < 0 \}, \\ I_{00}^-(x) &:= \{ i \in I_{00}(x) \mid \eta_i < 0 \}, \\ I_{+0}^-(x) &:= \{ i \in I_{+0}(x) \mid \eta_i < 0 \}, \\ I_{00}^+(x) &:= \{ i \in I_{00}(x) \mid \eta_i > 0 \}, \\ I_{+0}^+(x) &:= \{ i \in I_{+0}(x) \mid \eta_i > 0 \}, \\ I_{0-}^+(x) &:= \{ i \in I_{0-}(x) \mid \eta_i > 0 \}, \\ I_{+-}^+(x) &:= \{ i \in I_{+-}(x) \mid \eta_i > 0 \}. \end{aligned}$$

In fact, when the lower index of I contains one sigh, the upper index of I indicates the sigh of μ_i, and when the lower index of I contains two sighs, the upper index of I indicates the sigh of η_i.

The purpose of this section is to investigate weak, strong, and strictly converse duality results for $MWD(x)$, under the following condition which is generalization of a standard assumption in all existing literature on duality for MPVCs; see, e.g., [4, 6, 10].

Definition 3.1. We say that the `Condition A` holds at $(x,u) \in \mathcal{M} \times \mathbb{R}^n$ when the following functions are ∂_c-quasiconvex at u:

$$\begin{cases} H_i, & for \quad i \in I_0^-(x), \\ G_i, & for \quad i \in I_{00}^+(x) \cup I_{0-}^+(x) \cup I_{+0}^+(x) \cup I_{+-}^+(x), \\ -H_i, & for \quad i \in I_+^+(x) \cup I_0^+(x), \\ -G_i, & for \quad i \in I_{0+}^-(x) \cup I_{00}^-(x) \cup I_{+0}^-(x). \end{cases}$$

The following theorem presents the weak duality result for $MWD(x)$.

Theorem 3.2. *(Weak Duality) Let $x \in \mathcal{M}$ be a feasible point for MPVC (1) and $(u,\mu,\eta) \in \mathcal{M}(x)$ be a feasible points for $MWD(x)$. If the* `Condition A` *holds at (x,u), and f is ∂_c-pseudoconvex at u, then*

$$f(x) \geq f(u).$$

Proof. Owing to the feasibility of $(u, \lambda^H, \lambda^G)$ for $MWD(x)$, there exist some $\xi^f \in \partial_c f(u)$, $\xi_i^H \in \partial_c H_i(u)$, and $\xi_i^G \in \partial_c G_i(u)$ as $i \in I$, such that

$$\xi^f + \sum_{i \in I} \big(-\mu_i \xi_i^H + \eta_i \xi_i^G \big) = 0_n. \tag{4}$$

Considering $i \in I$, we have $i \in I_+(x)$ or $i \in I_0(x)$.

- If $i \in I_+(x)$, according to $\mu_i \geq 0$ by (3), we have

  $$\mu_i = 0 \quad \text{or} \quad \mu_i > 0.$$

  Clearly, $\langle -\mu_i \xi_i^H, x-u \rangle = 0$ when $\mu_i = 0$, and

  $$-\mu_i H_i(x) < 0 \leq -\mu_i H_i(u),$$

  when $\mu_i > 0$, where the above relation holds by (3). In recent case, we get $i \in I_+^+(x)$ and so, $\langle -\mu_i \xi_i^H, x-u \rangle \leq 0$ by `Condition A`. Consequently,

  $$\sum_{i \in I_+(x)} \langle -\mu_i \xi_i^H, x-u \rangle \leq 0. \tag{5}$$

  Owing to `Condition A`, (3), and repeating the above argument, we deduce that

  $$\sum_{i \in I_+(x)} \langle \eta_i \xi_i^G, x-u \rangle \leq 0.$$

The above inequality and (5) imply that

$$\sum_{i\in I_+(x)} \left\langle -\mu_i\xi_i^H + \eta_i\xi_i^G, x-u\right\rangle \leq 0. \tag{6}$$

- Repeating the process for $i \in I_0(x)$, we obtain that

$$\sum_{i\in I_0(x)} \left\langle -\mu_i\xi_i^H + \eta_i\xi_i^G, x-u\right\rangle \leq 0. \tag{7}$$

Adding (6) and (7), and according to (4), we get

$$\left\langle \xi^f, x-u\right\rangle + \underbrace{\left\langle \sum_{i\in I}\big(-\mu_i\xi_i^H + \eta_i\xi_i^G\big), x-u\right\rangle}_{\leq 0} = 0 \implies \left\langle \xi^f, x-u\right\rangle \geq 0.$$

Now, the ∂_c-pseudoinvexity of f at u concludes that $f(x) \geq f(u)$, as required. □

For stating the strong duality result for $MWD(\widetilde{x})$, the following definition and theorem are required from [16, 17].

Definition 3.3. Considering $\widetilde{x} \in \mathcal{M}$, put

$$\mathcal{A}_4(\widetilde{x}) := \Big(\bigcup_{i\in I_0} \partial_c H_i(\widetilde{x})\Big) \cup \Big(\bigcup_{i\in I_{0+}} -\partial_c H_i(\widetilde{x})\Big) \cup \Big(\bigcup_{i\in I_{+0}\cup I_{00}} \partial_c G_i(\widetilde{x})\Big).$$

We say that MPVC (1) satisfies the "generalized VC_4-Abadie constraint qualification" (GVC_4-ACQ, in short) at $\widetilde{x}$, if

$$\big\{y \in \mathbb{R}^n \mid \langle \xi, y\rangle \leq 0, \quad \forall \xi \in \mathcal{A}_4(\widetilde{x})\big\} \subseteq \Gamma(\mathcal{M}, \widetilde{x}),$$

and the convex cone of $\mathcal{A}_4(\widetilde{x})$ is a closed subset of $\mathbb{R}^n$, where $\Gamma(\mathcal{M}, \widetilde{x})$ denotes the Bouligand tangent cone of $\mathcal{M}$ at $\widetilde{x}$, defined as

$$\Gamma(\mathcal{M}, \widetilde{x}) := \{y \in \mathbb{R}^n \mid \exists t_k \downarrow 0, \, \exists y_k \to y \text{ such that } \widetilde{x} + t_k y_k \in \mathcal{M}, \, \forall k \in \mathbb{N}\}.$$

Theorem 3.4. *[16, Theorem 4(iii)] and [17, Theorem 11(b)] Suppose that $\widetilde{x}$ is a local solution of MPVC (1) and GVC_4-ACQ holds at $\widetilde{x}$. Then, there exist real numbers $\widetilde{\mu}_i$ and $\widetilde{\eta}_i$ as $i \in I$ such that*

$$\begin{cases} 0_n \in \partial_c f(\widetilde{x}) + \sum_{i\in I}\big(-\widetilde{\mu}_i\partial_c H_i(\widetilde{x}) + \widetilde{\eta}_i\partial_c G_i(\widetilde{x})\big), \\ \widetilde{\mu}_i = 0, \quad i \in I_+(\widetilde{x}); \quad \widetilde{\mu}_i \geq 0, \quad i \in I_{0-}(\widetilde{x}) \cup I_{00}(\widetilde{x}), \\ \widetilde{\eta}_i = 0, \quad i \in I_{+-}(\widetilde{x}) \cup I_{0+}(\widetilde{x}) \cup I_{0-}(\widetilde{x}); \quad \widetilde{\eta}_i \geq 0, \quad i \in I_{+0}(\widetilde{x}) \cup I_{00}(\widetilde{x}). \end{cases}$$

Now, the strong duality result for $MWD(x)$ can be stated as follows.

Theorem 3.5. *(Strong Duality) Suppose that GVC_4-ACQ is satisfied at the local solution $\widetilde{x} \in \mathcal{M}$ of MPVC (1). Then, we can find some $\widetilde{\mu} \in \mathbb{R}^m$ and $\widetilde{\eta} \in \mathbb{R}^m$ such that $(\widetilde{x}, \widetilde{\mu}, \widetilde{\eta}) \in \mathcal{M}(\widetilde{x})$. If, in addition, the* `Condition A` *holds at $(\widetilde{x}, \widetilde{x})$, and f is ∂_c-pseudoconvex at $\widetilde{x}$, then $(\widetilde{x}, \widetilde{\mu}, \widetilde{\eta})$ is a global solution for $MWD(\widetilde{x})$.*

Proof. According to Theorem 3.4, there exist some coefficients $\widetilde{\mu}_i$ and $\widetilde{\eta}_i$ as $i \in I$ such that

$$\begin{cases} 0 \in \partial_c f(\widetilde{x}) + \sum_{i \in I} \big(- \widetilde{\mu}_i \partial_c H_i(\widetilde{x}) + \widetilde{\eta}_i \partial_c G_i(\widetilde{x}) \big), \\ \widetilde{\mu}_i = 0, \quad i \in I_+(\widetilde{x}); \quad \widetilde{\mu}_i \geq 0, \quad i \in I_{0-}(\widetilde{x}) \cup I_{00}(\widetilde{x}), \\ \widetilde{\eta}_i = 0, \quad i \in I_{+-}(\widetilde{x}) \cup I_{0+}(\widetilde{x}) \cup I_{0-}(\widetilde{x}); \quad \widetilde{\eta}_i \geq 0, \quad i \in I_{+0}(\widetilde{x}) \cup I_{00}(\widetilde{x}). \end{cases} \tag{8}$$

Consequently $(\widetilde{x}, \widetilde{\mu}, \widetilde{\eta})$ is a feasible point for $MWD(\widetilde{x})$, in which $\widetilde{\mu} = (\widetilde{\mu}_1, \dots, \widetilde{\mu}_m)$ and $\widetilde{\eta} = (\widetilde{\eta}_1, \dots, \widetilde{\eta}_m)$. Owing to the weak duality result (Theorem 3.2), we have

$$f(\widetilde{x}) \geq f(u), \quad \forall (u, \mu, \eta) \in \mathcal{M}(\widetilde{x}),$$

and according to $(\widetilde{x}, \widetilde{\mu}, \widetilde{\eta}) \in \mathcal{M}(\widetilde{x})$, we conclude that $(\widetilde{x}, \widetilde{\mu}, \widetilde{\eta})$ is a global solution for the dual problem $MWD(\widetilde{x})$. ☐ Clearly, we can replace GVC_4-ACQ in Theorem 3.5 with some stronger constraint qualifications, defined in [13, 14, 16, 17].

Since (8) states that $\widetilde{\mu}_i = 0$ for $i \in I_+(\widetilde{x})$ and $\widetilde{\eta}_i = 0$ for $i \in I_{+-}(\widetilde{x}) \cup I_{0+}(\widetilde{x}) \cup I_{0-}(\widetilde{x})$, we can replace the `Condition A` with the following weaker condition:

Definition 3.6. We say that the `Condition B` holds at $(x, u) \in \mathcal{M} \times \mathbb{R}^n$ when the following functions are ∂_c-quasiconvex at u:

$$\begin{cases} H_i, & for \quad i \in I_0^-(x), \\ G_i, & for \quad i \in I_{00}^+(x) \cup I_{+0}^+(x), \\ -H_i, & for \quad i \in I_0^+(x), \\ -G_i, & for \quad i \in I_{00}^-(x) \cup I_{+0}^-(x). \end{cases}$$

It should be noted that if the functions H_i and G_i as $i \in I$ are linear, the `Conditions A and B` are automatically satisfied at all feasible points of MPVC (1).

Owing to the mentioned points, we can improve the strong duality result (Theorem 3.5) as follows.

Corollary 3.7. *Suppose that GVC_4-ACQ is satisfied at the local solution $\widetilde{x} \in \mathcal{M}$ of MPVC (1). Then, we can find some $\widetilde{\mu} \in \mathbb{R}^m$ and $\widetilde{\eta} \in \mathbb{R}^m$ such that $(\widetilde{x}, \widetilde{\mu}, \widetilde{\eta}) \in \mathcal{M}(\widetilde{x})$. If, in addition, the* `Condition B` *holds at $(\widetilde{x}, \widetilde{x})$, and f is ∂_c-pseudoconvex at $\widetilde{x}$, then $(\widetilde{x}, \widetilde{\mu}, \widetilde{\eta})$ is a global solution for $MWD(\widetilde{x})$.*

For stating the next result, the following condition which is stronger than `Condition A` is required.

Definition 3.8. We say that the `Condition C` holds at $u \in \mathbb{R}^n$ when the `Condition A` holds at all $(x, u) \in \mathcal{M} \times \mathbb{R}^n$. In the other word, the `Condition C` is satisfied at $u \in \mathbb{R}^n$ if the following functions are ∂_c-quasiconvex:

$$\begin{cases} H_i, & for \quad i \in \bigcup_{x \in \mathcal{M}} I_0^-(x), \\ G_i, & for \quad i \in \bigcup_{x \in \mathcal{M}} \left(I_{00}^+(x) \cup I_{+0}^+(x)\right), \\ -H_i, & for \quad i \in \bigcup_{x \in \mathcal{M}} I_0^+(x), \\ -G_i, & for \quad i \in \bigcup_{x \in \mathcal{M}} \left(I_{00}^-(x) \cup I_{+0}^-(x)\right). \end{cases}$$

Remark 3.9. Since it may find some $i^* \in I_0^-(x_1) \cap I_0^+(x_2)$ for two feasible points $x_1, x_2 \in \mathcal{M}$, the ∂_c-quasiconvexity of H_i and $-H_i$ as respectively $i \in \bigcup_{x \in \mathcal{M}} I_0^-(x)$ and $\bigcup_{x \in \mathcal{M}} I_0^+(x)$ at $u \in \mathbb{R}^n$ conclude that H_{i^*} is ∂_c-quasilinear at u. Thus, an important special case where the `Condition C` is satisfied at $u \in \mathbb{R}^n$ is the case that all constraints functions H_i and G_i as $i \in I$ are ∂_c-quasilinear at u.

The following theorem shows under what assumptions the feasible point $\widetilde{x} \in \mathcal{M}$ is an optimal solution for MPVC (1).

Theorem 3.10. *(Converse Duality) Let $(\widetilde{u}, \widetilde{\mu}, \widetilde{\eta}) \in \bigcap_{x \in \mathcal{M}} \mathcal{M}(x)$ be given. If the objective function f is ∂_c-pseudoinvex at $\widetilde{u}$ and the* ***`Condition C`*** *holds at $\widetilde{u}$, then $\widetilde{u}$ is a global solution for the MPVC (1).*

Proof. The feasibility of $(\widetilde{u}, \widetilde{\mu}, \widetilde{\eta})$ for $MWD(x)$ implies that

$$\widetilde{\xi}^f + \sum_{i\in I} \big(- \widetilde{\mu}_i \widetilde{\xi}_i^H + \widetilde{\eta}_i \widetilde{\xi}_i^G \big) = 0_n, \tag{9}$$

for some $\widetilde{\xi}^f \in \partial_c f(\widetilde{u})$, $\widetilde{\xi}_i^H \in \partial_c H_i(\widetilde{u})$, and $\widetilde{\xi}_i^G \in \partial_c G_i(\widetilde{u})$. Suppose that $\widetilde{x} \in \mathcal{M}$ is arbitrarily given. Since $(\widetilde{u}, \widetilde{\mu}, \widetilde{\eta}) \in \mathcal{M}(\widetilde{x})$, we have $-\widetilde{\mu}_i H_i(\widetilde{u}) \geq 0$ and $\widetilde{\eta}_i G_i(\widetilde{u}) \geq 0$ for all $i \in I$. From this, the feasibility of $\widetilde{x}$ for MPVC (1), and $(\widetilde{u}, \widetilde{\mu}, \widetilde{\eta}) \in \mathcal{M}(\widetilde{x})$, we give:

$$\begin{cases} \widetilde{\mu}_i\big(- H_i(\widetilde{x})\big) \leq 0 \leq -\widetilde{\mu}_i\big(- H_i(\widetilde{u})\big), & \forall i \in I_+^+(\widetilde{x}) \cup I_0^+(\widetilde{x}), \\ (-\widetilde{\mu}_i) H_i(\widetilde{x}) \leq 0 \leq (-\widetilde{\mu}_i) H_i(\widetilde{u}), & \forall i \in I_0^-(\widetilde{x}), \\ \widetilde{\eta}_i G_i(\widetilde{x}) \leq 0 \leq \widetilde{\eta}_i G_i(\widetilde{u}), & \forall i \in I_{00}^+(\widetilde{x}) \cup I_{0-}^+(\widetilde{x}) \cup I_{+0}^+(\widetilde{x}) \cup I_{+-}^+(\widetilde{x}), \\ (-\widetilde{\eta}_i)\big(- G_i(\widetilde{x})\big) \leq 0 \leq (-\widetilde{\eta}_i)\big(- G_i(\widetilde{u})\big), & \forall i \in I_{0+}^-(\widetilde{x}) \cup I_{00}^-(\widetilde{x}) \cup I_{+0}^-(\widetilde{x}). \end{cases}$$

We note that the fulfillment of `Condition C` at $\widetilde{u}$ implies the fulfillment of `Condition A` at $(\widetilde{x}, \widetilde{u})$. Thus, the above inequalities and the `Condition C` at $\widetilde{u}$ deduce that for all $\xi_i^H \in \partial_c H_i(\widetilde{u})$ and $\xi_i^G \in \partial_c G_i(\widetilde{u})$ as $i \in I$, one has

$$\Big\langle \sum_{i\in I} \big(- \widetilde{\mu}_i \xi_i^H + \widetilde{\eta}_i \xi_i^G \big), \widetilde{x} - \widetilde{u} \big) \Big\rangle \leq 0.$$

This equality and (9) yield $\langle \widetilde{\xi}^f, \widetilde{x} - \widetilde{u} \rangle \geq 0$, and hence, the ∂_c-pseudoinvexity of f at $\widetilde{u}$ concludes that

$$f(\widetilde{x}) \geq f(\widetilde{u}).$$

Since $\widetilde{x}$ was an arbitrary element of $\mathcal{M}$, the last inequality shows that $\widetilde{u}$ is a global solution for MPVC (1), and the proof is complete. □

In the following theorem, a sufficient condition for the optimality of a feasible point of MPVC (1) is proven.

Theorem 3.11. *(Restricted Converse Duality) Assume that $(\widetilde{u}, \widetilde{\mu}, \widetilde{\eta}) \in \bigcap_{x\in\mathcal{M}} \mathcal{M}(x)$ and there exists $\widetilde{x} \in \mathcal{M}$ such that $f(\widetilde{x}) = f(\widetilde{u})$. If the objective function f is ∂_c-pseudoinvex at $\widetilde{u}$ and the `Condition C` holds at $\widetilde{u}$, then $\widetilde{x}$ is a global solution for the MPVC (1).*

Proof. Suppose on the contrary that $\widetilde{x}$ is not a global solution for MPVC (1). Thus, there exists $x^* \in \mathcal{M}$ such that $f(x^*) < f(\widetilde{x})$. From this the assumption of theorem we obtain that

$$f(x^*) < f(\widetilde{u}). \tag{10}$$

Since $(\widetilde{u}, \widetilde{\mu}, \widetilde{\eta}) \in \mathcal{M}(x^*)$ and the `Condition A` holds at $(x^*, \widetilde{u})$, the weak duality Theorem 3.2 for $MWD(x^*)$ concludes that

$$f(\widetilde{u}) \leq f(x^*).$$

The last inequality contradicts (10), and the proof is complete. □

The following theorem obtains the condition for uniqueness of solutions of MPVC (1) and $MWD(\widetilde{x})$.

Theorem 3.12. *(Strict Converse Duality) Suppose that $\widetilde{x} \in \mathcal{M}$ is a local solution for MPVC (1), $(\widetilde{u}, \widetilde{\mu}, \widetilde{\eta}) \in \mathcal{M}(\widetilde{x})$ is a global solution for $MWD(\widetilde{x})$, and GVC_4-ACQ holds at $\widetilde{x}$. If the objective function f is ∂_c-strictly pseudoinvex at $\widetilde{u}$, and the `Condition A` holds $(\widetilde{x}, \widetilde{u})$, then*

$$\widetilde{x} = \widetilde{u}.$$

Proof. On the contrary, suppose that $\widetilde{x} \neq \widetilde{u}$. Owing to $(\widetilde{u}, \widetilde{\mu}, \widetilde{\eta}) \in \mathcal{M}(\widetilde{x})$, we have

$$\xi^f + \sum_{i \in I} \big(-\widetilde{\mu}_i {\xi_i}^H + \widetilde{\eta}_i {\xi_i}^G \big) = 0_n, \tag{11}$$

for some $\xi^f \in \partial_c f(\widetilde{u})$, $\xi_i^H \in \partial_c H_i(\widetilde{u})$, and $\xi_i^G \in \partial_c G_i(\widetilde{u})$. Repeating process of proof of Theorem 3.2, we conclude that

$$\left\langle \sum_{i \in I} \big(-\widetilde{\mu}_i \xi_i^H + \widetilde{\eta}_i \xi_i^G \big), \widetilde{x} - \widetilde{u} \right\rangle \leq 0.$$

The last inequality and (11) deduce that $\langle \xi^f, \widetilde{x} - \widetilde{u} \rangle \geq 0$, so the ∂_c-strictly pseudoinvexity of f at $\widetilde{u}$ obtains that

$$f(\widetilde{u}) < f(\widetilde{x}). \tag{12}$$

On the other hand, employing the strong duality Theorem 3.5, there exist some vectors $\mu^* \in \mathbb{R}^m$ and $\eta^* \in \mathbb{R}^m$ such that $(\widetilde{x}, \mu^*, \eta^*) \in \mathcal{M}(\widetilde{x})$ is a global solution for the problem $MWD(\widetilde{x})$. This is a contradiction with (12), because it states that the objective function of $MWD(\widetilde{x})$ has two different values at its two global solutions. □

4 Conclusion

In this paper we introduced a Mond-Weir type dual problem for nonsmooth mathematical optimization problem with vanishing constraints, and then, we present the weak, strong, converse, restricted converse, and strict converse duality results for this dual problem. The results are based on Clarke subdifferential.

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