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Original Research Paper

A Study on Harmonic Univalent Function with (p, q)-Calculus and Introducing (p, q)-Poisson Distribution Series

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Abstract. Looking at the history of fractional derivatives, it can be clearly seen that various generalizations have been presented for it regularly by researchers. Perhaps, in the meantime, the derivative of the q-fraction has received more attention due to the provision of discrete space and the entry of the computer into the computing scene. But recently, a new generalization has been presented for the q-derivative, namely (p, q)-derivative. In this research, we intend to define the (p, q)-Poisson Distribution of harmonic functions by using (p, q)-derivatives. Some numerical result provided for (p, q)-analogue to make our presentation more objective. Also, we gave (p, q)-analogue of exponential function, then we use it to express the (p, q)-Poisson Distribution. By that, we will check the conditions of Poisson Distribution for two subclasses of harmonic univalent functions.

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1 Introduction

As we know, the history of calculus is very noisy, fascinating and full of interesting generalizations. The concept of derivative as a tangent line has a very ancient history. This concept has undergone many changes since the time of the work of ancient Greek mathematicians to its current form. It may be a little unbelievable to say that we owe calculus and differential equations to the plague. The first foundations of calculus were laid by Newton during the years 1665 and 1666, when he was staying on his family farm to avoid the plague. Later, in the 19th century, with the presentation of the concept of limit and continuity by Cauchy, Newton's discoveries got a solid foundation, and thus calculus became an integral part of mathematics, physics, modeling, etc. Although the calculus is still safe and sound, but the matter did not end there and its most basic concepts, such as the derivative, have undergone changes and transformations today.

Needless to say, by examining the researches and articles published in the last decade in the field of science and engineering, we are more or less dealing with the subject of fractional calculus, fractional differential equations and concepts of this type. Today, fractional calculus plays a main and practical role in all mathematical modeling, engineering, physics, thermodynamics, etc. researches. The efforts of Podlubani and Kielbas led to the significant development and progress of fractional derivatives [30, 40]. Among the studies published in this regard, the following can be mentioned: transmission of infectious diseases[22], tumor growth[23], epidemics of hepatitis B[14, 15, 16], fractional preypredator model[24], discrete fractional order model for thermostat [7] (for more contributions see [8, 9, 12, 13, 31, 35, 36, 50]).

Another very famous generalization of the derivative concept is related to the works of the English mathematician Frank Hilton. In 1910, he removed the limit from the definition of the derivative and introduced the derivative known as the quantum derivative or q-derivative [27, 28]. His work led to the creation of quantum calculus and quickly covered the field of mathematics and physics and led to extensive research in this field. In the last decade, many articles have been published by different researchers in this field by combining fractional and quantum calculus(see [1, 2, 3, 17, 18, 19, 20, 34, 46, 47]).

In 2004, Remmel and Wachs, while researching the Stirling number, generalized the quantum derivative by taking the idea from Jackson's works and introduced the concept of (p, q)-derivative [43]. Later (p, q)-analogue for Bernestain operators were investigated by Mursaleen et al. [37]. Finally, in 2018, the basic theorem of (p, q)-calculus was presented by Sadjang [44]. With the introduction of the (p, q)-derivative in 2020 to fractional calculus, investigating the properties of this type of derivative has attracted the attention of researchers and numerous articles have been published in this field. one can see ([25, 41, 42, 45, 49]) for more details. Now, in this research, we want to use the basic concepts of (p, q)-derivative to combine it with statistics and analysis of complex functions and take advantage of its potential.

In 1990, Ísmail et al. investigated properties of the class of complex functions such that were analytic on the unit disk by using q-derivative [26]. Later, in 2018, Jahangiri applied the q-operator to complex harmonic functions for the first time to obtain sharp coefficient bound and distortion theorems [29]. Let $\mathfrak{D} = \{\kappa : \kappa \in \mathbb{C} \quad and \quad |\kappa| < 1\}$ be the unit disk and denote the class of analytical function on $\mathfrak{D}$ by $\mathfrak{B}$. A function harmonic in $\mathfrak{D}$ may be written as $\omega = \boldsymbol{h} + \bar{u}$ where $\boldsymbol{h}$ and u analytic in $\mathfrak{D}$. $\boldsymbol{h}$ called analytic and u called co-analytic part of ω. Now consider the complex valued harmonic function $\omega = \boldsymbol{h} + \bar{u}$, which $\boldsymbol{h}(0) = \boldsymbol{h'}(0) - 1 = 0$ with the following power series formulas

$$\boldsymbol{h}(\kappa) = \kappa + \sum_{n=2}^{\infty} a_n \kappa^n, \qquad and \qquad u(\kappa) = \sum_{n=1}^{\infty} b_n \kappa^n. \tag{1}$$

Then, following Clunie and Sheil-Small [11], denote the class of such functions which are harmonic, univalent and sense-preserving by $\mathcal{SH}$. Obviously, the sense-preserving feature means $|b_1| < 1$. Clunie and Sheil-Small in their study by using Lewy's Theorem (see [33]) showed that the sufficient condition for the harmonic function $\omega = \boldsymbol{h} + \bar{u}$ to be locally univalent and sense-preserving in $\mathfrak{D}$, is that

$$|v| = \left|\frac{u'}{\boldsymbol{h'}}\right| < 1,$$

which v is the second dilatation of ω, and $\boldsymbol{h'}(\kappa) \neq 0$.

Although many random phenomena in nature seem random, they have a specific pattern. Statistical distributions based on the laws of probability, try to show the characteristics of random phenomena and give us information about them. It is obvious that statistical distributions have been investigated from different aspects due to their various applications in describing random phenomena [5, 6, 21, 32, 38, 39, 51]. One of the most important statistical distributions that are widely used in the business is the Poisson Distribution. A variable x is said to (p,q)-Poisson Distribution if it takes the values 0,1,2,3,... with probabilities $e^r_{p,q}$, $\frac{r}{[1]_{p,q}!}e^r_{p,q}$, $\frac{r^2}{[2]_{p,q}!}e^r_{p,q}$, , , respectively.

In this study, we will introduce two functions by using $\mathcal{P}^{m,c}_{p,q}$ and $\mathcal{TP}^{m,c}_{p,q}$ (p,q) with Poisson Distribution. We aim to find the conditions for these functions to belong to the classes (p,q)-harmonic functions.

2 Preliminaries

Definition 2.1. [49] Let $\kappa \in \mathbb{R}$, and $p, q \in (0,1)$. Then the generalized quantum analogue of κ, namely $[\kappa]_{p,q}$, expressed as follows:

$$[\kappa]_{p,q} = \frac{p^\kappa - q^\kappa}{p-q} \qquad (\kappa = 1,2,3,...), \qquad [0]_{p,q} = 0, \qquad [1]_{p,q} = 1.$$

Note that, if we take $p = 1$, then we have the q-analogue of κ, which defined as follows

$$[\kappa]_q = \frac{1-q^\kappa}{1-q} \qquad (\kappa = 1,2,3,...) \qquad q \neq 1.$$

Definition 2.2. [49] For a natural number m, the (p,q)-analogue of factorial function $m!$, defined as follows

$$[m]_{p,q}! = [1]_{p,q} \times [2]_{p,q} \times ... \times [m]_{p,q}.$$

Using MATLAB Package (R2018b), we computed (p, q)-analogue value for some p and q values which $\kappa = 1, 2, 3, \ldots, 9, 10$, and presented the results in Table 1.

$[\kappa]_{p,q}$	$p = 0.1, q = 0.2$	$p = 0.3, q = 0.45$	$p = 0.5, q = 0.35$	$p = 0.92, q = 0.72$
$[1]_{p,q}$	1	1	1	1
$[2]_{p,q}$	0.3000	0.7500	0.8500	1.6600
$[3]_{p,q}$	0.0700	0.4250	0.5475	2.0748
$[4]_{p,q}$	0.0150	0.2194	0.3166	2.3140
$[5]_{p,q}$	0.0031	0.1068	0.1733	2.4288
$[6]_{p,q}$	6.3×10^{-4}	0.0505	0.0919	2.4564
$[7]_{p,q}$	1.27×10^{-4}	0.0235	0.0478	2.4241
$[8]_{p,q}$	2.55×10^{-5}	0.0108	0.0245	2.3517
$[9]_{p,q}$	5.11×10^{-6}	0.0049	0.0125	2.2534
$[10]_{p,q}$	1.023×10^{-6}	0.0022	0.0063	2.1397

Table 1: Numerical results for (p, q)-analogue in definition 2.1.

Definition 2.3. It is obvious that the (p, q)-analogue of exponential function e^m, formulated as follows

$$e^m_{p,q} = 1 + \frac{m}{[1]_{p,q}!} + \frac{m^2}{[2]_{p,q}!} + \ldots + \frac{m^n}{[n]_{p,q}!} + \ldots \quad = \sum_{n=0}^{\infty} \frac{m^n}{[n]_{p,q}!}. \tag{2}$$

Therefore we can express the (p, q)-Poisson Distribution by

$$\mathcal{P}^m_{p,q}(n, m) = \frac{m^n}{[n]_{p,q}!} e^{-m}_{p,q}.$$

Definition 2.4. In view of definition of (p, q)-Poisson Distribution, we introduce the following power series

$$\mathcal{P}^m_{p,q}(\kappa) = \kappa + \sum_{n=2}^{\infty} \frac{m^{n-1} e^{-m}_{p,q}}{[n-1]_{p,q}!} \kappa^n,$$

such that their coefficients are probabilities of the (p, q)-Poisson Distribution.

Definition 2.5. [49] Let $\boldsymbol{h}$ be a real valued function which defined on $[0,\mathcal{L}]$, then the (p,q)-derivative of $\boldsymbol{h}$ formulated as follows

$$D_{p,q}\boldsymbol{h}(\kappa)=\frac{\boldsymbol{h}(p\kappa)-\boldsymbol{h}(q\kappa)}{(p-q)\kappa},\qquad p\neq q\qquad \kappa\in[0,\mathcal{L}].$$

where, $D_{p,q}\boldsymbol{h}(0)=\boldsymbol{h'}(0)$.

Remark 2.6. According to the definition 2.5, for functions $\boldsymbol{h}$ and u in (1), we obtain

$$D_{p,q}\boldsymbol{h}(\kappa)=1+\sum_{n=2}^{\infty}[n]_{p,q}a_n\kappa^{n-1},$$

and

$$D_{p,q}u(\kappa)=\sum_{n=1}^{\infty}[n]_{p,q}b_n\kappa^{n-1}.$$

Definition 2.7. Let $\omega=\boldsymbol{h}+\bar{u}$ be a harmonic function, as defined in (1), we called this function (p,q)-harmonic, locally univalent and sense-preserving if and only if following inequality holds true

$$|v_{p,q}(z)|=\left|\frac{D_{p,q}u(\kappa)}{D_{p,q}\boldsymbol{h}(\kappa)}\right|<1.$$

Also, $\mathcal{SH}_{p,q}$ represents the class of such functions.

Remark 2.8. [4] Note that as $p\to 1^-$ and $q\to 1^-$, $\mathcal{SH}_{p,q}$ reduces to $\mathcal{SH}$.

Now, for $\omega=\boldsymbol{h}+\bar{u}\in\mathcal{SH}_{p,q}$, we define a new class $\mathcal{SH}_{p,q}(\gamma,\lambda,\beta)$ as follows.

Definition 2.9. [48] Let $\omega=\boldsymbol{h}+\bar{u}\in\mathcal{SH}_{p,q}$. Then $\omega\in\mathcal{SH}_{p,q}(\gamma,\lambda,\beta)$ if

$$\Re\left[1+\frac{1}{\gamma}\left\{(1-\lambda)\frac{\omega(\kappa)}{\kappa}+\lambda(\partial_{p,q}\omega(\kappa))-1\right\}\right]\geq\beta$$

where $0\leq\beta<1$, $\gamma\in\mathbb{C}-\{0\}$ and $0\leq\lambda\leq 1$.

Definition 2.10. We define $\mathcal{TSH}_{p,q}(\gamma,\lambda,\beta)=\mathcal{SH}_{p,q}(\gamma,\lambda,\beta)\cap\mathcal{T}$, where $\mathcal{T}$ denote the subclass of $\mathcal{SH}$ consisting of functions of the type $\omega(\kappa)=\boldsymbol{h}(\kappa)+\bar{u}(\kappa)$, where

$$\boldsymbol{h}(\kappa)=\kappa-\sum_{n=2}^{\infty}|a_n|\kappa^n,$$

and

$$u(\kappa)=\sum_{n=1}^{\infty}|b_n|\kappa^n,\qquad |b_1|<1. \tag{3}$$

We say that an analytic function ϕ is subordinate to an analytic function ψ and write $\phi\prec\psi$, if there exists a complex-valued function ω which maps $\mathfrak{D}$ onto itself with $\omega(0)=0$, such that $\phi(\kappa)=\psi(\omega(\kappa))$ $(\kappa\in\mathfrak{D})$. Furthermore, if the function ψ is univalent in $\mathfrak{D}$, then we have the following equivalence

$$\phi(\kappa)\prec\psi(\kappa)\Longleftrightarrow\phi(0)=\psi(0)\quad\textit{and}\quad\phi(\kappa)\subset\psi(\kappa).$$

Definition 2.11. [10] Let $\omega=\boldsymbol{h}+\bar{u}\in\mathcal{SH}_{p,q}$. Then $\omega\in\mathcal{SH}_{p,q}(\gamma,\mu,\nu)$ if

$$\frac{\kappa D_{p,q}\boldsymbol{h}(\kappa)-\overline{\kappa D_{p,q}u(\kappa)}}{\gamma(\kappa D_{p,q}\boldsymbol{h}(\kappa)-\overline{\kappa D_{p,q}u(\kappa)})+(1-\gamma)(\boldsymbol{h}(\kappa)+\overline{u(\kappa)})}\prec\frac{1+\mu\kappa}{1+\nu\kappa},$$

such that $-1\le\nu<0$, $0<\mu\le1$, $0\le\gamma<1$.

Lemma 2.12. *[48] Let $\omega=\boldsymbol{h}+\bar{u}$, satisfies following inequality*

$$\sum_{n=2}^{\infty}\{\{1+([n]_{p,q}-1)\lambda\}\left[|a_n|+|b_n|\right]\}\le(1-\beta)\gamma, \tag{4}$$

where $\beta\in[0,1)$, $\gamma\in\mathbb{C}\backslash0$, $\lambda\in[0,1]$. Then $\omega\in\mathcal{SH}_{p,q}(\gamma,\lambda,\beta)$.

Remark 2.13. Let $\omega=\boldsymbol{h}+\bar{u}$ be given by (3). Then $\omega\in\mathcal{TSH}_{p,q}(\gamma,\lambda,\beta)$ if and only if the coefficient condition (4) is satisfied.

Lemma 2.14. *[10] Let $\omega=\boldsymbol{h}+\bar{u}$ be of the form in (1). If the following inequality holds true*

$$\sum_{n=2}^{\infty}(\varphi_n|a_n|+\psi_n|b_n|)\le\mu-\nu \tag{5}$$

which

$$\varphi_n = (\mu\gamma - \nu)[n]_{p,q} + (1-\gamma)([n]_{p,q} - 1 + \mu),$$

$$\psi_n = (\mu\gamma - \nu)[n]_{p,q} + (1-\gamma)([n]_{p,q} + 1 - \mu),$$

and $-1 \leq \nu \leq 0, \quad 0 < \mu \leq 1, \quad 0 \leq \gamma < 1$. *Then* $\omega \in \mathcal{SH}_{p,q}(\gamma,\mu,\nu)$.

Remark 2.15. Let $\omega = \boldsymbol{h} + \bar{u}$ be given by (3). Then $\omega \in \mathcal{TSH}_{p,q}(\gamma,\mu,\nu)$ if and only if the coefficient condition (5) is satisfied.

3 Main Results

Now, in light of the discussions raised, we are expressing our main results. Firstly, let us in view of definition 2.4, introduce harmonic functions $\mathcal{P}^{m,c}_{p,q}$ and $\mathcal{TP}^{m,c}_{p,q}$ as follows

$$\mathcal{P}^{m,c}_{p,q}(\kappa) = \mathcal{P}^{m}_{p,q}(\kappa) + \overline{\mathcal{P}^{c}_{p,q}(\kappa) - \kappa}$$

and

$$\mathcal{TP}^{m,c}_{p,q} = 2\kappa - \mathcal{P}^{m}_{p,q}(\kappa) + \overline{\mathcal{P}^{c}_{p,q}(\kappa) - \kappa}. \tag{6}$$

Theorem 3.1. *Let* $m, c > 0$, $p, q \in (0,1)$, $0 \leq \beta < 1$, $0 \leq \lambda \leq 1$, $\gamma \in \mathbb{C} - \{0\}$ *and the following inequality*

$$\begin{aligned}
&\lambda(\frac{p+q}{2})(m+c) + \frac{\lambda}{2}e_{p,q}^{-m(1-p)} + \frac{\lambda}{2}e_{p,q}^{-m(1-q)} + \frac{\lambda}{2}e_{p,q}^{-c(1-p)} + \frac{\lambda}{2}e_{p,q}^{-c(1-q)} \\
&\leq (1-\beta)\gamma + e_{p,q}^{-m} + e_{p,q}^{-c} + 2(\lambda - 1),
\end{aligned} \tag{7}$$

is satisfied then $\mathcal{P}^{m,c}_{p,q} \in \mathcal{SH}_{p,q}(\gamma,\lambda,\beta)$.

Proof. Referring Lemma 2.12, it is sufficient to show that the following inequality

$$\sum_{n=2}^{\infty}\left\{\{1+([n]_{p,q}-1)\lambda\}\frac{m^{n-1}e_{p,q}^{-m}}{[n-1]_{p,q}!}\right\} + \sum_{n=2}^{\infty}\left\{\{1+([n]_{p,q}-1)\lambda\}\frac{c^{n-1}e_{p,q}^{-c}}{[n-1]_{p,q}!}\right\} \leq (1-\beta)\gamma \tag{8}$$

holds true. To do this, we shall show that the function

$$\mathcal{P}^{m}_{p,q}(\kappa) + \overline{\mathcal{P}^{c}_{p,q}(\kappa) - \kappa} = \mathcal{P}^{m,c}_{p,q}(\kappa) \in \mathcal{SH}_{p,q}(\gamma,\lambda,\beta). \tag{9}$$

By using (2), we obtain

$$\sum_{n=2}^{\infty}\left\{\{1+([n]_{p,q}-1)\lambda\}\frac{m^{n-1}e_{p,q}^{-m}}{[n-1]_{p,q}!}+\{1+([n]_{p,q}-1)\lambda\}\frac{c^{n-1}e_{p,q}^{-c}}{[n-1]_{p,q}!}\right\}$$
$$=\sum_{n=2}^{\infty}\left\{1+\left((\frac{p+q}{2})[n-1]_{p,q}+\frac{p^{n-1}+q^{n-1}}{2}-1\right)\lambda\right\}\frac{m^{n-1}e_{p,q}^{-m}}{[n-1]_{p,q}!}$$
$$+\sum_{n=2}^{\infty}\left\{1+\left((\frac{p+q}{2})[n-1]_{p,q}+\frac{p^{n-1}+q^{n-1}}{2}-1\right)\lambda\right\}\frac{c^{n-1}e_{q}^{-c}}{[n-1]_{p,q}!}$$
$$=\sum_{n=2}^{\infty}\frac{m^{n-1}e_{p,q}^{-m}}{[n-1]_{p,q}!}+\lambda(\frac{p+q}{2})\sum_{n=2}^{\infty}\frac{m^{n-1}e_{p,q}^{-m}}{[n-2]_{p,q}!}+\frac{\lambda}{2}\sum_{n=2}^{\infty}\frac{p^{n-1}m^{n-1}e_{p,q}^{-m}}{[n-1]_{p,q}!}$$
$$+\frac{\lambda}{2}\sum_{n=2}^{\infty}\frac{q^{n-1}m^{n-1}e_{p,q}^{-m}}{[n-1]_{p,q}!}-\lambda\sum_{n=2}^{\infty}\frac{m^{n-1}e_{p,q}^{-m}}{[n-1]_{p,q}!}+\sum_{n=2}^{\infty}\frac{c^{n-1}e_{p,q}^{-c}}{[n-1]_{p,q}!}+\lambda(\frac{p+q}{2})\sum_{n=2}^{\infty}\frac{c^{n-1}e_{p,q}^{-c}}{[n-2]_{p,q}!}$$
$$+\frac{\lambda}{2}\sum_{n=2}^{\infty}\frac{p^{n-1}c^{n-1}e_{p,q}^{-c}}{[n-1]_{p,q}!}+\frac{\lambda}{2}\sum_{n=2}^{\infty}\frac{q^{n-1}c^{n-1}e_{p,q}^{-c}}{[n-1]_{p,q}!}-\lambda\sum_{n=2}^{\infty}\frac{c^{n-1}e_{p,q}^{-c}}{[n-1]_{p,q}!}$$
$$=e_{p,q}^{-m}\sum_{n=1}^{\infty}\frac{m^{n}}{[n]_{p,q}!}+\lambda me_{p,q}^{-m}(\frac{p+q}{2})\sum_{n=0}^{\infty}\frac{m^{n}}{[n]_{p,q}!}+\frac{\lambda}{2}e_{p,q}^{-m}\sum_{n=1}^{\infty}\frac{p^{n}m^{n}}{[n]_{p,q}!}+\frac{\lambda}{2}e_{p,q}^{-m}\sum_{n=1}^{\infty}\frac{q^{n}m^{n}}{[n]_{p,q}!}$$

$$-\lambda e_{p,q}^{-m}\sum_{n=1}^{\infty}\frac{m^{n}}{[n]_{p,q}!}+e_{p,q}^{-c}\sum_{n=1}^{\infty}\frac{c^{n}}{[n]_{p,q}!}+\lambda ce_{p,q}^{-c}(\frac{p+q}{2})\sum_{n=0}^{\infty}\frac{c^{n}}{[n]_{p,q}!}+\frac{\lambda}{2}e_{p,q}^{-c}\sum_{n=1}^{\infty}\frac{p^{n}c^{n}}{[n]_{p,q}!}$$
$$+\frac{\lambda}{2}e_{p,q}^{-c}\sum_{n=1}^{\infty}\frac{q^{n}c^{n}}{[n]_{p,q}!}-\lambda e_{p,q}^{-c}\sum_{n=1}^{\infty}\frac{c^{n}}{[n]_{p,q}!}$$
$$=e_{p,q}^{-m}(e_{p,q}^{m}-1)+\lambda(\frac{p+q}{2})me_{p,q}^{-m}e_{p,q}^{m}+\frac{\lambda}{2}e_{p,q}^{-m}(e_{p,q}^{pm}-1)+\frac{\lambda}{2}e_{p,q}^{-m}(e_{p,q}^{qm}-1)$$
$$-\lambda e_{p,q}^{-m}(e_{p,q}^{m}-1)+e_{p,q}^{-c}(e_{p,q}^{c}-1)+\lambda(\frac{p+q}{2})ce_{p,q}^{-c}e_{p,q}^{c}+\frac{\lambda}{2}e_{p,q}^{-c}(e_{p,q}^{pc}-1)$$
$$+\frac{\lambda}{2}e_{p,q}^{-c}(e_{p,q}^{qc}-1)-\lambda e_{p,q}^{-c}(e_{p,q}^{c}-1).$$

Therefore, inequality (7) holds true if

$$\lambda(\frac{p+q}{2})(m+c)+\frac{\lambda}{2}e_{p,q}^{-m(1-p)}+\frac{\lambda}{2}e_{p,q}^{-m(1-q)}+\frac{\lambda}{2}e_{p,q}^{-c(1-p)}+\frac{\lambda}{2}e_{p,q}^{-c(1-q)}$$
$$\leq(1-\beta)\gamma+e_{p,q}^{-c}+e_{p,q}^{-m}-2+2\lambda,$$

which is equivalent to (8). This completes the proof. □

Corollary 3.2. *Let $m,c>0$, $0\leq\beta<1$, $0\leq\lambda\leq 1$, $\gamma\in\mathbb{C}-\{0\}$ then the function $\mathcal{TP}_{p,q}^{m,c}$ defined by (6) belongs to the class $\mathcal{TSH}_{p,q}(\gamma,\lambda,\beta)$ if and only if satisfied inequality (8).*

Theorem 3.3. *Let $m, c > 0$, $p, q \in (0,1)$, $-1 \leq \nu \leq 0$, $0 < \mu \leq 1$, $0 \leq \gamma < 1$, $\gamma \in \mathbb{C} - \{0\}$, and the following inequality*

$$\begin{aligned}
&(\mu\gamma - \nu + 1 - \gamma)(\frac{p+q}{2})(m+c) + \frac{(\mu\gamma - \nu + 1 - \gamma)}{2} e_{p,q}^{-m(1-p)} \\
&+ \frac{(\mu\gamma - \nu + 1 - \gamma)}{2}) e_{p,q}^{-m(1-q)} + \frac{(\mu\gamma - \nu + 1 - \gamma)}{2} e_{p,q}^{-c(1-p)} + \frac{(\mu\gamma - \nu + 1 - \gamma)}{2} e_{p,q}^{-c(1-q)} \\
&\leq \mu - \nu + (\mu - \nu) e_{p,q}^{-m} + (2\mu\gamma - 2\gamma + 2 - \mu - \nu) e_{p,q}^{-c}
\end{aligned} \tag{10}$$

holds true. Then $\mathcal{P}_{p,q}^{m,c} \in \mathcal{SH}_{p,q}(\gamma, \mu, \nu)$.

Proof. In view of Lemma 2.14, it is sufficient to show that the following inequality

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left\{ \{(\mu\gamma - \nu + 1 - \gamma)([n]_{p,q}) + (1-\gamma)(\mu - 1)\} \frac{m^{n-1} e_{p,q}^{-m}}{[n-1]_{p,q}!} \right\} \\
&+ \sum_{n=2}^{\infty} \left\{ \{(\mu\gamma - \nu + 1 - \gamma)([n]_{p,q}) + (1-\gamma)(1-\mu)\} \frac{c^{n-1} e_{p,q}^{-c}}{[n-1]_{q}!} \right\} \\
&\leq \mu - \nu
\end{aligned} \tag{11}$$

is satisfied. For this purpose, similar to the previous case, we shall show that the function $\mathcal{P}_{p,q}^{m,c}(\kappa) \in \mathcal{SH}_{p,q}(\gamma, \mu, \nu)$. By using (2), we obtain

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left\{ \{(\mu\gamma - \nu + 1 - \gamma)([n]_{p,q}) + (1-\gamma)(\mu - 1)\} \frac{m^{n-1} e_{p,q}^{-m}}{[n-1]_{p,q}!} \right\} \\
&+ \sum_{n=2}^{\infty} \left\{ \{(\mu\gamma - \nu + 1 - \gamma)([n]_{p,q}) + (1-\gamma)(1-\mu)\} \frac{c^{n-1} e_{p,q}^{-c}}{[n-1]_{p,q}!} \right\} \\
&= \sum_{n=2}^{\infty} \left\{ \left\{ (\mu\gamma - \nu + 1 - \gamma) \left((\frac{p+q}{2})[n-1]_{p,q} + \frac{p^{n-1} + q^{n-1}}{2} \right) + (1-\gamma)(\mu - 1) \right\} \frac{m^{n-1} e_{p,q}^{-m}}{[n-1]_{p,q}!} \right\} \\
&+ \sum_{n=2}^{\infty} \left\{ \left\{ (\mu\gamma - \nu + 1 - \gamma) \left((\frac{p+q}{2})[n-1]_{p,q} + \frac{p^{n-1} + q^{n-1}}{2} \right) + (1-\gamma)(1-\mu) \right\} \frac{c^{n-1} e_{p,q}^{-c}}{[n-1]_{q}!} \right\}
\end{aligned}$$

$$= (\mu\gamma-\nu+1-\gamma)(\frac{p+q}{2})\sum_{n=2}^{\infty}\frac{m^{n-1}e_{p,q}^{-m}}{[n-2]_{p,q}!}+\frac{(\mu\gamma-\nu+1-\gamma)}{2}\sum_{n=2}^{\infty}\frac{p^{n-1}m^{n-1}e_{p,q}^{-m}}{[n-1]_{p,q}!}$$
$$+\frac{(\mu\gamma-\nu+1-\gamma)}{2}\sum_{n=2}^{\infty}\frac{q^{n-1}m^{n-1}e_{p,q}^{-m}}{[n-1]_{p,q}!}+(1-\gamma)(\mu-1)\sum_{n=2}^{\infty}\frac{m^{n-1}e_{p,q}^{-m}}{[n-1]_{p,q}!}$$
$$+(\mu\gamma-\nu+1-\gamma)(\frac{p+q}{2})\sum_{n=2}^{\infty}\frac{c^{n-1}e_{p,q}^{-c}}{[n-2]_{p,q}!}+\frac{(\mu\gamma-\nu+1-\gamma)}{2}\sum_{n=2}^{\infty}\frac{p^{n-1}c^{n-1}e_{p,q}^{-c}}{[n-1]_{p,q}!}$$
$$+\frac{(\mu\gamma-\nu+1-\gamma)}{2}\sum_{n=2}^{\infty}\frac{q^{n-1}c^{n-1}e_{p,q}^{-c}}{[n-1]_{p,q}!}+(1-\gamma)(1-\mu)\sum_{n=2}^{\infty}\frac{c^{n-1}e_{p,q}^{-c}}{[n-1]_{p,q}!}$$
$$= (\mu\gamma-\nu+1-\gamma)(\frac{p+q}{2})me_{p,q}^{-m}\sum_{n=0}^{\infty}\frac{m^{n}}{[n]_{p,q}!}+\frac{(\mu\gamma-\nu+1-\gamma)}{2}e_{p,q}^{-m}\sum_{n=1}^{\infty}\frac{p^{n}m^{n}}{[n]_{p,q}!}$$
$$+\frac{(\mu\gamma-\nu+1-\gamma)}{2}e_{p,q}^{-m}\sum_{n=1}^{\infty}\frac{q^{n}m^{n}}{[n]_{p,q}!}+(1-\gamma)(\mu-1)e_{p,q}^{-m}\sum_{n=1}^{\infty}\frac{m^{n}}{[n]_{p,q}!}$$
$$+(\mu\gamma-\nu+1-\gamma)(\frac{p+q}{2})ce_{p,q}^{-c}\sum_{n=0}^{\infty}\frac{c^{n}}{[n]_{p,q}!}+\frac{(\mu\gamma-\nu+1-\gamma)}{2}e_{p,q}^{-c}\sum_{n=1}^{\infty}\frac{p^{n}c^{n}}{[n]_{p,q}!}$$

$$+\frac{(\mu\gamma-\nu+1-\gamma)}{2}e_{p,q}^{-c}\sum_{n=1}^{\infty}\frac{q^{n}c^{n}}{[n]_{p,q}!}+(1-\gamma)(1-\mu)e_{p,q}^{-c}\sum_{n=1}^{\infty}\frac{c^{n}}{[n]_{p,q}!}$$
$$= (\mu\gamma-\nu+1-\gamma)(\frac{p+q}{2})m+\frac{(\mu\gamma-\nu+1-\gamma)}{2}e_{p,q}^{-m}(e_{p,q}^{pm}-1)$$
$$+\frac{(\mu\gamma-\nu+1-\gamma)}{2}e_{p,q}^{-m}(e_{p,q}^{qm}-1)+(1-\gamma)(\mu-1)e_{p,q}^{-m}(e_{p,q}^{m}-1)$$
$$+(\mu\gamma-\nu+1-\gamma)(\frac{p+q}{2})c+\frac{(\mu\gamma-\nu+1-\gamma)}{2}e_{p,q}^{-c}(e_{p,q}^{pc}-1)$$
$$+\frac{(\mu\gamma-\nu+1-\gamma)}{2}e_{p,q}^{-c}(e_{p,q}^{qc}-1)+(1-\gamma)(1-\mu)e_{p,q}^{-c}(e_{p,q}^{c}-1).$$

Therefore, the inequality (10) holds true if

$$(\mu\gamma-\nu+1-\gamma)(\frac{p+q}{2})(m+c)+\frac{(\mu\gamma-\nu+1-\gamma)}{2}e_{p,q}^{-m(1-p)}$$
$$+\frac{(\mu\gamma-\nu+1-\gamma)}{2}e_{p,q}^{-m(1-q)}+\frac{(\mu\gamma-\nu+1-\gamma)}{2}e_{p,q}^{-c(1-p)}$$
$$+\frac{(\mu\gamma-\nu+1-\gamma)}{2}e_{p,q}^{-c(1-q)}-(\mu-\nu)e_{p,q}^{-m}-(2\mu\gamma-2\gamma+2-\mu-\nu)e_{p,q}^{-c}$$
$$\leq \mu-\nu,$$

which is equivalent to (11). This completes the proof. □

Corollary 3.4. *Let $m, c > 0$, $-1 \leq \nu \leq 0$, $0 < \mu \leq 1$, $\gamma \in \mathbb{C} - \{0\}$, then the function $\mathcal{TP}_{p,q}^{m,c}$ defined by (6) belongs to the class $\mathcal{TSH}_{p,q}(\gamma, \mu, \nu)$ if and only if satisfied inequality (11).*

4 Conclusion

In this work, for first time we combine the generalized quantum calculus namely, (p, q)-calculus with Poisson Distribution for harmonic functions. For this purpose, at first we generalized the power series for exponential function by (p, q)-analogue of factorial function. Afterwards by using (p, q)-derivative and harmonic analysis, we introduce two subclass of harmonic univalent functions in complex space and investigate the status of desired distribution in such subclasses.

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