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Results on Hyper Equality Algebras

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Abstract. Equality algebras as a generalization of EQ-algebras, introduced by Jenei [8]. Hyper equality algebras introduced and studied in [4], as a generalization of equality algebras. Now, in this paper, we investigate relations among hyper equality algebras and other hyper algebraic structures such as hyper $K(BE,\ MV)$-algebras and hyper hoops. Specially, we prove that any linearly ordered hyper MV-algebra is a strongly commutative symmetric hyper equality algebra and under some conditions, any strongly commutative involutive hyper equality algebra is a hyper MV-algebra.

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1. Introduction

Jenei introduced a new class of logical algebras in [8] and called them equality algebras. This class is a generalization of EQ-algebras defined in [11], where the product operation in EQ-algebras is replaced by another binary operation smaller or equal than the original product. An equality algebra consisting of two binary operations meet and equivalence, and constant 1. An equality algebra $\mathcal{E} = \langle X, \sim, \wedge, 1\rangle$ is an algebra of type $(2,2,0)$ such that, for all $x, y, z \in X$, the following axioms are fulfilled:

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(E1) $\langle X, \wedge, 1\rangle$ is a commutative idempotent integral monoid ($\wedge$-semilattice with top element 1).
(E2) $x \sim y = y \sim x$.
(E3) $x \sim x = 1$.
(E4) $x \sim 1 = x$.
(E5) $x \leqslant y \leqslant z$ implies $x \sim z \leqslant y \sim z$ and $x \sim z \leqslant x \sim y$.
(E6) $x \sim y \leqslant (x \wedge z) \sim (y \wedge z)$.
(E7) $x \sim y \leqslant (x \sim z) \sim (y \sim z)$.

The operation $\wedge$ is called *meet (infimum)* and $\sim$ is *an equality operation*. We write $x \leqslant y$ if and only if $x \wedge y = x$. Define the following two derived operations, by using the implication and the equivalence operation of the equality algebra $\langle X, \sim, \wedge, 1\rangle$ by $x \to y = x \sim (x \wedge y)$ and $x \leftrightarrow y = (x \to y) \wedge (y \to x)$. Relations of equality algebras with the other logical algebras are studied in many works. In [5, 6, 9] authors proved that any equality algebra is corresponding with BCK-meet-semilattice and any BCK-meet-semilattice with distributivity property is corresponding with equality algebra. In [15] authors have proved that there are relations among equality algebras and some of other logical algebras such as residuated lattice, MTL-algebra, BL-algebra, MV-algebra, Hertz-algebra, Heyting-algebra, Boolean-algebra, EQ-algebra and hoop.

Hyper equality algebras are introduced and studied in [4]. Authors in [4] are provided many basic properties of this class of hyper algebras and they investigated relation of hyper equality algebras with hyper BCK-algebras. Now, in this paper some results on hyper equality algebras are obtained and we give some definitions such as comparable and commutative hyper equality algebras. Some examples for clarifying the differences between various versions of hyper equality algebras are given. Finally, we investigated relation among hyper equality algebras with hyper $K(BE, MV)$-algebra and hyper hoops.

2. Preliminaries

In this section, we review some elementary aspects that are necessary for this paper.

Let H be a non-empty set. Then the function $\circ : H \times H \longrightarrow P(H)^*$ is said to be a *hyper operation* on H. For two subsets A and B of H, denote by $A \circ B$ the set $\bigcup_{a \in A, b \in B} a \circ b$.

Definition 2.1. *[4] A* hyper equality algebra $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ *is a non-empty set H endowed with a binary operation $\wedge$, a binary hyper operation $\sim$ and a top element 1 such that, for all $x, y, z \in H$, the following axioms are fulfilled:*
(HE1) $\langle H, \wedge, 1\rangle$ *is a meet-semilattice with top element 1.*
(HE2) $x \sim y \ll y \sim x$.
(HE3) $1 \in x \sim x$.
(HE4) $x \in 1 \sim x$.
(HE5) $x \leqslant y \leqslant z$ *implies* $x \sim z \ll y \sim z$ *and* $x \sim z \ll x \sim y$.
(HE6) $x \sim y \ll (x \wedge z) \sim (y \wedge z)$.
(HE7) $x \sim y \ll (x \sim z) \sim (y \sim z)$.
where $x \leqslant y$ if and only if $x \wedge y = x$, and for any non-empty subsets $A, B \subseteq H$, $A \ll B$ is defined by, for all $x \in A$, there exists $y \in B$ such that $x \leqslant y$. A hyper equality algebra $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ is called good *if, for all $x \in H$, $x = 1 \sim x$. A hyper equality algebra $\mathcal{H}$ is called* bounded, *if there is a bottom element 0 in $\mathcal{H}$. Define the following two derived operations, the "implication" and the "equivalence operation" of hyper equality algebra $\langle H, \sim, \wedge, 1\rangle$, for any $x, y \in H$, by*

$$x \to y = x \sim (x \wedge y) \quad \textit{and} \quad x \leftrightarrow y = (x \to y) \wedge (y \to x)$$

If hyper equality algebra $\mathcal{H}$ is bounded, then the unary operation $*$ on $\mathcal{H}$ which, for all $x \in H$, is defined by $x^* = x \sim 0$ is called a negation. *It is clear that $x^* = x \to 0$.*

Proposition 2.2. *[4] Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a hyper equality algebra. Then for all $x, y, z \in H$, the following statements are equivalent:*
(HE5) $x \leqslant y \leqslant z$ *implies* $x \sim z \ll y \sim z$ *and* $x \sim z \ll x \sim y$.
(HE5a) $x \sim (x \wedge y \wedge z) \ll x \sim (x \wedge y)$.
(HE5b) $x \to (y \wedge z) \ll x \to y$.

Proposition 2.3. *[4] Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a hyper equality algebra. Then, for all $x, y, z \in H$ and $A, B, C \subseteq H$, we have the following prop-*

erties:
(i) $x \leqslant y$ *and* $y \leqslant x$*, imply* $x = y$.
(ii) $1 \in x \to x,\ 1 \in x \to 1,\ x \ll x \sim 1,\ x \in 1 \to x$ *and* $1 \in x \leftrightarrow x$.
(iii) $x \sim y \ll x \to y$ *and* $x \sim y \ll y \to x$.
(iv) $x \leqslant y$ *implies* $1 \in x \to y$.
(v) $x \leqslant y \leqslant z$ *implies* $z \sim x \ll z \sim y$ *and* $z \sim x \ll y \sim x$.
(vi) $x \ll y \to x$ *and* $A \ll B \to A$.
(vii) $x \leqslant y$ *implies* $z \to x \ll z \to y$ *and* $y \to z \ll x \to z$.
(viii) $A \ll B$ *implies* $C \to A \ll C \to B$ *and* $B \to C \ll A \to C$.
(ix) $x \leqslant y$ *implies* $x \ll y \sim x$.
(x) $y \ll (x \to y) \to y$.
(xi) $x \to y \ll (y \to z) \to (x \to z)$.

Proposition 2.4. *[4] Let* $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ *be a hyper equality algebra. If* $1 \sim y = y$ *or* $x \leqslant y$*, then, for all* $x, z \in H$*, the following statements hold:*
(i) $x \ll (x \sim y) \sim y$.
(ii) $x \ll (x \to y) \to y$.
(iii) $x \ll y \to z$ *if and only if* $y \ll x \to z$.
(iv) $x \to (y \to z) \ll y \to (x \to z)$.

Corollary 2.5. *[4] Let* $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ *be a good hyper equality algebra. Then, for all* $x, y, z \in H$*, the following statements hold:*
(i) $x \ll (x \sim y) \sim y$.
(ii) $x \ll (x \to y) \to y$.
(iii) $x \ll y \to z$ *if and only if* $y \ll x \to z$.
(iv) $x \to (y \to z) \ll y \to (x \to z)$.

Proposition 2.6. *[4, Proposition 4] Let* $\mathcal{H} = \langle H; \sim, \wedge, 1, 0\rangle$ *be a bounded hyper equality algebra. Then, for all* $x, y \in H$*, the following statements hold:*
(i) $0 \in 1^*$ *and* $1 \in 1^{**}$.
(ii) $1 \in 0^*$ *and* $0 \in 0^{**}$.
(iii) $x \sim y \ll x^* \sim y^*$.
(iv) $x \to y \ll y^* \to x^*$.
(v) $x \leqslant y$ *implies* $y^* \ll x^*$.

3. Some Results on Hyper Equality Algebras

In this section, we define some new notions on hyper equality algebras and give some results and examples related to these notions.

Definition 3.1. *Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a hyper equality algebra and $A, B \subseteq H$. We say that A and B are* comparable *subsets of H, if $A \ll B$ or $B \ll A$. We say that H is a* comparable hyper equality algebra *respect to $\sim$, if, for any $x, y, z, w \in H$, $x \sim y$ and $z \sim w$ are comparable. If there is not any doubt about $\sim$, we say that H is a* comparable hyper equality algebra.

Note that, according to the Definition 2.1, for any $x, y \in H$, $x \sim y$ and $y \sim x$ are comparable.

Example 3.2. Let $H = \{0, a, 1\}$ be a poset such that $0 < a < 1$. For any $x, y \in H$, we define the operations $\wedge$ and $\sim$ on H as follows,

$x \wedge y = \min\{x, y\}$ and

$\sim$	0	a	1
0	$\{1\}$	$\{0,1\}$	$\{0,1\}$
a	$\{0,1\}$	$\{1\}$	$\{a,1\}$
1	$\{0,1\}$	$\{a,1\}$	$\{1\}$

Then $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ is a comparable hyper equality algebra.

Definition 3.3. *[4] Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a hyper equality algebra. Then H is called a symmetric hyper equality algebra, if $x \sim y = y \sim x$, for all $x, y \in H$.*

Example 3.4. (i) Let $H = \{0, a, 1\}$ be a poset such that $0 < a < 1$. For any $x, y \in H$, we define the operations $\wedge$ and $\sim$ on H as follows:

$x \wedge y = \min\{x, y\}$ and

$\sim$	0	a	1
0	$\{1\}$	$\{0,a\}$	$\{0,a\}$
a	$\{0,a\}$	$\{1\}$	$\{a\}$
1	$\{0,a\}$	$\{0,a\}$	$\{1\}$

Then, $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ is a hyper equality algebra, which is not symmetric. Because $a \sim 1 \neq 1 \sim a$.

(ii) Let $H = \{0, a, b, 1\}$ be a poset such that $0 \leqslant a, b \leqslant 1$. For any $x, y \in H$, we define the operations $\wedge$ and $\sim$ on H as follows:

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\sim$	0	a	b	1
0	$\{1\}$	$\{1\}$	$\{b,1\}$	$\{0,a\}$
a	$\{1\}$	$\{1\}$	$\{a,1\}$	$\{a\}$
b	$\{b,1\}$	$\{a,1\}$	$\{1\}$	$\{b,1\}$
1	$\{0,a\}$	$\{a\}$	$\{b,1\}$	$\{1\}$

Then $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ is a symmetric hyper equality algebra.

(iii) Let $H = \{0, a, b, 1\}$ be a poset such that $0 < a < b < 1$. For any $x, y \in H$, we define the operations $\wedge$ and $\sim$ on H as follows:

$x \wedge y = \min\{x, y\}$ and

$\sim$	0	a	b	1
0	$\{1\}$	$\{0,a\}$	$\{0\}$	$\{0\}$
a	$\{0,a\}$	$\{1\}$	$\{0,a,b\}$	$\{a\}$
b	$\{0\}$	$\{0,a,b\}$	$\{1\}$	$\{b\}$
1	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$

Then $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ is a symmetric hyper equality algebra.

Proposition 3.5. *Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a symmetric hyper equality algebra. Then, for all $x, y, z \in H$, the following statements hold:*
(i) $x \sim y \ll (z \to x) \sim (z \to y)$.
(ii) $x \sim y \ll (z \to x) \to (z \to y)$.
(iii) $x \to y \ll (z \to x) \to (z \to y)$.

Proof. (i) Since $\mathcal{H}$ is symmetric, by (HE6) and (HE7), for any $x, y, z \in H$, we have

$$\begin{aligned} x \sim y &\ll (x \wedge z) \sim (y \wedge z) \\ &\ll ((x \wedge z) \sim z) \sim ((y \wedge z) \sim z) \\ &= (z \sim (x \wedge z)) \sim (z \sim (y \wedge z)) \\ &= (z \to x) \sim (z \to y). \end{aligned}$$

(ii) By (i) and Proposition 2.3 (iii), the proof is clear.

(iii) For any $x, y, z \in H$, by (ii) and Proposition 2.3 (vii),

$$x \to y = x \sim (x \wedge y) \ll (z \to x) \to (z \to (x \wedge y)) \ll (z \to x) \to (z \to y) \quad \square$$

Definition 3.6. *Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a hyper equality algebra. We say that $\mathcal{H}$ is commutative, if for all $x, y \in H$, $(x \to y) \to y$ and*

$(y \to x) \to x$ *are comparable. Also, we say it is* strongly commutative*, if, for all* $x, y \in H$*,* $(x \to y) \to y = (y \to x) \to x$*.*

Example 3.7. (i) Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a hyper equality algebra as in Example 3.2. Then H is a commutative hyper equality algebra, which is not strongly commutative. Because,

$$(0 \to a) \to a = \{0, a, 1\} \neq (a \to 0) \to 0 = \{0, 1\}.$$

(ii) Let $H = \{0, a, b, 1\}$ be a poset such that $0 < a < b < 1$. For any $x, y \in H$, we define the operations $\wedge$ and $\sim$ on H as follows:

$x \wedge y = \min\{x, y\}$ and

$\sim$	0	a	b	1
0	$\{1\}$	$\{b\}$	$\{b\}$	$\{0\}$
a	$\{b\}$	$\{1\}$	$\{b\}$	$\{0, a\}$
b	$\{b\}$	$\{b\}$	$\{1\}$	$\{0, b\}$
1	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$

Thus, $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ is a strongly commutative hyper equality algebra.

Remark 3.8. *Let* $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ *be a hyper equality algebra such that, for any* $x, y \in H$*,*

$$(x \to y) \to y \ll (y \to x) \to x$$

Then $\mathcal{H}$ *is a commutative hyper equality algebra, since by changing* x *and* y *by each others, we obtain that*

$$(y \to x) \to x \ll (x \to y) \to y$$

Proposition 3.9. *Let* $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ *be a separated hyper equality algebra. Then, for any* $x, y \in H$*,* $1 \in x \to y$ *if and only if* $x \leqslant y$*.*

Proof. By Proposition 2.3 (iv), for any $x, y \in H$, if $x \leqslant y$ then $1 \in x \to y$. Now, let $1 \in x \to y$. Then $1 \in x \sim (x \wedge y)$ and so $x \wedge y = x$. Hence, $x \leqslant y$. □

4. Relations Among Hyper Equality Algebras and Other Hyper Logical Algebras

In this section, we investigate the relation between hyper equality algebras and hyper K, hyper BE, hyper hoop and hyper MV-algebras. For this, first we define some conditions and give some related examples.

Notation 4.1. Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a hyper equality algebra. We say that H has (I) or (II) properties, if, for any $x, y \in H$, H satisfies the following conditions:

$$x \to (y \to z) = y \to (x \to z) \qquad , \qquad \text{(I)}$$

$$x \to y \ll (z \to x) \to (z \to y) \qquad , \qquad \text{(II)}$$

Example 4.2. (i) Let $H = \{0, a, 1\}$ be a poset such that $0 < a < 1$. For any $x, y \in H$, we define the operations $\wedge$ and $\sim$ on H as follows:

$x \wedge y = \min\{x, y\}$ and

$\sim$	0	a	1
0	$\{1\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{1\}$	$\{a,1\}$
1	$\{0\}$	$\{a,1\}$	$\{1\}$

Then $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ is a hyper equality algebra with properties (I) and (II).

(ii) Let $H = \{0, a, b, 1\}$ be a poset such that $0 < a < b < 1$. For any $x, y \in H$, we define the operations $\wedge$ and $\sim$ on H as follows:

$x \wedge y = \min\{x, y\}$ and

$\sim$	0	a	b	1
0	$\{1\}$	$\{b\}$	$\{0\}$	$\{0\}$
a	$\{0,b\}$	$\{1\}$	$\{b\}$	$\{0,a\}$
b	$\{0\}$	$\{b\}$	$\{1\}$	$\{0,b\}$
1	$\{0\}$	$\{0,a\}$	$\{b\}$	$\{1\}$

Then $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ is a hyper equality algebra, which has not properties (I) and (II). Because,

$$b \to (1 \to a) = \{0, b\} \neq \{b\} = 1 \to (b \to a)$$

and

$$a \to 0 = \{0, b\} \not\ll \{0\} = (b \to a) \to (b \to 0)$$

(iii) Let $H = \{0, a, b, 1\}$ be a poset such that $0 < a < b < 1$. For any $x, y \in H$, we define the operations $\wedge$ and $\sim$ on H as follows:

$x \wedge y = \min\{x, y\}$ and

$\sim$	0	a	b	1
0	$\{1\}$	$\{b\}$	$\{0\}$	$\{0\}$
a	$\{0, b\}$	$\{1\}$	$\{b\}$	$\{0, a\}$
b	$\{0\}$	$\{b\}$	$\{1\}$	$\{0, b\}$
1	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$

Then $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ is a hyper equality with property (I). But it has not property (II). Because,

$$a \to 0 = \{0, b\} \neq \{0\} = (b \to a) \to (b \to 0)$$

(iv) Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a hyper equality algebra as Examples 3.4 (i) or (iii). Then $\mathcal{H}$ has property (II), but does not satisfy in (I).

4.1 Relation with hyper K-algebras

In this section, we investigate the relation between hyper equality algebras and hyper K-algebras.

Definition 4.1.1. *[1] A hyper K-algebra is a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0 such that, for all $x, y, z \in H$, it satisfying the following conditions*
(HK1) $(x \circ z) \circ (y \circ z) \ll x \circ y$.
(HK2) $(x \circ z) \circ y = (x \circ y) \circ z$.
(HK3) $x \leqslant x$.
(HK4) If $x \leqslant y$ *and* $y \leqslant x$, *then* $x = y$.
(HK5) $x \geqslant 0$.
where, for any $x, y \in H$, $x \leqslant y$ if and only if $0 \in x \circ y$, and for any $A, B \subseteq H$, $A \ll B$ means that there exist $a \in A$ and $b \in B$ such that $a \leqslant b$.

Definition 4.1.2. *[4] Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a hyper equality algebra. Then H is called separated, if $1 \in x \sim y$ implies $x = y$, for all $x, y \in H$.*

Theorem 4.1.3 *Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a separated hyper equality algebra with properties (I) and (II). Then $(H, \circ, 0)$ is a hyper K-algebra, where, for any $x, y \in H$, $x \circ y = y \sim (y \wedge x)$.*

Proof. Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a separated hyper equality algebra. For any $x, y \in H$, we define $x \circ y := y \to x$, $0 := 1$ and $0 \in x \circ y$ if and only if $x \leqslant' y$. Also, for any $A, B \subseteq H$, we define $A \ll' B$ if and only if there exist $a \in A$ and $b \in B$ such that $a \leqslant' b$. We show that $(H, \circ, 0)$ is a hyper K-algebra. By (II), for all $x, y, z \in H$, we get that $x \to y \ll (z \to x) \to (z \to y)$. Thus, there are $a \in x \to y$ and $b \in (z \to x) \to (z \to y)$ such that $a \leqslant b$. Then by Proposition 3.9, $1 \in a \to b$. Hence, $0 \in b \circ a$ if and only if $b \leqslant' a$. This means that (HK1) holds. By (I), the proof of (HK2) is clear. By Proposition 2.3 (ii), for any $x \in H$, $1 \in x \to x = x \circ x$ and so we have (HK3). Now, assume that, for any $x, y \in H$, $x \leqslant' y$ and $y \leqslant' x$. Then $0 \in x \circ y$ and $0 \in y \circ x$. Thus, $1 \in x \to y$ and $1 \in y \to x$. By Proposition 3.9, $x \leqslant y$ and $y \leqslant x$. Then by Proposition 2.3 (i), $x = y$. Hence, (HK4) holds. Moreover, by Proposition 2.3 (ii), for all $x \in H$, $1 \in x \to 1$. Then, $0 \in 0 \circ x$, and so $x \geqslant' 0$. Hence, $(H, \circ, 0)$ satisfies in (HK5). Therefore, $(H, \circ, 0)$ is a hyper K-algebra. □

In the following example we show that the converse of Theorem 4.1.3 may not be true, in general.

Example 4.1.4. Let $H = \{0, a, b\}$ be a poset such that $0 \leqslant a \leqslant b$. For any $x, y \in H$, we define hyperoperation "$\circ$" on H as follows:

$\circ$	0	a	b
0	$\{0, a\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, a\}$	$\{a\}$
b	$\{b\}$	$\{0, b\}$	$\{0, a, b\}$

Then $(H, \circ, 0)$ is a hyper K-algebra which is a join-semilattice. By Theorem 4.1.3, if, for any $x, y \in H$, we define $x \sim y = x \to y := y \circ x$, then $x \wedge y = x$ if and only if $1 \in x \sim y$ and $1 := 0$. Then it is clear that $(H, \wedge)$ becomes a meet-semilattice with the top element 1. But it is not a hyper equality algebra, because $1 \sim a = 1 \to a = a \circ 0 = \{a\}$ and $a \sim 1 = a \to 1 = 0 \circ a = \{0\} = \{1\}$. Hence (HE2) does not hold.

Corollary 4.1.5. *Every separated and symmetric hyper equality algebra that satisfies in the condition (I), is a hyper K-algebra.*

Proof. Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a separated and symmetric hyper equality algebra. Then by Proposition 3.5 (iii), $\mathcal{H}$ satisfies in (II). Thus, by Theorem 4.1.3, $\mathcal{H}$ is a hyper K-algebra. □

Remark 4.1.6. *Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a hyper equality algebras as in Examples 3.4 (i) or (ii). Since $\mathcal{H}$ does not satisfy in (I), the axiom (HK2) does not hold. Thus, it is not a hyper K-algebra.*

Example 4.1.7. [4] Let $H = \{0, a, 1\}$ be a set. For any $x, y \in H$, we define the operations $\wedge$ and $\sim$ on H as follows:

$x \wedge y = \min\{x, y\}$ and

$\sim$	0	a	1
0	$\{1\}$	$\{0, a\}$	$\{0\}$
a	$\{0, a\}$	$\{1\}$	$\{a\}$
1	$\{0\}$	$\{a\}$	$\{1\}$

Then $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ is a separated and symmetric hyper equality algebra that satisfies in (I). Thus, by Corollary 4.1.5, $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ is a hyper K-algebra.

4.2 Relation with hyper BE-algebras

Now, in this section, we investigate the relation between hyper equality algebras and hyper BE-algebras.

A *hyper BE-algebra* is a hypergroupoid $(H, \circ)$ with a constant 1, such that, for all $x, y, z \in H$, it satisfies in the following conditions:
(HBE1) $x \leqslant 1$ and $x \leqslant x$.
(HBE2) $x \circ (y \circ z) = y \circ (x \circ z)$.
(HBE3) $x \in 1 \circ x$.
(HBE4) $1 \leqslant x$ implies $x = 1$.
where, $x \leqslant y$ if and only if $1 \in x \circ y$ and for any $A, B \subseteq H$, $A \ll B$ means that there exist $a \in A$ and $b \in B$ such that $a \leqslant b$ (See [12]).

Theorem 4.2.1. *Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a separated hyper equality algebra that satisfies in the condition (I). Then $(H, \circ, 1)$ is a hyper BE-algebra, where, for any $x, y \in H$, $x \circ y := x \to y$.*

Proof. By Proposition 2.3 (ii), for any $x \in H$, $1 \in x \to x$ and $1 \in x \to 1$. Then by Proposition 3.9, for all $x \in H$, $x \leqslant x$ and $x \leqslant 1$. Also, by (I), for all $x, y, z \in H$, $x \to (y \to z) = y \to (x \to z)$ and so, $x \circ (y \circ z) = y \circ (x \circ z)$. Moreover, by Proposition 2.3 (ii), for all $x \in H$, $x \in 1 \to x$. Then $x \in 1 \circ x$. Finally, for $x \in H$, suppose $1 \leqslant x$. By (HBE1), we have $x \leqslant 1$. Then by Proposition 2.3 (i), we get that $x = 1$. Hence, $(H, \circ, 1)$ is a hyper BE-algebra. □

4.3 Relation with hyper hoop

Now, in this section, we get the relation between hyper equality algebras and hyper hoops.

Definition 4.3.1. *[2] A hyper hoop is a non-empty set H endowed with two binary hyperoperations $\odot$, $\to$ and a constant 1 such that, for all $x, y, z \in H$, the following conditions hold:*
(HHA1) $(H, \odot, 1)$ *is a commutative semihypergroup with 1 as the unit.*
(HHA2) $1 \in x \to x$.
(HHA3) $(x \to y) \odot x = (y \to x) \odot y$.
(HHA4) $x \to (y \to z) = (x \odot y) \to z$.
(HHA5) $1 \in x \to 1$.
(HHA6) If $1 \in x \to y$ *and* $1 \in y \to x$, *then* $x = y$.
(HHA7) If $1 \in x \to y$ *and* $1 \in y \to z$, *then* $1 \in x \to z$.
where, we have $x \leqslant y$ if and only if $1 \in x \to y$, and for any $A, B \subseteq H$, $A \ll B$ means that there exist $a \in A$ and $b \in B$ such that $a \leqslant b$. If $\mathcal{H} = (H, \odot, \to, 1)$ is a hyper hoop, then $\leqslant$ is a partial order relation on H. Moreover, for all $A, B \subseteq H$, we define $A \ll B$ if there exist $a \in A$ and $b \in B$ such that $a \leqslant b$ and $A \leqslant B$, if for all $a \in A$ there exists $b \in B$ such that $a \leqslant b$. A hyper hoop $\mathcal{H}$ is bounded if, for all $x \in H$, there is an element $0 \in H$ such that $0 \leqslant x$. For a bounded hyper hoop, for any $x \in H$, we consider $x' = x \to 0$.

Definition 4.3.2. *Let H be a bounded hyper equality algebra. We say that H is an involutive, if for all $x \in H$, $x^{**} = \{x\}$, where $x^* = x \sim 0$.*

Theorem 4.3.3. *Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be an involutive separated hyper equality algebra such that, for all $x, y \in H$, $x \to y = y^* \to x^*$ and* *satisfy in condition (I). Then* $(H, \odot, \to, 1)$ *is a hyper hoop, where for all* $x, y \in H$, $x \odot y = (x \to y^*)^*$ *and* $(x \to y) \odot x = x \wedge y$.

Proof. According to definition of $\odot$, since H is an involutive separated hyper equality algebra, by condition (I), we have:

$$\begin{aligned} x \odot (y \odot z) &= (x \to (y \to z^*)^{**})^* = (x \to (y \to z^*))^* \\ &= (x \to (z \to y^*))^* = (z \to (x \to y^*))^* \\ &= ((x \to y^*)^* \to z^*)^* \\ &= (x \odot y) \odot z \end{aligned}$$

Hence, $(H, \odot)$ is a semihypergroup. Now, we show that $\odot$ is commutative. Let $x, y \in H$, since $x \to y = y^* \to x^*$ and H is involutive, we have

$$x \odot y = (x \to y^*)^* = (y \to x^*)^* = y \odot x$$

Then, $(H, \odot)$ is a commutative semihypergroup. Finally, we show that 1 is the unit. For any $x \in H$,

$$1 \odot x = (1 \to x^*)^* = (x \to 1^*)^* = (x \to 0)^* = x^{**} = \{x\}$$

Similarly, for all $x \in H$, $x \odot 1 = \{x\}$. Thus, for all $x \in H$, $x \in (1 \odot x) \cap (x \odot 1)$. By Proposition 2.3 (ii), it is easy to see that (HHA2) holds. By assumption, for all $x, y \in H$, $(x \to y) \odot x = x \wedge y$. Then, for all $x, y \in H$, $(x \to y) \odot x = (y \to x) \odot y$. Moreover, for all $x, y \in H$, since $x \to y = y^* \to x^*$, by (I), we get that

$$\begin{aligned} x \to (y \to z) &= x \to (z^* \to y^*) = z^* \to (x \to y^*) = (x \to y^*)^* \to z \\ &= (x \odot y) \to z. \end{aligned}$$

Hence, by Proposition 2.3 (ii), (HHA5) holds. If for any $x, y \in H$, $1 \in x \to y$ and $1 \in y \to x$, then by Proposition 3.9, $x \leqslant y$ and $y \leqslant x$. Thus, by Proposition 2.3 (i), $x = y$. Moreover, if, for any $x, y, z \in H$, $1 \in x \to y$ and $1 \in y \to z$, then by Proposition 3.9, $x \leqslant y$ and $y \leqslant z$. Thus, $x \leqslant z$ and so $1 \in x \to z$. Therefore, $(H, \odot, \to, 1)$ is a hyper hoop. □

Example 4.3.4 (i) Let $H = \{0, a, 1\}$ be a poset such that $0 < a < 1$. For any $x, y \in H$, we define the operations $\wedge$ and $\sim$ on H as follows,

$x \wedge y = \min\{x, y\}$ and

$\sim$	0	a	1
0	$\{1\}$	$\{a\}$	$\{0\}$
a	$\{a\}$	$\{1\}$	$\{0, a\}$
1	$\{0\}$	$\{a\}$	$\{1\}$

Then $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ is an involutive hyper equality algebra that satisfies in (I) and (II). For any $x, y \in H$, we define the operations $\odot$ on H as follows,

$\odot$	0	a	1
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{0\}$	$\{0\}$	$\{a\}$
1	$\{0\}$	$\{a\}$	$\{1\}$

Then we can see that $x \odot y = (x \rightarrow y^*)^*$ and $(x \rightarrow y) \odot x = x \wedge y$, for any $x, y \in H$. Thus, by Theorem 4.1.3, $(H, \odot, \rightarrow, 1)$ is a hyper hoop.
(ii) Let $H = \{0, a, 1\}$ be a poset such that $0 < a < 1$. For any $x, y \in H$, we define the operations $\wedge$ and $\sim$ on H as follows,

$x \wedge y = \min\{x, y\}$ and

$\sim$	0	a	1
0	$\{1\}$	$\{a\}$	$\{0\}$
a	$\{a\}$	$\{a, 1\}$	$\{0, a\}$
1	$\{0\}$	$\{a\}$	$\{1\}$

Then $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ is an involutive seperated hyper equality algebra that does not satisfy in (I), because $0 \rightarrow (a \rightarrow a) \neq a \rightarrow (0 \rightarrow a)$. Define $x \odot y = (x \rightarrow y^*)^*$. Then, $(H, \odot, \rightarrow, 1)$ is not a hyper hoop, because the axiom (HHA1) does not hold. Indeed, $0 \odot a \neq a \odot 0$.

Proposition 4.3.5. [2] *Let $\mathcal{H} = (H, \odot, \rightarrow, 1)$ be a hyper hoop. Then, for all $x, y, z, t \in H$ and $A, B, C \subseteq H$, the following statements hold:*
(i) If $1 \in 1 \rightarrow x$, then $x = 1$.
(ii) $x \in 1 \rightarrow x$ and x is the maximum element of $1 \rightarrow x$.
(iii) If $\mathcal{H}$ is bounded, then $0 \in x \odot 0$.
(iv) If $A \ll B \leqslant C$, then $A \ll C$ and $\{x\} \leqslant A \leqslant \{y\}$.
(v) If $A \ll \{x\} \ll B$, then $A \ll B$ and if $A \ll \{x\} \leqslant B$, then $A \ll B$.
(vi) If $x \leqslant y$, then $z \rightarrow x \leqslant z \rightarrow y$.
(vii) If $x \leqslant y$, then $y \rightarrow z \leqslant x \rightarrow z$.

(viii) $z \to y \leqslant (y \to x) \to (z \to x)$.
(ix) $z \to y \ll (x \to z) \to (x \to y)$.

Notation 4.3.6. Let H be a hyper hoop. We say that H satisfies in condition (A), if for any $x, y, z \in H$,

$$x \to ((y \to z) \odot y) = ((x \to y) \to (x \to z)) \odot (x \to y) \ , \ \text{(A)}$$

Example 4.3.7. [3] Let $H = \{1, a, b\}$ be a poset such that $b < a < 1$. We define the hyperoperations $\odot$ and $\to$ on H as follows:

$\odot$	1	a	b
1	$\{1\}$	$\{1, a\}$	$\{1, a, b\}$
a	$\{1, a\}$	$\{a, b\}$	$\{1, a, b\}$
b	$\{1, a, b\}$	$\{1, a, b\}$	$\{1, a, b\}$

$\to$	1	a	b
1	$\{1, a, b\}$	$\{a, b\}$	$\{b\}$
a	$\{1, b\}$	$\{1, a\}$	$\{a, b\}$
b	$\{1, a, b\}$	$\{1, a, b\}$	$\{1, a, b\}$

Then $(H, \odot, \to)$ is a hyper hoop. It is easy to see that H satisfies in the condition (A).

By the following example we show that every hyper hoop does not satisfy in the condition (A).

Example 4.3.8. Let $H = \{1, a, b, c\}$ be a set. For any $x, y \in H$, define the hyperoperations $\odot$ and $\to$ on H as follows:

$\odot$	1	a	b	c
1	$\{1\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, c\}$
b	$\{b\}$	$\{a, b\}$	$\{b, c\}$	$\{b, c\}$
c	$\{c\}$	$\{a, c\}$	$\{b, c\}$	$\{c\}$

$\to$	1	a	b	c
1	$\{1\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{1\}$	$\{1, a\}$	$\{b\}$	$\{1, c\}$
b	$\{1\}$	$\{a\}$	$\{1, b, c\}$	$\{b, c\}$
c	$\{1\}$	$\{a\}$	$\{b\}$	$\{1, b, c\}$

By routine calculations, we can see that $(H, \odot, \rightarrow, 1)$ is a hyper hoop, but it does not satisfy the condition (A), because,

$$a \rightarrow ((b \rightarrow c) \odot b) = \{1, b, c\} \neq \{b, c\} = ((a \rightarrow b) \rightarrow (a \rightarrow c)) \odot (a \rightarrow b)$$

Theorem 4.3.9. *Every linearly ordered hyper hoop $(H, \odot, \rightarrow, 1)$ that satisfying the condition (A), is a separated hyper equality algebra.*

Proof. Define $\sim := \leftrightarrow$, where $\leftrightarrow$ is the equivalence operation of H, i.e.,

$$x \sim y = x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x) \ , \ \forall x, y \in H$$

Also we consider $\ll' = \leqslant$. Now, we show that $\langle H; \leftrightarrow, \wedge, 1\rangle$ is a hyper equality algebra. Note that $(H, \leqslant)$ is a $\wedge$-semilattice with the top element 1. Thus, (HE1) holds. The proof of (HE2) is clear. By (HHA2), we have the proof of (HE3). By Proposition 4.3.5 (ii) and the axiom (HHA5), for any $x \in H$, $x \in 1 \rightarrow x$ and $1 \in x \rightarrow 1$. Then, for any $x \in H$, $x \in x \leftrightarrow 1$ and so we have (HE4). Let $x, y, z \in H$ such that $x \leqslant y \leqslant z$. Then by Proposition 4.3.5 (vi), we have $z \rightarrow x \ll' z \rightarrow y$. Thus, $\{1\} \wedge (z \rightarrow x) \ll' \{1\} \wedge (z \rightarrow y)$. Also, by Proposition 4.3.5 (vii), $y \rightarrow z \ll' x \rightarrow z$. Hence,

$$(x \rightarrow z) \wedge (z \rightarrow x) \ll' (y \rightarrow z) \wedge (z \rightarrow y)$$

and so $x \leftrightarrow z \ll' y \leftrightarrow z$. By the Similar way, if $y \leqslant z$, then by Proposition 4.3.5 (vii), $z \rightarrow x \ll' y \rightarrow x$. Thus, $\{1\} \wedge (z \rightarrow x) \ll' \{1\} \wedge (y \rightarrow x)$. Now, by Proposition 4.3.5 (vi),

$$(x \rightarrow z) \wedge (z \rightarrow x) \ll' (x \rightarrow y) \wedge (y \rightarrow x)$$

and so $x \leftrightarrow z \ll' x \leftrightarrow y$. Hence we have (HE5). Since $\mathcal{H}$ satisfies the condition (A), for all $x, y, z \in H$, we have

$$x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z) \tag{1}$$

Then by (1), for any $x, y, z \in H$, we get that

$$((x \wedge z) \rightarrow y) \wedge ((x \wedge z) \rightarrow z) = (x \wedge z) \rightarrow (y \wedge z) \tag{2}$$

and

$$((y \wedge z) \rightarrow x) \wedge ((y \wedge z) \rightarrow z) = (y \wedge z) \rightarrow (x \wedge z) \tag{3}$$

Thus, by (2) and (3), for any $x, y, z \in H$, we obtain that

$$\begin{aligned}
x \leftrightarrow y &= (x \to y) \wedge (y \to x) \\
&= (x \to y) \wedge \{1\} \wedge (y \to x) \wedge \{1\} \\
&\ll' ((x \wedge z) \to y) \wedge ((x \wedge z) \to z) \wedge ((y \wedge z) \to x) \wedge ((y \wedge z) \to z) \\
&= ((x \wedge z) \to (y \wedge z)) \wedge ((y \wedge z) \to (x \wedge z)) \\
&= (x \wedge z) \leftrightarrow (y \wedge z)
\end{aligned}$$

and so we get (HE6).

Finally, for any $x, y, z \in H$, by Proposition 4.3.5 (vi) and (vii), we have

$$\begin{aligned}
x \leftrightarrow y &= (x \to y) \wedge (y \to x) \\
&= (x \to y) \wedge (y \to x) \wedge (y \to x) \wedge (x \to y) \\
&\ll' [((y \to z) \wedge (z \to y)) \to (x \to z)] \\
&\quad \wedge [((z \to y) \wedge (y \to z)) \to (z \to x)] \\
&\quad \wedge [((x \to z) \wedge (z \to x)) \to (y \to z)] \\
&\quad \wedge [((z \to x) \wedge (x \to z)) \to (z \to y)] \\
&= (x \leftrightarrow z) \leftrightarrow (y \leftrightarrow z)
\end{aligned}$$

Hence we get (HE7). Therefore, $\langle H; \leftrightarrow, \wedge, 1\rangle$ is a hyper equality algebra and by (HHA6), it is separated. □

4.4 Relation with hyper MV-algebras

Finally, in this section, we get the relation between hyper equality algebras and hyper MV-algebras.

A *hyper MV-algebra* is a non-empty set H endowed with a hyper operation " $\oplus$ ", a unary operation $*$ and a constant 0 such that, for any $x, y, z \in H$, the following statements hold:

(HMV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.

(HMV2) $x \oplus y = y \oplus x$.

(HMV3) $(x^*)^* = x$.

(HMV4) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

(HMV5) $0^* \in x \oplus 0^*$.

(HMV6) $0^* \in x \oplus x^*$.
(HMV7) $x \leqslant y$ and $y \leqslant x$ implies $x = y$.
where $x \leqslant y$ is defined by $0^* \in x^* \oplus y$. For any $A, B \subseteq H$, $A \leqslant B$ means that there exist $a \in A$ and $b \in B$ such that $a \leqslant b$ and $A \ll B$ means that for all $a \in A$ there exists $b \in B$ such that $a \leqslant b$. Also, we set $0^* = 1$ (See [7]). We summarize some results on hyper MV-algebras as follows:

Proposition 4.4.1. [7, 13] *Let $\mathcal{M} = (H; \oplus, *, 0)$ be a hyper MV-algebra. Then, for all $x, y, z \in H$ and for all non-empty subset A, B and C of H, the following statements hold:*
(i) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.
(ii) $0 \leqslant x$.
(iii) if $x \leqslant y$, then $y^* \leqslant x^*$ and if $A \leqslant B$, then $B^* \leqslant A^*$.
(iv) $x \leqslant 1$.
(v) $x \leqslant x \oplus y$ and $A \leqslant A \oplus B$.
(vi) $x \in x \oplus 0$.
(vii) if $x \leqslant y$, then $x \oplus z \ll y \oplus z$.
(viii) if $x \leqslant y$, then $x \wedge z \ll y \wedge z$.

Theorem 4.4.2. *Every linearly ordered hyper MV-algebra is a strongly commutative symmetric hyper equality algebra.*

Proof. Let $\mathcal{M} = (H; \oplus, *, 0)$ be a linearly ordered hyper MV-algebra. or any $x, y \in H$, we define the operation $\sim$ on H as follows:

$$x \sim y = x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)$$

where $x \rightarrow y = x^* \oplus y$. Now, we show that $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ is a hyper equality algebra. By Proposition 4.4.1 (iv), $(H, \leqslant)$ is a $\wedge$-semilattice with top element 1. Hence we have (HE1).
(HE2): Let $x, y \in H$. Then by (HE1), we have

$$\begin{aligned} x \sim y &= (x \rightarrow y) \wedge (y \rightarrow x) \\ &= (x^* \oplus y) \wedge (y^* \oplus x) && (4) \\ &= (y^* \oplus x) \wedge (x^* \oplus y) && (5) \\ &= (y \rightarrow x) \wedge (x \rightarrow y) && (6) \\ &= y \sim x && (7) \end{aligned}$$

(HE3): By (HMV2) and (HMV6), for any $x \in H$, we have $1 = 0^* \in (x^* \oplus x) \wedge (x \oplus x^*) = x \sim x$.
(HE4): By Proposition 4.4.1 (vi), for any $x \in H$, we have $x \in x \oplus 0$. Then by (HMV2), $x \in 0 \oplus x = 1^* \oplus x = 1 \to x$. Also, $x \to 1 = x^* \oplus 1 = x^* \oplus 0^*$. By (HMV5), for any $x \in H$, we have $1 \in x \to 1$. Hence,

$$x \in (1 \to x) \wedge 1 \leqslant (1 \to x) \wedge (x \to 1) = 1 \sim x$$

(HE5): Let $x, y, z \in H$ such that $x \leqslant y \leqslant z$. By Proposition 4.4.1 (iii), $z^* \leqslant y^* \leqslant x^*$. Then by (HMV6) and Proposition 4.4.1 (vii), we have $1 \in z^* \oplus z \ll y^* \oplus z \ll x^* \oplus z$. Thus, for any $A \subseteq H$,

$$(x^* \oplus z) \wedge A = (y^* \oplus z) \wedge A \tag{8}$$

Also, by Proposition 4.4.1 (vii), since $x \leqslant y$, we have

$$z^* \oplus x \ll z^* \oplus y \tag{9}$$

Then, for any $x, y, z \in H$, by (9), (8) and Proposition 4.4.1 (viii), we have

$$\begin{aligned} x \sim z &= (x \to z) \wedge (z \to x) \\ &= (x^* \oplus z) \wedge (z^* \oplus x) \\ &\ll (x^* \oplus z) \wedge (z^* \oplus y) \\ &= (y^* \oplus z) \wedge (z^* \oplus y) \\ &= y \sim z \end{aligned}$$

Since $z^* \leqslant y^*$, by Proposition 4.4.1 (vii), we have

$$z^* \oplus x \ll y^* \oplus x \tag{10}$$

Moreover, by (HMV6) and Proposition 4.4.1 (vii), since $x \leqslant y \leqslant z$, we get that

$$1 \in x \oplus x^* \ll y \oplus x^* \ll z \oplus x^*$$

So, for any $A \subseteq H$,

$$(x^* \oplus z) \wedge A = (x^* \oplus y) \wedge A \tag{11}$$

Then, for any $x, y, z \in H$, by (10), (11) and Proposition 4.4.1 (vii), we have

$$\begin{aligned} x \sim z &= (x \to z) \wedge (z \to x) \\ &= (x^* \oplus z) \wedge (z^* \oplus x) \\ &\ll (x^* \oplus z) \wedge (y^* \oplus x) \\ &= (x^* \oplus y) \wedge (z^* \oplus y) \\ &= x \sim y. \end{aligned}$$

Also, it is clear that (HE6) holds, because $\mathcal{M} = (H; \oplus, *, 0)$ is a linearly ordered hyper MV-algebra.

(HE7) From (HMV5), for any $x, y, z \in H$, we obtain that

$$\begin{aligned} 0^* = 1 &\in 0^* \oplus (z \oplus x^*)^* && \text{by (HMV6)} \\ &\subseteq ((x^* \oplus y)^* \oplus (x^* \oplus y)) \oplus (z \oplus x^*)^* && \text{by Proposition 4.4.1 (i)} \\ &= (x^* \oplus y)^* \oplus ((z \oplus x^*)^* \oplus (x^* \oplus y)) && \text{by Proposition 4.4.1 (i)} \\ &= (x^* \oplus y)^* \oplus (((z \oplus x^*)^* \oplus x^*) \oplus y)) && \text{by (HMV4)} \\ &= (x^* \oplus y)^* \oplus (((x \oplus z^*)^* \oplus z^*) \oplus y)) && \text{by Proposition 4.4.1 (i)} \\ &= (x^* \oplus y)^* \oplus ((x \oplus z^*)^* \oplus (z^* \oplus y)) \end{aligned}$$

and

$$\begin{aligned} 0^* = 1 &\in 0^* \oplus (y \oplus z^*)^* && \text{by (HMV6)} \\ &\subseteq ((x^* \oplus y)^* \oplus (x^* \oplus y)) \oplus (y \oplus z^*)^* && \text{by Proposition 4.4.1 (i)} \\ &= (x^* \oplus y)^* \oplus ((y \oplus z^*)^* \oplus (x^* \oplus y)) && \text{by Proposition 4.4.1 (i)} \\ &= (x^* \oplus y)^* \oplus (((y \oplus z^*)^* \oplus y) \oplus x^*)) && \text{by (HMV4)} \\ &= (x^* \oplus y)^* \oplus (((z \oplus y^*)^* \oplus z) \oplus x^*)) && \text{by Proposition 4.4.1 (i)} \\ &= (x^* \oplus y)^* \oplus ((y^* \oplus z)^* \oplus (x^* \oplus z)) \end{aligned}$$

Hence, for any $x, y, z \in H$,

$$x^* \oplus y \ll (z \oplus y^*)^* \oplus (z^* \oplus y) \quad \text{and} \quad x^* \oplus y \ll (y^* \oplus z)^* \oplus (x^* \oplus z)$$

Then, for any $x, y, z \in H$,

$$x \to y \ll (z \to x) \to (z \to y) \tag{12}$$

and

$$x \to y \ll (y \to z) \to (x \to z) \tag{13}$$

Without loss of generality, suppose $x, y, z \in H$ such that $x \leqslant y \leqslant z$. Since $\mathcal{M}$ is linearly ordered. Thus, $1 \in x \to z$ and $1 \in y \to z$. Then, by (12), we have

$$\begin{aligned} x \to y &\ll (z \to x) \to (z \to y) \\ &\ll ((z \to x) \wedge (x \to z)) \to ((z \to y) \wedge (y \to z)) \\ &= (x \sim z) \to (y \sim z) \end{aligned} \tag{14}$$

and similarly by (13), we get that

$$x \to y \ll (y \sim z) \to (x \sim z) \tag{15}$$

Thus, by (14) and (15), for all $x, y, z \in H$, we obtain that

$$x \to y \ll ((x \sim z) \to (y \sim z)) \wedge ((y \sim z) \to (x \sim z)) = (x \sim z) \sim (y \sim z)$$

Hence, (HE7) holds. Therefore, $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ is a hyper equality algebra. Moreover, let $x, y \in H$, then

$$(x \to y) \to y = (x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x = (y \to x) \to x \tag{16}$$

Hence, by (4) and (16), $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ is a strongly commutative, symmetric hyper equality algebra. $\square$

Theorem 4.4.3. *Every strongly commutative involutive hyper equality algebra $\mathcal{H} = \langle H, \sim, \wedge, 1\rangle$ is a hyper MV-algebra, if it satisfies in the condition (I) and, for all $x, y \in H$, $x \to y = y' \to x'$, where $'$ is the negation on H.*

Proof. Let $x, y \in H$ and define $x \oplus y = x' \to y$ and $x^* = x'$. We prove that $\mathcal{M} = (H; \oplus, *, 0)$ is a hyper MV-algebra.
(HMV1): Let $x, y, z \in H$. Then, by the condition (I), we have

$$\begin{aligned} x \oplus (y \oplus z) &= x' \to (y' \to z) = x' \to (z' \to y) = z' \to (x' \to y) \\ &= (x' \to y)' \to z = (x \oplus y) \oplus z. \end{aligned}$$

(HMV2): Let $x, y \in H$. Since $\mathcal{H}$ is involutive, by $x \to y = y' \to x'$, we have

$$x \oplus y = x' \to y = y' \to x = y \oplus x$$

Moreover, for any $x \in H$, it is clear that $x'' = x^{**} = \{x\}$. Then (HMV3) holds.

(HMV4): Since $\mathcal{H}$ is strongly commutative, for any $x, y \in H$, we have

$$(x^* \oplus y)^* \oplus y = (x \to y) \to y = (y \to x) \to x = (y^* \oplus x)^* \oplus x$$

(HMV5): By (HE3), for all $x \in H$,

$$0^* = 1 \in 0 \sim 0 = 0 \sim (0 \wedge x) = 0 \to x = (0')' \to x = 0^* \oplus x$$

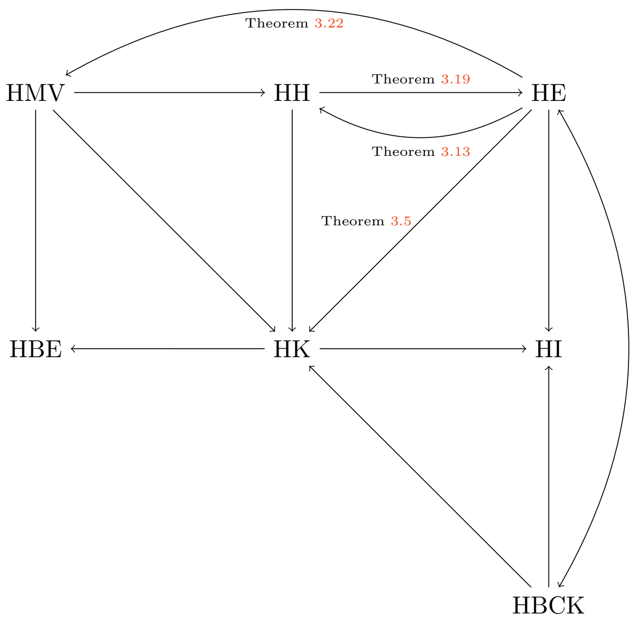
By (HE3) and Proposition 2.3 (i), clearly, (HMV6) and (HMV7) hold. Therefore, $(H; \oplus, *, 0)$ is a hyper MV-algebra. $\square$

Example 4.4.4. Let $\mathcal{H} = \langle H; \sim, \wedge, 1\rangle$ be a hyper equality algebra as in Example 4.3.4. It is a strongly commutative, involutive hyper equality algebra that satisfies in the conditions of Theorem 4.4.3. Hence, $\mathcal{M} = (H; \oplus, *, 0)$ is a hyper MV-algebra, where, for any $x, y \in H$, $x \oplus y = x' \to y$, $x^* = x'$ and $'$ is the negation in $\mathcal{H}$.

5. Conclusions and Future Works

In this paper, we investigate relation of hyper equality algebras and hyper K, BE, hoop and MV-algebras. Here, we briefly, recall all results that are obtained before and results that we have gave in this paper. Every hyper MV-algebra is a hyper K-algebra [7] and every hyper K-algebra is hyper BE-algebra [3]. Thus, these results imply that every hyper MV-algebra is a hyper BE-algebra. Also, it is easy to see that every hyper K-algebra is a hyper I-algebra. We show that every hyper equality algebra that satisfies the conditions (I) and (II) is a hyper K-algebra and hyper BE-algebra. This implies that every hyper equality algebra that satisfies (I) and (II) is a hyper I-algebra. We prove that under some conditions, every hyper equality algebra is a hyper hoop and vice versa. Also, every hyper hoop is a hyper K-algebra [3]. From [4], every hyper equality algebra that satisfies in (I) and (II), is a hyper BCK-meet-semilattice and the converse holds if the hyper BCK-meet-semilattice is linearly ordered and for all $x, y, z \in H$, $x \to y \ll (y \to z) \to (x \to z)$ and $x \leqslant y$ imply that $z \to x \leqslant z \to y$. Finally, we prove similar results between hyper MV-algebras and hyper equality algebras. In [3], authors showed that, a hyper MV-algebra $(M, +, *, 0)$, such that, for all $x, y, z \in M$, satisfies $0^* \in x^* + y$ and $0^* \in y^* + z$ imply $0^* \in x^* + z$, $(M, \odot, \to, 1)$ is a bounded hyper hoop. The states and homomorphisms on hyper equality algebras, some results on quotient structure and filter theory and positive implicative hyper equality algebras could be topics for our next task.

By the above results and our results in this paper, we summarize the obtained results by the following diagram.



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References

[1] R. A. Borzooei, A. Hasankhani, M. M. Zahedi, and Y. B. Jun, On hyper K-algebras, *Mathematica Japonicae*, 52 (1) (2000), 113-121.

[2] R. A. Borzooei, H. R. Varasteh, and K. Borna, On hyper hoop-algebras, *Ratio Mathematica*, 30 (2016), 67-81.

[3] R. A. Borzooei, H. R. Varasteh, K. Borna, and A. Radfar, Results on hyper hoop-algebras, *New Mathematics and Natural Computation*, 13 (1) (2017), 61-74.

[4] X. Y. Cheng, X. L. Xin, and Y. B. Jun, Hyper equality algebras, *Quantitative Logic and Soft Computing, Advances in Intelligent Systems and Computing*, 510 (2016), 415-428.

[5] L. C. Ciungu, On pseudo-equality algebras, *Archive for Mathematical Logic*, 53 (2014), 561-570.

[6] L. C. Ciungu, Internal states on equality algebras, *Soft Computing*, 19 (2015), 939-953.

[7] Sh. Ghorbani, A. Hasankhani, and E. Eslami, Hyper MV-algebras, *Set-Valued Mathematics Applications*, 1 (2008), 205-222.

[8] S. Jenei, Equality algebras, *Studia Logica*, 100 (2012), 1201-1209.

[9] S. Jenei and L. Kóródi, Pseudo equality algebras, *Archive for Mathematical Logic*, 52 (2013), 469-481.

[10] Y. B. Jun, M. M. Zahedi, X. L. Xin, and R. A. Borzooei, On hyper BCK-algebras, *Italian Journal of Pure and Applied Mathematics*, 8 (2000), 127-136.

[11] V. Novák and B. De Baets, EQ-algebras, *Fuzzy Sets and Systems*, 160 (2009), 2956-2978.

[12] A. Radfar, A. Rezaei, and A. Borumand Saeid, Hyper BE-algebras, *Novi Sad Journal of Mathematics*, 44 (2) (2014), 137-147.

[13] L. Torkzadeh and A. Ahadpanah, Hyper MV-ideals in hyper MV-algebras, *Mathematical Logic Quarterly*, 56 (1) (2010), 51-62.

[14] M. M. Zahedi, L. Torkzadeh, and R. A. Borzooei, Hyper I-algebras and polygroups, *Quasigroups and Related Systems*, 11 (2004), 103-113.

[15] F. Zebardast, R. A. Borzooei, and M. Aaly Kologani, Results on equality algebras, *Information Sciences*, 381 (2017), 270-282.

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