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Coincidence-Best Proximity Points of Contraction Pairs in Uniformly Convex Metric Spaces

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Abstract. A new class of contraction pairs in uniformly convex metric spaces is introduced. The coincidence and best proximity points of this class of mappings is studied, and finally, illustrative examples are given to support the new results.

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1. Introduction

Let (X,d) be a metric space, and A,B be subsets of X. A mapping $T : A \cup B \to A \cup B$ is called *cyclic* if $T(A) \subseteq B$ and $T(B) \subseteq A$; similarly, a mapping $S : A \cup B \to A \cup B$ is called *noncyclic* if $S(A) \subseteq A$ and $S(B) \subseteq B$. We begin with the following generalization of Banach's contraction principle.

Theorem 1.1. *(See [17]) Assume that (X,d) is a complete metric space, and that A and B are two nonempty, closed subsets of X. Assume further*

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that T is a cyclic mapping that satisfies

$$d(Tx,Ty) \leqslant \alpha\, d(x,y),$$

where $\alpha \in (0,1)$ and $x \in A,\ y \in B$. Then T has a unique fixed point in $A \cap B$.

Again, for given nonempty subsets A and B in X, a mapping $T : A\cup B \to A\cup B$ is called a *cyclic contraction* if T is cyclic and there is an $\alpha \in (0,1)$ such that

$$d(Tx,Ty) \leqslant \alpha\, d(x,y) + (1-\alpha)\mathrm{dist}(A,B);$$

where $x \in A,\ y \in B$, and

$$\mathrm{dist}(A,B) := \inf\{d(x,y) : (x,y) \in A \times B\}.$$

Given $T : A\cup B \to A\cup B$, we call an element $x \in A\cup B$ a best proximity point of T if

$$d(x,Tx) = \mathrm{dist}(A,B).$$

The following result for contractions was proved in [6].

Theorem 1.2. *(See [6]) Let X be a Banach space that is uniformly convex, and let A and B be two nonempty, closed, convex subsets in X. Assume further that $T : A \cup B \to A \cup B$ is a cyclic contraction mapping, and that $x_0 \in A$ is given. Put $x_{n+1} := Tx_n$ where $n \geqslant 0$. Then there is a unique element $x \in A$ such that x_{2n} tends to x, moreover*

$$\|x - Tx\| = \mathrm{dist}(A,B).$$

In the study of best proximity points, we usually consider a cyclic mapping T. The objective here is to minimize the expression $d(x,Tx)$ where x runs through the domain of T; that is $A \cup B$. In other words, we want to find

$$\min\{d(x,Tx) : x \in A \cup B\}.$$

If A and B intersect, the solution is clearly a fixed point of T; otherwise we have

$$d(x,Tx) \geqslant \mathrm{dist}(A,B), \qquad \forall x \in A \cup B,$$

so that the point at which the equality occurs is called a best proximity point of T. This point of view dominates the literature.

Recently, N. Shahzad, M. Gabeleh, and O. Olela Otafudu [24] considered two mappings T and S simultaneously and established the following result. Here T is assumed to be cyclic and S is assumed to be noncyclic. According to [24], for a nonempty pair of subsets (A, B) in X, and a cyclic-noncyclic pair $(T; S)$ on $A \cup B$ (that is, $T : A \cup B \to A \cup B$ is cyclic and $S : A \cup B \to A \cup B$ is noncyclic); they call a point $p \in A \cup B$ a *coincidence-best proximity point* for $(T; S)$ provided that

$$d(Sp, Tp) = \mathrm{dist}(A, B).$$

In the especial case that S equals the identity mapping, the point p will become a best proximity point of T. In the case that $\mathrm{dist}(A, B) = 0$, the point p is called a *coincidence point* of $(T; S)$ (see [10] and [13] for more information). With the definition just given, and depending on the situation as to whether S equals the identity mapping, or if the distance between A and B equals zero, one obtains a best proximity point of T, or a coincidence point of T and S. This was in fact the philosophy behind the phrase "coincidence-best proximity point". We start by recalling the notion of a cyclic-noncyclic contraction.

Definition 1.3. *(See [24]) Assume that (A, B) are nonempty subsets of a metric space (X, d) and that $T, S : A \cup B \to A \cup B$ are two mappings. We call the pair $(T; S)$ a cyclic-noncyclic contraction pair if it satisfies the following conditions:*

(1) $(T; S)$ is cyclic-noncyclic on $A \cup B$.

(2) There is a number $r \in (0, 1)$ for which

$$d(Tx, Ty) \leqslant rd(Sx, Sy) + (1 - r)\mathrm{dist}(A, B), \quad \forall x \in A,\ y \in B.$$

To state the main result of [24], we need to recall the notion of convexity in the framework of metric spaces. This concept was introduced by Takahashi [27], (see also [25]).

Definition 1.4. *Consider a metric space (X, d) and the interval $[0, 1]$. A mapping $\mathcal{W} : X \times X \times I \to X$ is said to be a convex structure on X if for every $(x, y; \lambda) \in X \times X \times I$ and $u \in X$,*

$$d(u, \mathcal{W}(x, y; \lambda)) \leqslant \lambda d(u, x) + (1 - \lambda)d(u, y).$$

A metric space (X, d) together with a convex structure $\mathcal{W}$, is called a *convex metric space*. In this case, we shall at times write $(X, d, \mathcal{W})$ is a convex structure. For example, all Banach spaces and the convex subsets of the Banach spaces satisfy this property. A subset K of $(X, d, \mathcal{W})$ is said to be convex if for each x and y in K and all $\lambda \in [0, 1]$ we have $\mathcal{W}(x, y; \lambda) \in K$.

In a similar fashion, one may define a uniformly convex subset of (X, d). Indeed, we say that the space $(X, d, \mathcal{W})$ is uniformly convex if given $\varepsilon > 0$, there is a number α depending on ε such that for $r > 0$ and $x, y, z \in X$ with $d(z, x) \leqslant r,\ d(z, y) \leqslant r$ and $d(x, y) \geqslant r\varepsilon$,

$$d(z, \mathcal{W}(x, y; \frac{1}{2})) \leqslant r(1 - \alpha) < r.$$

As a typical example of a uniformly convex metric space, we refer to uniformly convex Banach spaces.

Definition 1.5. (See [24]) *Assume that (X, d) is a metric space, and that A, B are two nonempty subsets of X. We call $S : A \cup B \to A \cup B$ a relatively anti-Lipschitzian mapping if there is a number $c > 0$ for which*

$$d(x, y) \leqslant c\, d(Sx, Sy), \quad x \in A,\ y \in B.$$

Theorem 1.6. *(See [24]) Let $(X, d, \mathcal{W})$ be a complete uniformly convex metric space, and let (A, B) be a nonempty, closed pair of subsets of X such that A is convex. Assume that $(T; S)$ is a cyclic-noncyclic contraction pair on $A \cup B$ such that $T(A) \subseteq S(B)$ and $T(B) \subseteq S(A)$, and that S is continuous on A and relatively anti-Lipschitzian on $A \cup B$. Then $(T; S)$ has a coincidence-best proximity point in A. Furthermore, if $x_0 \in A$ and $Sx_{n+1} := Tx_n$, then (x_{2n}) is convergent to the coincidence-best proximity point of $(T; S)$.*

Existence results related to best proximity pairs was first studied in [7]. They used a geometric property which is called *proximal normal structure*, indeed, they studied noncyclic relatively nonexpansive mappings. For details on the theory we refer the reader to [1, 2, 3, 4, 5, 8, 9, 12, 15, 22, 23, 26] and the references therein.

In the current paper, we study sufficient conditions which ensure the existence of coincidence-best proximity point for a pair of m-contraction mappings in convex metric spaces. We call a pair $(T;S)$ an m-contraction pair if there exists a nonnegative integer m satisfying the following conditions:

(1) $(T;S)$ is a cyclic-noncyclic pair on $A\cup B$.

(2) There exists a real number $\alpha\in(0,1)$ such that for each $(x,y)\in A\times B$ we have

$$d(Tx,Ty)\leqslant \alpha\, d(S^m x,S^m y)+(1-\alpha)\mathrm{dist}(A,B).$$

If $m=1$, we get back to Definition 1.3, but there are m-contraction pairs that are not cyclic-noncyclic contraction pairs (see Example 2.2 below). Therefore, we get a true generalization of known results.

2. The Main Result

We begin this section with the following definition.

Definition 2.1. *Assume that (X,d) is a metric space and that A,B are nonempty subsets of X. Assume further that the mappings $T,S: A\cup B\to A\cup B$ are given. We call $(T;S)$ an m-contraction pair if there exists a nonnegative integer m satisfying the following conditions:*

(1) (T,S) is a cyclic-noncyclic pair on $A\cup B$.

(2) There exists a real number $\alpha\in(0,1)$ such that for each $(x,y)\in A\times B$ we have

$$d(Tx,Ty)\leqslant \alpha\, d(S^m x,S^m y)+(1-\alpha)\mathrm{dist}(A,B).$$

If we define $S^0=I$, then every cyclic contraction pair is a 0-contraction pair. If $m=1$, the class of 1-contraction pairs reduces to the class of cyclic-noncyclic pairs. In the following, we verify that the new class is much wider.

Example 2.2. Let $X:=\mathbb{R}$ be equipped with $d(x,y)=|x-y|$. For $A=(-\infty,-3]$ and $B=[3,+\infty)$ we define $T,S: A\cup B\to A\cup B$ by

$$Tx:=-2x,\ \forall x\in A\cup B\ \ and\ \ Sx:=\begin{cases}2x+1 & if\ \ x\in A\\ 2x-1 & if\ \ x\in B.\end{cases}$$

Then $(T;S)$ is a 3-contraction pair with $\alpha = \frac{1}{4}$. Indeed,

$$S^2x := \begin{cases} 4x+3 \ if \ x \in A \\ 4x-3 \ if \ x \in B \end{cases} \quad and \quad S^3x := \begin{cases} 8x+7 \ if \ x \in A \\ 8x-7 \ if \ x \in B. \end{cases}$$

Thus, for all $(x,y) \in A \times B$ we have

$$\begin{aligned} |Tx - Ty| = (2y - 2x) &\leqslant \frac{1}{4}(8y - 8x - 14) + \frac{3}{4}(6) \\ &= \alpha|S^3x - S^3y| + (1-\alpha)\text{dist}(A,B). \end{aligned}$$

But, for each $\alpha \in (0,1)$, we have

$$\begin{aligned} |Tx - Ty| = (2y - 2x) &> 2y - 2x - 2 \\ &= |Sx - Sy| \\ &\geqslant \alpha|Sx - Sy| + (1-\alpha)\text{dist}(A,B), \end{aligned}$$

which means that $(T;S)$ is not a cyclic-noncyclic contraction pair.

Remark 2.3. *Notice that the condition (2) of the above definition implies that*

$$d(Tx,Ty) \leqslant d(S^m x, S^m y), \quad \forall (x,y) \in A \times B.$$

Moreover, if S is a noncyclic relatively nonexpansive mapping; which means that

$$d(Sx,Sy) \leqslant d(x,y), \quad \forall (x,y) \in A \times B,$$

then

$$d(S^m x, S^m y) \leqslant d(x,y), \quad \forall (x,y) \in A \times B,$$

that is, T is a cyclic contraction. In addition, if in the above definition S is assumed to be continuous, then T would be continuous as well.

We begin with the following lemma in which m is a fixed nonnegative integer.

Lemma 2.4. *Let (A,B) be a nonempty pair of subsets of a metric space (X,d) and let $(T;S)$ be a cyclic-noncyclic pair. Assume that $T(A) \subseteq S^m(B)$ and $T(B) \subseteq S^m(A)$. Then there exists sequences $\{x_{m,n}\}$ and*

$\{x_{1,n}\}$ in X such that for all $n \geqslant 0$ we have $Tx_{m,n} = S^m x_{m,n+1} = Sx_{1,n+1}$. Moreover, $\{x_{m,2n}\}$, $\{x_{1,2n}\}$ are sequences in A, and $\{x_{m,2n-1}\}$, $\{x_{1,2n-1}\}$ are sequences in B.

Proof. Let $x_{1,0} = x_{m,0} \in A$. Since $Tx_{m,0} \in S^m(B) \subseteq S(B)$, there exists $x_{m,1}, x_{1,1} \in B$ such that $Tx_{m,0} = S^m x_{m,1} = Sx_{1,1}$. Again, since $Tx_{m,1} \in S^m(A) \subseteq S(A)$, there exists $x_{m,2}, x_{1,2} \in A$ such that $Tx_{m,1} = S^m x_{m,2} = Sx_{1,2}$.

Continuing this process, we obtain sequences $\{x_{m,n}\}$, $\{x_{1,n}\}$ such that $\{x_{m,2n}\}$, $\{x_{1,2n}\}$ are in A and $\{x_{m,2n+1}\}$, $\{x_{1,2n+1}\}$ are in B, and $Tx_{m,n} = S^m x_{m,n+1} = Sx_{1,n+1}$ for all $n \geqslant 0$. $\square$

In the setting of the above lemma, it is clear that for each $m \in \mathbb{N}$ we have

$$S^m(A) \subseteq S(A), \quad S^m(B) \subseteq S(B),$$

because, the mapping S is noncyclic, moreover

$$S^m(A) \subseteq S^{m-1}(A) \subseteq \cdots \subseteq S^2(A) \subseteq S(A) \subseteq A,$$

$$S^m(B) \subseteq S^{m-1}(B) \subseteq \cdots \subseteq S^2(B) \subseteq S(B) \subseteq B.$$

Lemma 2.5. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be an m-contraction pair. Assume that $T(A) \subseteq S^m(B)$ and $T(B) \subseteq S^m(A)$. For $x_{1,0} = x_{m,0} \in A$, define $Tx_{m,n} = S^m x_{m,n+1} = Sx_{1,n+1}$ for each $n \geqslant 0$. Then we have*

$$d(Sx_{1,2n}, Sx_{1,2n+1}) \to \operatorname{dist}(A, B),$$

$$d(S^m x_{m,2n}, S^m x_{m,2n+1}) \to \operatorname{dist}(A, B).$$

Proof. We note that

$$\begin{aligned} d(Sx_{1,2n+1}, Sx_{1,2n+2}) &= d(S^m x_{m,2n+1}, S^m x_{m,2n+2}) \\ &= d(Tx_{m,2n}, Tx_{m,2n+1}) \\ &\leqslant \alpha\, d(S^m x_{m,2n}, S^m x_{m,2n+1}) + (1-\alpha)\mathrm{dist}(A,B) \\ &= \alpha\, d(Tx_{m,2n-1}, Tx_{m,2n}) + (1-\alpha)\mathrm{dist}(A,B) \\ &\leqslant \alpha\, [\alpha\, d(S^m x_{m,2n-1}, S^m x_{m,2n}) \\ &+ (1-\alpha)\mathrm{dist}(A,B)] + (1-\alpha)\mathrm{dist}(A,B) \\ &= \alpha^2 d(S^m x_{m,2n-1}, S^m x_{m,2n}) + (1-\alpha^2)\mathrm{dist}(A,B) \\ &= \alpha^2 d(Tx_{m,2n-2}, Tx_{m,2n-1}) + (1-\alpha^2)\mathrm{dist}(A,B) \\ &\leqslant \cdots \\ &\leqslant \alpha^{2n} d(Tx_{m,0}, Tx_{m,1}) + (1-\alpha^2)\mathrm{dist}(A,B). \end{aligned}$$

Now, if $n \to \infty$, we conclude that

$$d(Sx_{1,2n}, Sx_{1,2n+1}) \to \mathrm{dist}(A,B),$$

$$d(S^m x_{m,2n}, S^m x_{m,2n+1}) \to \mathrm{dist}(A,B). \quad \square$$

Lemma 2.6. Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be an m-contraction pair such that $T(A) \subseteq S^m(B)$ and $T(B) \subseteq S^m(A)$, moreover, assume that T and S commute on $A \cup B$. For $x_{1,0} = x_{m,0} \in A$, define $Tx_{m,n} = S^m x_{m,n+1} = Sx_{1,n+1}$ for each $n \geqslant 0$. Then we have

$$d(S^m x_{m,2n}, S^m x_{1,2n+1}) \to \mathrm{dist}(A,B).$$

Proof. We note that

$$
\begin{aligned}
d(S^m x_{m,2n}, S^m x_{1,2n+1}) &= d(S^m x_{m,2n}, S^{m-1}(Sx_{1,2n+1})) \\
&= d(S^m x_{m,2n}, S^{m-1}(Tx_{m,2n})) \\
&= d(S^m x_{m,2n}, T(S^{m-1}x_{m,2n})) \\
&= d(Tx_{m,2n-1}, T(S^{m-1}x_{m,2n})) \\
&\leqslant \alpha\, d(S^m x_{m,2n-1}, S^m(S^{m-1}x_{m,2n})) \\
&+ (1-\alpha)\mathrm{dist}(A,B) \\
&= \alpha\, d(S^m x_{m,2n-1}, S^{m-1}(S^m x_{m,2n})) \\
&+ (1-\alpha)\mathrm{dist}(A,B) \\
&= \alpha\, d(S^m x_{m,2n-1}, S^{m-1}(Tx_{m,2n-1})) \\
&+ (1-\alpha)\mathrm{dist}(A,B) \\
&= \alpha\, d(S^m x_{m,2n-1}, T(S^{m-1}x_{m,2n-1})) \\
&+ (1-\alpha)\mathrm{dist}(A,B) \\
&= \alpha\, d(Tx_{m,2n-2}, T(S^{m-1}x_{m,2n-1})) \\
&+ (1-\alpha)\mathrm{dist}(A,B) \\
&\leqslant \alpha\,[\alpha\, d(S^m x_{m,2n-2}, S^m(S^{m-1}x_{m,2n-1})) \\
&+ (1-\alpha)\mathrm{dist}(A,B)] + (1-\alpha)\mathrm{dist}(A,B) \\
&= \alpha^2 d(S^m x_{m,2n-2}, S^{m-1}(S^m x_{m,2n-1})) \\
&+ (1-\alpha^2)\mathrm{dist}(A,B)
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha^2 d(S^m x_{m,2n-2}, S^{m-1}(Tx_{m,2n-2})) \\
&+ (1-\alpha^2)\mathrm{dist}(A,B) \\
&\leqslant \cdots \\
&\leqslant \alpha^{2n} d(S^m x_{m,0}, S^{m-1}(Tx_{m,0})) \\
&+ (1-\alpha^{2n})\mathrm{dist}(A,B) \\
&= \alpha^{2n} d(S^m x_{m,0}, S^{m-1}(Sx_{1,1})) \\
&+ (1-\alpha^{2n})\mathrm{dist}(A,B) \\
&= \alpha^{2n} d(S^m x_{m,0}, S^m x_{1,1}) \\
&+ (1-\alpha^{2n})\mathrm{dist}(A,B).
\end{aligned}
$$

Now, if we let $n \to \infty$ in the above relation, we conclude that

$$d(S^m x_{m,2n}, S^m x_{1,2n+1}) \to \operatorname{dist}(A, B). \quad \square$$

Theorem 2.7. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be an m-contraction pair defined on $A \cup B$. Assume that $T(A) \subseteq S^m(B)$ and $T(B) \subseteq S^m(A)$ and S is continuous on A and assume T and S commute on $A \cup B$. For $x_{1,0} = x_{m,0} \in A$, define $Tx_{m,n} = S^m x_{m,n+1} = Sx_{1,n+1}$ for each $n \geqslant 0$. If $\{x_{1,2n}\}$ has a convergent subsequence in A, then the pair $(T; S)$ has a coincidence-best proximity point in A; that is, there exists $p \in A$ such that*

$$d(Sp, Tp) = \operatorname{dist}(A, B).$$

Proof. Let $\{x_{1,2n_k}\}$ be a subsequence of $\{x_{1,2n}\}$ such that $x_{1,2n_k} \to p \in A$. Then by Lemma 2.6 we have

$$\begin{aligned}
\operatorname{dist}(A, B) &\leqslant d(Sp, Tp) \\
&\leqslant d(Sp, Tx_{m,2n_k-1}) + d(Tx_{m,2n_k-1}, Tp) \\
&= d(Sp, Sx_{1,2n_k}) + d(Tp, Tx_{m,2n_k-1}) \\
&\leqslant d(Sp, Sx_{1,2n_k}) + d(S^m p, S^m x_{m,2n_k-1}) \\
&\to \operatorname{dist}(A, B),
\end{aligned}$$

that is,

$$d(Sp, Tp) = \operatorname{dist}(A, B). \quad \square$$

Lemma 2.8. *Let (A, B) be a nonempty pair of subsets of a metric space (X, d) and let $(T; S)$ be an m-contraction pair defined on $A \cup B$. Assume that $T(A) \subseteq S^m(B)$ and $T(B) \subseteq S^m(A)$ and assume T and S commute on $A \cup B$. For $x_{1,0} = x_{m,0} \in A$, define $Tx_{m,n} = S^m x_{m,n+1} = Sx_{1,n+1}$ for each $n \geqslant 0$. Then $\{S^m x_{m,2n}\}$, $\{S^m x_{1,2n}\}$ and $\{Sx_{1,2n}\}$ are bounded sequences in A and $\{S^m x_{m,2n+1}\}$, $\{S^m x_{1,2n+1}\}$ and $\{Sx_{1,2n+1}\}$ are bounded sequences in B.*

Proof. Since for each $n \geqslant 0$, $S^m x_{m,n} = Sx_{1,n}$, and since

$$d(Sx_{1,2n}, Sx_{1,2n+1}) \to \operatorname{dist}(A, B),$$

$$d(S^m x_{m,2n}, S^m x_{1,2n+1}) \to \operatorname{dist}(A,B),$$

and

$$d(S^m x_{m,2n}, S^m x_{m,2n+1}) \to \operatorname{dist}(A,B),$$

it suffices to show that $\{S^m x_{m,2n}\}$ is bounded in A. Assume to the contrary that there exists $N_0 \in \mathbb{N}$ such that

$$d(T(S^m x_{m,1}), S^m x_{m,2N_0+1}) > M,$$

$$d(T(S^m x_{m,1}), S^m x_{m,2N_0-1}) \leqslant M,$$

where,

$$\begin{aligned} M > \max\{&\frac{\alpha^2}{1-\alpha^2} d(S^m(S^m x_{m,0}), T(S^m x_{m,1})) \\ &+ \operatorname{dist}(A,B),\ d(T(S^m x_{m,1}), T x_{m,0})\}. \end{aligned}$$

By the above assumption, we have

$$\begin{aligned} \frac{M - \operatorname{dist}(A,B)}{\alpha^2} &+ \operatorname{dist}(A,B) \\ &< \frac{d(T(S^m x_{m,1}), S^m x_{m,2N_0+1}) - \operatorname{dist}(A,B)}{\alpha^2} \\ &+ \operatorname{dist}(A,B) \\ &\leqslant \frac{d(T(S^m x_{m,1}), S^m x_{m,2N_0+1})}{\alpha^2} \\ &+ \frac{(\alpha^2-1) d(T(S^m x_{m,1}), S^m x_{m,2N_0+1})}{\alpha^2} \\ &= d(T(S^m x_{m,1}), S^m x_{m,2N_0+1}) \\ &= d(T(S^m x_{m,1}), T x_{m,2N_0}) \\ &\leqslant d(S^m(S^m x_{m,1}), S^m x_{m,2N_0}) \\ &= d(S^m(T x_{m,0}), T x_{m,2N_0-1}) \\ &= d(T(S^m x_{m,0}), T x_{m,2N_0-1}) \\ &\leqslant d(S^m(S^m x_{m,0}), S^m x_{m,2N_0-1}) \\ &\leqslant d(S^m(S^m x_{m,0}), T(S^m x_{m,1})) \\ &+ d(T(S^m x_{m,1}), S^m x_{m,2N_0-1}) \\ &\leqslant d(S^m(S^m x_{m,0}), T(S^m x_{m,1})) + M. \end{aligned}$$

This implies that

$$\frac{M - \operatorname{dist}(A,B)}{\alpha^2} + \operatorname{dist}(A,B) < d(S^m(S^m x_{m,0}), T(S^m x_{m,1})) + M,$$

hence,

$$M - (1-\alpha^2)\operatorname{dist}(A,B) < \alpha^2[d(S^m(S^m x_{m,0}), T(S^m x_{m,1})) + M],$$

and therefore

$$(1-\alpha^2)M < \alpha^2 d(S^m(S^m x_{m,0}), T(S^m x_{m,1})) + (1-\alpha^2)\operatorname{dist}(A,B).$$

It now follows that

$$M < \frac{\alpha^2}{1-\alpha^2} d(S^m(S^m x_{m,0}), T(S^m x_{m,1})) + \operatorname{dist}(A,B),$$

which is a contradiction by the choice of M. □

We recall that a subset $A \subseteq X$ is said to be boundedly compact if the closure of every bounded subset of A is compact and is contained in A.

Theorem 2.9. *Let (A,B) be a nonempty pair of subsets of a metric space (X,d) such that A is boundedly compact and let $(T;S)$ be an m-contraction pair defined on $A \cup B$. Assume that $T(A) \subseteq S^m(B)$ and $T(B) \subseteq S^m(A)$ and that T and S commute on $A \cup B$. If S is relatively anti-Lipschitzian and continuous on A, then there exists $p \in A$ such that*

$$d(Sp, Tp) = \operatorname{dist}(A,B).$$

Proof. For $x_{1,0} = x_{m,0} \in A$, define $Tx_{m,n} = S^m x_{m,n+1} = Sx_{1,n+1}$ for each $n \geqslant 0$. Since by Lemma 2.8, $\{Sx_{1,2n}\}$ is bounded on A, and A is boundedly compact, there exists a subsequence $\{Sx_{1,2n_k}\}$ of $\{Sx_{1,2n}\}$ such that

$$Sx_{1,2n_k} \to Sp,$$

for some $p \in A$. We know that S is a relatively anti-Lipschitzian, therefore

$$d(x_{1,2n_k}, p) \leqslant c d(Sx_{1,2n_k}, Sp) \to 0, \quad k \to \infty.$$

This implies that $\{x_{1,2n_k}\}$ is a convergent subsequence of $\{x_{1,2n}\}$. Now, the results follows from Theorem 2.7. □

Remark 2.10. *Under the above conditions, the pair $(T; S^m)$ is a cyclic-noncyclic contraction, so that the Theorem 1.6 can be invoked to guarantee the existence of $p \in A \cup B$ such that*

$$d(S^m p, Tp) = \operatorname{dist}(A, B).$$

This, in general, does not imply that $d(Sp, Tp) = \operatorname{dist}(A, B)$ unless we already know that $S^m p = Sp$. This latter happens if p is the unique fixed point of S. We emphasize that we have made no assumption on the pair (A, B) to satisfy the so called P-property. The following examples reveal that the conclusion of Theorem 2.9 is not a consequence of Theorem 1.6.

Example 2.11. Consider $X := \mathbb{R}$ with the usual metric. For $A = (-\infty, -1]$ and $B = [1, +\infty)$ define $T, S : A \cup B \to A \cup B$ by

$$Tx := -x, \quad \forall x \in A \cup B \quad and \qquad Sx := \begin{cases} 2x + 1 \ if \ x \in A \\ 2x - 1 \ if \ x \in B. \end{cases}$$

Then $(T; S)$ is a 3-contraction pair with $\alpha = \frac{1}{8}$. Indeed,

$$S^2 x := \begin{cases} 4x + 3 \ if \ x \in A \\ 4x - 3 \ if \ x \in B \end{cases} \qquad and \quad S^3 x := \begin{cases} 8x + 7 \ if \ x \in A \\ 8x - 7 \ if \ x \in B. \end{cases}$$

Thus, for all $(x, y) \in A \times B$ we have

$$\begin{aligned} |Tx - Ty| = (y - x) &\leqslant \frac{1}{8}(8y - 8x - 14) + \frac{7}{8}(2) \\ &= \alpha |S^3 x - S^3 y| + (1 - \alpha)\operatorname{dist}(A, B). \end{aligned}$$

Also, $T(A) = B \subseteq S^3(B)$ and $T(B) = A \subseteq S^3(A)$. Moreover, S is continuous on A (note that the theorem just requires the continuity of S on A, not on the whole domain) and A is boundedly compact in X.

Besides, S is relatively anti-Lipschitzian on $A \cup B$ with $c = 1$. In fact, for all $(x, y) \in A \times B$ we have

$$|Sx - Sy| = 2y - 2x - 2 \geqslant |x - y|.$$

Finally, for each $x \in A$ we have

$$T(Sx) = -(2x + 1) = -2x - 1 = S(-x) = S(Tx),$$

and for each $x \in B$ we have

$$T(Sx) = -(2x-1) = -2x+1 = S(-x) = S(Tx),$$

that is, T and S commute on $A \cup B$. Therefore, the existence of coincidence-best proximity point of the pair $(T;S)$ follows from Theorem 2.9, that is, there exists $p \in A$ such that

$$|Tp - Sp| = \operatorname{dist}(A,B) = 2 \quad or \quad -p-(2p+1) = 2,$$

which implies that $p = -1$. In this case, p is a fixed point of S and so, p is a best proximity point of T.

The following example shows that there exists a coincidence-best proximity point that is not a fixed point for S^m and S. This means that the result of the above theorem does not follow from Theorem 1.6, in general.

Example 2.12. Let $X := \mathbb{R}$ be equipped with the usual metric. For $A = (-\infty, 1]$ and $B = [-1, +\infty)$ define $T, S : A \cup B \to A \cup B$ by

$$Tx := -x, \ \forall x \in A \cup B \quad and \quad Sx := \begin{cases} 2x-1 \ if \ x \in A \\ 2x+1 \ if \ x \in B. \end{cases}$$

Then $(T;S)$ is a 2-contraction pair with $\alpha = \frac{1}{4}$. Indeed,

$$S^2x := \begin{cases} 4x-3 \ if \ x \in A \\ 4x+3 \ if \ x \in B. \end{cases}$$

Thus, for all $(x,y) \in A \times B$ we have

$$\begin{aligned} |Tx - Ty| = |y-x| &\leqslant \frac{1}{4}|4y-4x+6| + \frac{3}{4}(0) \\ &= \alpha|S^2y - S^2x| + (1-\alpha)\operatorname{dist}(A,B). \end{aligned}$$

Also, $T(A) = B \subseteq S^2(B)$ and $T(B) = A \subseteq S^2(A)$. Moreover, S is continuous on A and A is boundedly compact in X.

Besides, S is relatively anti-Lipschitzian on $A \cup B$ with $c = 1$. In fact, for all $(x,y) \in A \times B$ we have

$$|Sy - Sx| = |2y - 2x + 2| \geqslant |x-y|.$$

Finally, for each $x \in A$ we have

$$T(Sx) = -(2x-1) = -2x+1 = S(-x) = S(Tx)$$

and for each $x \in B$ we have

$$T(Sx) = -(2x+1) = -2x-1 = S(-x) = S(Tx),$$

that is, T and S commute on $A \cup B$. Thus, the existence of coincidence-best proximity point of the pair $(T;S)$ follows from Theorem 2.9, that is, there exists $p \in A$ such that

$$|Tp - Sp| = \operatorname{dist}(A,B) = 0 \quad or \quad -p-(2p-1) = 0,$$

which implies that $p = \frac{1}{3}$. In this case, p is not a fixed point of the mapping S and so, p is a best proximity point of the cyclic mapping T.

3. Uniformly Convex Metric Spaces

In this section we prove the same result in the setting of uniformly convex metric spaces. We begin with the following lemma.

Lemma 3.1. *Let (A,B) be a nonempty pair of subsets of a uniformly convex metric space $(X,d,\mathcal{W})$ such that A is convex. Let $(T;S)$ be an m-contraction pair defined on $A \cup B$ such that $T(A) \subseteq S^m(B)$ and $T(B) \subseteq S^m(A)$. For $x_{1,0} = x_{m,0} \in A$, define $Tx_{m,n} = S^m x_{m,n+1} = Sx_{1,n+1}$ for each $n \geqslant 0$. Then*

$$d(Sx_{1,2n+2}, Sx_{1,2n}) \to 0,$$

$$d(Sx_{1,2n+3}, Sx_{1,2n+1}) \to 0,$$

$$d(S^m x_{m,2n+2}, S^m x_{m,2n}) \to 0,$$

and

$$d(S^m x_{m,2n+3}, S^m x_{m,2n+1}) \to 0.$$

Proof. We prove that $d(S^m x_{m,2n+2}, S^m x_{m,2n}) \to 0$. Assume to the contrary that there exists $\varepsilon_0 > 0$ such that for each $k \geqslant 1$, there exists $n_k \geqslant k$ for which

$$d(S^m x_{m,2n_k+2}, S^m x_{m,2n_k}) \geqslant \varepsilon_0.$$

Choose $0 < \gamma < 1$ such that $\frac{\varepsilon_0}{\gamma} > \operatorname{dist}(A, B)$ and choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \min\left\{\frac{\varepsilon_0}{\gamma} - \operatorname{dist}(A, B), \frac{\operatorname{dist}(A, B)\alpha(\gamma)}{1 - \alpha(\gamma)}\right\}.$$

By Lemma 2.5, since $d(S^m x_{m,2n_k}, S^m x_{m,2n_k+1}) \to \operatorname{dist}(A, B)$, there exists $N \in \mathbb{N}$ such that

$$d(S^m x_{m,2n_k}, S^m x_{m,2n_k+1}) \leqslant \operatorname{dist}(A, B) + \varepsilon,$$

$$d(S^m x_{m,2n_k+2}, S^m x_{m,2n_k+1}) \leqslant \operatorname{dist}(A, B) + \varepsilon,$$

and

$$d(S^m x_{m,2n_k}, S^m x_{m,2n_k+2}) \geqslant \varepsilon_0 > \gamma(\operatorname{dist}(A, B) + \varepsilon).$$

It now follows from the uniform convexity of X and the convexity of A that

$$\begin{aligned}
\operatorname{dist}(A, B) &\leqslant d(S^m x_{m,2n_k+1}, \mathcal{W}(S^m x_{m,2n_k}, S^m x_{m,2n_k+2}, \frac{1}{2})) \\
&\leqslant (\operatorname{dist}(A, B) + \varepsilon)(1 - \alpha(\gamma)) \\
&< \operatorname{dist}(A, B) + \frac{\operatorname{dist}(A, B)\alpha(\gamma)}{1 - \alpha(\gamma)}(1 - \alpha(\gamma)) \\
&= \operatorname{dist}(A, B),
\end{aligned}$$

which is a contradiction. Similarly, we see that $d(S^m x_{m,2n+3}, S^m x_{m,2n+1}) \to 0$. Since for all $n \geqslant 0$ we have

$$S^m x_{m,n} = S x_{1,n},$$

the result follows. □

The following Theorem guarantees the existence and convergence of coincidence-best proximity points for m-contraction mappings in uniformly convex metric spaces.

Theorem 3.2. *Let (A, B) be a nonempty, closed pair of subsets of a complete uniformly convex metric space $(X, d; \mathcal{W})$ such that A is convex. Let $(T; S)$ be an m-contraction pair defined on $A \cup B$ such that $T(A) \subseteq S^m(B)$ and $T(B) \subseteq S^m(A)$, and that S is continuous on A*

and relatively anti-Lipschitzian on $A \cup B$. Assume further that T and S commute on $A \cup B$. Then there exists $p \in A$ such that

$$d(Sp, Tp) = \operatorname{dist}(A, B).$$

Further, if $x_{1,0} = x_{m,0} \in A$ and $Tx_{m,n} = S^m x_{m,n+1} = Sx_{1,n+1}$, then $\{x_{1,2n}\}$ converges to the coincidence-best proximity point of $(T; S)$.

Proof. For $x_{1,0} = x_{m,0} \in A$ define $Tx_{m,n} = S^m x_{m,n+1} = Sx_{1,n+1}$ for each $n \geqslant 0$. We prove that $\{S^m x_{m,2n}\}$ and $\{S^m x_{m,2n+1}\}$ are Cauchy sequences. At first, we verify that for each $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that

$$d(S^m x_{m,2l}, S^m x_{m,2n+1}) < \operatorname{dist}(A, B) + \varepsilon, \quad \forall l > n \geqslant N_0. \qquad (*)$$

Assume to the contrary that there exists $\varepsilon_0 > 0$ such that for each $k \geqslant 1$ there exists $l_k > n_k \geqslant k$ satisfying

$$d(S^m x_{m,2l_k}, S^m x_{m,2n_k+1}) \geqslant \operatorname{dist}(A, B) + \varepsilon_0,$$

$$d(S^m x_{m,2l_k-2}, S^m x_{m,2n_k+1}) < \operatorname{dist}(A, B) + \varepsilon_0.$$

We now have

$$\begin{aligned}
\operatorname{dist}(A, B) + \varepsilon_0 &\leqslant d(S^m x_{m,2l_k}, S^m x_{m,2n_k+1}) \\
&\leqslant d(S^m x_{m,2l_k}, S^m x_{m,2l_k-2}) \\
&\quad + d(S^m x_{m,2l_k-2}, S^m x_{m,2n_k+1}) \\
&\leqslant d(S^m x_{m,2l_k}, S^m x_{m,2l_k-2}) + \operatorname{dist}(A, B) + \varepsilon_0.
\end{aligned}$$

Letting $k \to \infty$, we obtain

$$d(S^m x_{m,2l_k}, S^m x_{m,2n_k+1}) \to \operatorname{dist}(A, B) + \varepsilon_0.$$

Besides, we have

$$\begin{aligned}
\operatorname{dist}(A, B) + \varepsilon_0 &\leqslant d(S^m x_{m,2l_k}, S^m x_{m,2n_k+1}) \\
&= d(Tx_{m,2l_k-1}, Tx_{m,2n_k}) \\
&\leqslant \alpha d(S^m x_{m,2l_k-1}, S^m x_{m,2n_k}) + (1-\alpha)\operatorname{dist}(A, B) \\
&= \alpha d(Tx_{m,2l_k-2}, Tx_{m,2n_k-1}) + (1-\alpha)\operatorname{dist}(A, B) \\
&\leqslant \alpha d(S^m x_{m,2l_k-2}, S^m x_{m,2n_k-1}) + (1-\alpha)\operatorname{dist}(A, B).
\end{aligned}$$

Therefore, letting $k \to \infty$ we obtain

$$\begin{aligned}\operatorname{dist}(A,B)+\varepsilon_0 &\leqslant \alpha(\operatorname{dist}(A,B)+\varepsilon_0)+(1-\alpha)\operatorname{dist}(A,B)\\ &\leqslant \operatorname{dist}(A,B)+\varepsilon_0.\end{aligned}$$

This implies that $\alpha = 1$, which is a contradiction. That is, $(*)$ holds. Now, assume $\{S^m x_{m,2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon_0 > 0$ such that for each $k \geqslant 1$ there exists $l_k > n_k \geqslant k$ for which

$$d(S^m x_{m,2l_k}, S^m x_{m,n_k}) \geqslant \varepsilon_0.$$

Choose $0 < \gamma < 1$ such that $\frac{\varepsilon_0}{\gamma} > \operatorname{dist}(A,B)$ and choose $\varepsilon > 0$ such that

$$0 < \varepsilon < \min\left\{\frac{\varepsilon_0}{\gamma} - \operatorname{dist}(A,B), \frac{\operatorname{dist}(A,B)\alpha(\gamma)}{1-\alpha(\gamma)}\right\}.$$

Let $N \in \mathbb{N}$ be such that

$$d(S^m x_{m,2n_k}, S^m x_{m,2n_k+1}) \leqslant \operatorname{dist}(A,B)+\varepsilon, \quad \forall n_k \geqslant N$$

and

$$d(S^m x_{m,2l_k}, S^m x_{m,2n_k+1}) \leqslant \operatorname{dist}(A,B)+\varepsilon, \quad \forall l_k > n_k \geqslant N.$$

Now, by the Uniform convexity of X we deduce that

$$\begin{aligned}\operatorname{dist}(A,B) &\leqslant d(S^m x_{m,2n_k+1}, \mathcal{W}(S^m x_{m,2n_k}, S^m x_{m,2l_k}, \frac{1}{2}))\\ &\leqslant (\operatorname{dist}(A,B)+\varepsilon)(1-\alpha(\gamma))\\ &< \operatorname{dist}(A,B),\end{aligned}$$

which is a contradiction. Therefore, $\{S^m x_{m,2n}\}$ is a Cauchy sequence in A and since for all $n \geqslant 0$ we have $S^m x_{m,n} = Sx_{1,n}$, we conclude that $\{Sx_{1,2n}\}$ is a Cauchy sequence in A. By the fact that S is continuous on A and anti-Lipschitzian on $A \cup B$, we have

$$d(x_{1,2l}, x_{1,2n}) \leqslant c\, d(Sx_{1,2l}, Sx_{1,2n}) \to 0, \quad l,n \to \infty,$$

that is, $\{x_{1,2n}\}$ is a Cauchy sequence. Since A is complete, there exists $p \in A$ such that $x_{1,2n} \to p$. Now, the result follows from a similar argument as in the proof of Theorem 2.7. $\square$

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