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Eventual Stability with Respect to Part of Variables of Nonlinear Differential Equations with Non-instantaneous Impulses

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Abstract. The eventual stability with respect to part of variables of a nonlinear differential equation with non-instantaneous impulses is studied using Lyapunov like functions. In these differential equations there are impulses, which start abruptly at some points and their action continue on given finite intervals. Sufficient conditions for eventual stability, uniform eventual stability and eventual asymptotic uniform stability with respect to part of variables of the zero solution are established. Examples are given to illustrate the results.

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1. Introduction

In the real world life there are many processes and phenomena that are characterized by rapid changes in their state. In the literature there are two popular types of impulses:

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- *instantaneous impulses*-the duration of these changes is relatively short compared to the overall duration of the whole process. The model is given by impulsive differential equations (see, for example, [1], [4], [5], [8]-[10], the monographs [7], [11] and the cited references therein);

- *non-instantaneous impulses* - an impulsive action, which starts at an arbitrary fixed point and remains active on a finite time interval. E. Hernandez and D. O'Regan ([6]) introduced this new class of abstract differential equations where the impulses are not instantaneous and they investigated the existence of mild and classical solutions. For recent work we refer the reader to [13]-[16]. An overview of the main properties of the presence of non-instantaneous impulses to ordinary differential equations, to fractional differential equations is given in the book [3].

In this paper the impulses start abruptly at some points and their action continue on given finite intervals. As a motivation for the study of these systems we consider the following simplified situation concerning the hemodynamical equilibrium of a person. In the case of a decompensation (for example, high or low levels of glucose) one can prescribe some intravenous drugs (insulin). Since the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes, we can interpret the situation as an impulsive action which starts abruptly and stays active on a finite time interval. The model of this situation is the so called non-instantaneous impulsive differential equation.

The concept of eventual stability of a point was introduced by La Salle and Rath [12] for ordinary differential equations to overcome the fact that for many perturbation and adaptive control problems the point in question may not be an equilibrium (invariant) point.

In this paper different types of eventual stability with respect to part of variables of solutions of non-instantaneous impulsive nonlinear differential equations are defined and studied. Several sufficient conditions for eventual stability, uniform eventual stability and eventual asymptotic uniform stability with respect to part of variables are obtained. Some examples illustrating our results are given. Note non-instantaneous impulsive differential equations are natural generalizations of impulsive differential equations and the eventual stability with respect to part of variables is a generalization of the eventual stability.

2. Statement of the Problem and Auxiliary Results

In this paper we will assume two increasing sequences of points $\{t_i\}_{i=1}^{\infty}$ and $\{s_i\}_{i=0}^{\infty}$ are given such that $s_0 = 0 < s_i \leqslant t_i < s_{i+1}$, $i = 1, 2, \dots$, and $\lim_{k\to\infty} s_k = \infty$.

Remark 2.1. *The intervals* $(s_k, t_k]$*,* $k = 1, 2, \dots$ *are called intervals of non-instantaneous impulses.*

Let $t_0 \in R_+$ be a given arbitrary point. Keeping in mind the meaning of t_0 as an initial time of the modeled the rate of change of the process, we will assume everywhere in the paper that the initial time t_0 is not in an interval of non-instantaneous impulses. Without loss of generality we will assume that $0 \leqslant t_0 < s_1$.

Consider the initial value problem for the system of *non-instantaneous impulsive differential equations* (NIDE)

$$\begin{aligned} &x' = f(t, x) \text{ for } t \in (t_k, s_{k+1}],\ k = 0, 1, 2, \dots \\ &x(t) = \phi_i(t, x(t_i - 0)) \quad \text{for } t \in (s_i, t_i],\ i = 1, 2, \dots, \\ &x(t_0) = x_0, \end{aligned} \tag{1}$$

where $x, x_0 \in R^n$, $f : \cup_{k=0}^{\infty}[t_k, s_{k+1}] \times R^n \to R^n$, $\phi_i : [s_i, t_i] \times R^n \to R^n$, $(i = 1, 2, 3, \dots)$.

Remark 2.2. *The functions* $\phi_k(t, x,$ $k = 1, 2, \dots,$ *are called non-instantaneous impulsive functions.*

Remark 2.3. *If* $t_k = s_k$*,* $k = 1, 2, \dots$ *then the IVP for NIDE (1) reduces to an IVP for impulsive differential equations (for example see the monographs [7], [11] and the cited references therein). In this case at any point of instantaneous impulse* t_k *the amount of jump of the solution* $x(t)$ *is given by* $I_k = \phi_k(t_k, x(t_k - 0)) - x(t_k - 0)$.

Let $J \subset R_+$ be a given interval. Introduce the following classes of functions

$$\begin{aligned} &PC^1(J) = \{u : J \to R^n :\ u \in C^1(J \cap \big(\cup_{k=0}^{\infty} (s_k, t_{k+1}]\big), R^n) : \\ &u(t_k) = u(t_k - 0) = \lim_{t\uparrow t_k} u(t) < \infty, \\ &\quad u(t_k + 0) = \lim_{t\downarrow t_k} u(t) < \infty, \quad k :\ t_k \in J\}, \end{aligned}$$

$PC(J) = \{u : J \to R^n : \ u \in C(J \cap \big(\cup_{k=0}^{\infty} (s_k, t_{k+1}]\big), R^n) :$
$u(t_k) = u(t_k - 0) = \lim_{t \uparrow t_k} u(t) < \infty,$
$u(t_k + 0) = \lim_{t \downarrow t_k} u(t) < \infty, \quad k : \ t_k \in J\}.$

Remark 2.4. *Any solution of (1) is from the class $PC^1([t_0, b))$, $b \leqslant \infty$, i.e. any solution might have a discontinuity at any point $t_k, k = 1, 2, \dots$.*

Let m, l are non-negative integers such that $n = m + l$. Let $x \in R^n$: $x = (y, z), \ y \in R^m, z \in R^l$. Then the system (1) could be written in the form

$$
\begin{aligned}
& y' = F(t, y, z) \qquad z' = G(t, y, z) \quad \text{for } t \in (t_k, s_{k+1}], \ k = 0, 1, 2, \dots \\
& y(t) = \Phi_i(t, y(t_i - 0), z(t_i - 0)) \qquad z(t) = \Psi_i(t, y(t_i - 0), z(t_i - 0)) \\
& \text{for } t \in (s_i, t_i], \ i = 1, 2, \dots, \qquad\qquad (2) \\
& y(t_0) = y_0, \quad z(t_0) = z_0
\end{aligned}
$$

where $F : \cup_{k=0}^{\infty} [t_k, s_{k+1}] \times R^n \to R^m$, $G : \cup_{k=0}^{\infty} [t_k, s_{k+1}] \times R^n \to R^l$,

$\Phi_i : [s_i, t_i] \times R^n \times R^n \to R^m, \Psi_i : [s_i, t_i] \times R^n \times R^n \to R^l$, $(i = 1, 2, 3, \dots)$.

Consider the corresponding IVP for ODE

$$
y' = F(t, y, z) \qquad z' = G(t, y, z) \quad \text{for } t \in \ [\tau, s_{p+1}] \quad \text{with } \ x(\tau) = \tilde{x}_0, \quad (3)
$$

where $\tau \in [t_p, s_{p+1})$, $p = 0, 1, 2, \dots$.

We will say condition (H1) is satisfied if

(H1) The function $F \in C([0, s_1] \bigcup \cup_{k=1}^{\infty} [t_k, s_{k+1}] \times R^n, R^m)$ and the function $G \in C([0, s_1] \bigcup \cup_{k=1}^{\infty} [t_k, s_{k+1}] \times R^n, R^l)$ are such that for any initial point $(\tau, \tilde{x}_0) : \ t_p \leqslant \tau < s_{p+1}, \ \tilde{x}_0 \in R^n$, the IVP for the system of ODE (3) has a solution $x(t; \tau, \tilde{x}_0) \in C^1([\tau, s_{p+1}], R^n)$.

We give a definition for different types of stability with respect to part of the variables of the zero solution. In the definition below we denote by $x(t; t_0, x_0) \in PC^1([t_0, \infty), R^n)$ any solution of (1), respectively, (2) with $x(t; t_0, x_0) = (y(t; t_0, y_0, z_0), z(t; t_0, y_0, z_0))$.

Definition 2.5. *The zero solution of the IVP for NIDE (1) is said to be*

• *eventually stable w.r.t. y if for every $\epsilon > 0$ and $t_0 \in [0, s_0) \bigcup \cup_{k=1}^{\infty} [t_k, s_{k+1})$ there exist $\delta = \delta(\epsilon, t_0) > 0$ and $\tau = \tau(\varepsilon) > 0$ such that for any $x_0 \in R^n$*

the inequality $||x_0|| < \delta$ *implies* $||y(t;t_0,x_0)|| < \epsilon$ *for* $t \geqslant t_0 \geqslant \tau$*;*

• *eventually uniformly stable w.r.t.* y *if for every* $\epsilon > 0$ *there exist* $\delta = \delta(\epsilon) > 0$ *and* $\tau = \tau(\epsilon) > 0$ *such that for any initial point* $t_0 \in [0,s_0)\bigcup\cup_{k=1}^{\infty}[t_k,s_{k+1})$: $t \geqslant \tau$ *and any initial value* $x_0 \in R^n$ *with* $||x_0|| < \delta$ *the inequality* $||y(t;t_0,x_0)|| < \epsilon$ *holds for* $t \geqslant t_0$*;*

• *eventually uniformly attractive w.r.t.* y *if there exists* $\beta > 0$ *such that for every* $\epsilon > 0$ *there exists* $T = T(\epsilon) > 0$ *and* $\tau = \tau(\epsilon) > 0$ *such that for any initial point* $t_0 \in [0,s_0)\bigcup\cup_{k=1}^{\infty}[t_k,s_{k+1})$: $t \geqslant \tau$ *and any initial value* $x_0 \in R^n$ *with* $||x_0|| < \beta$ *the inequality* $||x(t;t_0,x_0)|| < \epsilon$ *holds for* $t \geqslant t_0 + T$*;*

• *eventually uniformly asymptotically stable w.r.t.* y *if the zero solution is eventually uniformly stable w.r.t.* y *and eventually uniformly attractive w.r.t.* y.

In this paper we will use the followings sets:

$$\begin{aligned}
\mathcal{K} &= \{a \in C[\mathbb{R}_+,\mathbb{R}_+] : a \text{ is strictly increasing and } a(0) = 0\},\\
S(A) &= \{y \in R^m : ||y|| \leqslant A\}, \quad A > 0.
\end{aligned}$$

We now introduce the class Λ of Lyapunov-like functions which will be used to investigate the stability of the zero solution of the system NIDE (1).

Definition 2.6. *Let* $J \subset \mathbb{R}_+$ *be a given interval, and* $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$ *be a given set. We will say that the function* $V(t,x) : J \times \Delta \to \mathbb{R}_+$, $V(t,0) \equiv 0$ *belongs to the class* $\Lambda(J,\Delta)$ *if*

1. The function $V(t,x)$ *is continuous on* $J/\{s_k \in J\} \times \Delta$ *and it is locally Lipschitz with respect to its second argument;*

2. For each $s_k \in J$ *and* $x \in \Delta$ *there exist finite limits*

$$V(s_k - 0, x) = \lim_{t \uparrow s_k} V(t,x), \qquad \text{and} \;\; V(s_k + 0, x) = \lim_{t \downarrow s_k} V(t,x)$$

and $V(s_k - 0, x) = V(s_k, x)$.

In this paper we will use piecewise continuous Lyapunov functions from the introduced above class $\Lambda([t_0,T),\Delta)$. We will define the *generalized*

Dini derivative of the function $V(t,x) \in \Lambda([t_0,T),\Delta)$ along trajectories of solutions of IVP for the system NIDE (1) by:

$$ {}_{(1)}D_+V(t,x) = \limsup_{h\to 0^+} \frac{1}{h}\Big\{V(t,x) - V(t-h, x-hf(t,x))\Big\} \quad (4) $$

for $t \in (t_k, s_{k+1}),\ k = 0,1,2,\dots,$

where $x \in \Delta$, and for any $t \in (t_k, s_{k+1})$ there exists $h_t > 0$ such that $t-h \in (t_k, s_{k+1})$, $x - hf(t,x) \in \Delta$ for $0 < h \leqslant h_t$.

We will use some comparison results for NIDE (1) by applying Lyapunov functions.

We will use the following comparison results for NIDE (1).

Lemma 2.7. (*Comparison result for NIDE)[2]. Assume the following conditions are satisfied:*

1. *The function $x^*(t) = x(t;t_0,x_0) \in PC^1([t_0,T],\Delta)$ is a solution of the NIDE (1) where $t_0 \in [0,s_1), T \leqslant \infty$ are given constants, $\Delta \subset R^n$, $0 \in \Delta$.*

2. *The function $V \in \Lambda([t_0,T],\Delta)$ and*

   (i) the inequality ${}_{(1)}D_+V(t,x^*(t)) \leqslant 0$ for $t \in [t_0,T] \cap \Big(\cup_{k=0}^{\infty} (t_k, s_{k+1})\Big)$ holds;

   (ii) for any $k = 1,2,3,\dots$ the inequalities

   $$ V(t,x^*(t)) \leqslant V(s_k - 0, x^*(s_k - 0)) \quad \text{for } t \in [t_0,T] \cap (s_k, t_k] $$

   hold.

Then the inequality $V(t,x^*(t)) \leqslant V(t_0,x_0)$ holds on $[t_0,T]$.

Note in the case $T = \infty$ the interval is $[t_0,\infty)$.

Lemma 2.8. *(Comparison result for NIDE, negative Dini derivtive). [2]. Assume the following conditions are satisfied:*

1. *The function $x^*(t) = x(t;t_0,x_0) \in PC^1([t_0,T],\Delta)$ is a solution of the NIDE (1) where $\Delta \subset R^n,\ 0 \in \Delta$, $x_0 \in \Delta$ and $t_0,\ T \in R_+$, $t_0 < T,\ 0 \leqslant t_0 < s_1$ are given numbers.*

2. The function $V \in \Lambda([t_0, T], \Delta)$ *and*

(i) the inequality ${}_{(1)}D_+V(t, x^*(t)) \leqslant -c(||x^*(t)||)$ *for* $t \in \Big(\cup_{k=0}^{\infty} (t_k, s_{k+1})\Big) \cap [t_0, T]$ *holds where* $c \in \mathcal{K}$*;*

(ii) for any $k = 1, 2 \dots$ *the inequalities*

$$V(t, x^*(t)) \leqslant V(t_k - 0, x^*(s_k - 0)) \quad \text{for } t \in [t_0, T] \cap (s_k, t_k]$$

hold.

$$V(t, x^*(t)) \leq \begin{cases} V(t_0, x_0) - \int_{t_0}^{t} c(||x^*(s)||)ds \quad \text{for } t \in [t_0, s_1], \\ V(t_0, x_0) - \sum_{i=0}^{k-1} \int_{t_i}^{s_{i+1}} c(||x^*(s)||)ds \\ \qquad \text{for } t \in (s_k, t_k] \cap [t_0, T],\ k \geq 1 \\ V(t_0, x_0) - \Big(\sum_{i=0}^{k-1} \int_{t_i}^{s_{i+1}} c(||x^*(s)||)ds + \int_{t_k}^{t} c(||x^*(s)||)ds \Big) \\ \qquad \text{for } t \in (t_k, s_{k+1}] \cap [t_0, T],\ k \geq 1 \end{cases}$$

holds.

3. Main Results

We study the eventual stability properties w.r.t. part of variables of the zero solution of nonlinear differential equations with non-instantaneous impulses.

We say condition (H2) is satisfied if :

(H2) The functions $\Phi_k \in C([s_k, t_k] \times R^n \times R^n, R^m)$, $\Psi_k \in C([s_k, t_k] \times R^n \times R^n, R^l)$, and $\Phi_k(t, 0, 0) \equiv 0$, $\Psi_k(t, 0, 0) \equiv 0$ for $t \in [s_k, t_k]$, $k = 1, 2, \dots$.

Remark 3.1. *Conditions (H1) and (H2) guarantee the existence of a solution* $x(t; t_0, x_0) \in PC^1([t_0, \infty), R^n)$ *of NIDE (1) for any initial data* $(t_0, x_0) \in \cup_{k=0}^{\infty} [s_k, t_{k+1}) \times R^n$*. If additionally,* $f(t, 0) \equiv 0$ *for* $t \in [0, s_1] \bigcup \cup_{k=1}^{\infty} [s_k, t_{k+1}]$*, then (1) (with* $x_0 = 0$*) has a zero solution.*

Theorem 3.2. *(Eventual stability w.r.t. part of variables). Let the following conditions be satisfied:*

1. *Conditions (H1) and (H2) are satisfied and* $f(t, 0) \equiv 0$ *for* $t \in [0, s_1] \bigcup \cup_{k=1}^{\infty} [t_k, s_{k+1}]$.
2. *There exists a function* $V \in \Lambda(R_+, R^n)$ *such that* $V(t, 0) = 0$ *and a decreasing function* $\Theta \in C(R_+, R_+)$ *such that*
   - (i) *for any* $t \in [0, s_1) \bigcup \cup_{k=1}^{\infty} (t_k, s_{k+1})$ *and* $x \in R^n : \ x = (y, z)$ *such that* $t \geqslant \Theta(r) : \ 0 < r \leqslant ||y||$ *the inequality is satisfied*
   $$_{(1)}D_+V(t, x) \leqslant 0; \quad (5)$$
   - (ii) *for any* $x \in R^n$ *and any* $t \in (s_k, t_k]$, $k = 1, 2, 3, \ldots$ *such that* $t \geqslant \Theta(r)$ *for* $0 < r \leqslant ||y||$ *the inequality* $V(t, \phi_k(t, x)) \leqslant V(s_k - 0, x)$ *holds;*
   - (iii) *for any points* $x \in R^n : \ x = (y, z)$ *and* $t \geqslant \Theta(r)$ *for* $0 < r \leqslant ||y||$ *the inequality* $b(||y||) \leqslant V(t, x)$ *holds where* $x = (y, z), y \in R^m$ *and* $b \in \mathcal{K}$.

Then the zero solution of the NIDE (2) is eventually stable w.r.t. y.

Proof. Let $\epsilon \in (0, \lambda]$ and $t_0 \in [0, s_1) \bigcup \cup_{k=1}^{\infty} [s_k, t_{k+1})$ be such that $t_0 \geqslant \tau(\varepsilon)$ where $\tau(\varepsilon) = \Theta(b(\varepsilon))$. Without loss of generality assume $t_0 \in [0, s_1)$ (otherwise we can re-numerate the subscripts of the points t_k, s_k.

Since $V(t_0, 0) = 0$ there exists $\delta_1 = \delta_1(t_0, \varepsilon) : \ 0 < \delta_1 \leqslant \varepsilon$ such that $V(t_0, x) < b(\varepsilon)$ for $||x|| < \delta_1$. Let $x_0 \in R^n$ with $||x_0|| < \delta_1$. Then $V(t_0, x_0) < b(\varepsilon)$. Consider any solution
$x^*(t) = x(t; t_0, x_0) \in PC^1([t_0, \infty), R^n); \ x^*(t) = (y^*(t), z^*(t))$ of NIDE (2).

We will prove that
$$||y^*(t)|| < b(\varepsilon), \quad t \geqslant t_0. \tag{6}$$
Assume the opposite, i.e. there exists a point $t > t_0$ such that
$$||y^*(t)|| < b(\varepsilon) \ \text{ for } t \in [t_0, t^*) \quad \text{ and } \ ||y^*(t^)|| = b(\varepsilon). \tag{7}$$
Therefore, or any $r : \ 0 < r \leqslant ||y^*(t)|| \leqslant b(\varepsilon)$ on $[t_0, t^*]$ and for $t \in [t_0, t^*]$ the inequality $t \geqslant t_0 > \tau(\varepsilon) = \Theta(b(\varepsilon) \geqslant \Theta(r)$. Therefore, from condition 2(i) of Theorem 1 the inequality $_{(1)}D_+V(t, x^*(t)) \leqslant 0$ is satisfied on $[t_0, t^*]$. Therefore, the condition 2(i) of Lemma 1 is satisfied on

$[t_0, t^*]$. Also, the condition 2(ii) of Lemma 1 is satisfied on $[t_0, t^*]$ since $V(t, \phi_k(t, x^*(s_k - 0))) \leqslant V(s_k - 0, x^*(s_k - 0))$ holds, i.e. conditions of Lemma 1 are satisfied for $T = t^*$.

According to Lemma 1 applied to the solution $x^*(t)$ for $T = t^*$ and $\Delta = R_+ \times R^l$ we get $V(t, x^*(t)) \leqslant V(t_0, x_0)$ on $[t_0, t^*]$. Then from condition 2 (iii) and the choice of the initial value x_0 we obtain $b(\varepsilon) = b(||y^*(t^*)||) \leqslant V(t^*, x^*(t^*)) \leqslant V(t_0, x_0) < b(\varepsilon)$. The contradiction proves (6) and therefore, applying condition 2(iii) the zero solution of NIDE (1) is eventually stable w.r.t. y. □

Theorem 3.3. *(Eventual uniform stability w.r.t. part of variables). Let the following conditions be satisfied:*

1. *Conditions (H1) and (H2) are satisfied and $f(t, 0) \equiv 0$ for $t \in [0, s_1] \bigcup \cup_{k=1}^{\infty} [t_k, s_{k+1}]$.*
2. *There exists a function $V \in \Lambda(R_+, S(\lambda) \times R^l)$ and a decreasing function $\Theta \in C(S(\lambda), R_+)$, $\lambda > 0$ is a given number such that*
   - *(i) for any $t \in [0, s_1) \bigcup \cup_{k=1}^{\infty} (t_k, s_{k+1})$ and $x \in R^n : \ x = (y, z)$ such that $t \geqslant \Theta(r) : \ 0 < r \leqslant ||y|| < \lambda$ the inequality*

     $$_{(1)}D_+V(t, x) \leqslant 0$$

     holds;
   - *(ii) for any $x \in R^n$ and any $t \in (s_k, t_k]$, $k = 1, 2, 3, \ldots$ such that $t \geqslant \Theta(r)$ for $0 < r \leqslant ||y|| < \lambda$ the inequality $V(t, \phi_k(t, x)) \leqslant V(s_k - 0, x)$*
     holds;
   - *(iii) for any points $x \in R^n : \ x = (y, z)$ and $t \geqslant \Theta(r)$ for $0 < r \leqslant ||y|| < \lambda$ the inequality $b(||y||) \leqslant V(t, x) \leqslant a(||x||)$ holds where $a, b \in \mathcal{K}$.*

Then the zero solution of (2) is eventually uniformly stable w.r.t. y.

Proof. Let $\epsilon \in (0, \lambda]$ and $t_0 \in [0, s_1) \bigcup \cup_{k=1}^{\infty} [t_k, s_{k+1})$ be such that $t_0 \geqslant \tau(\varepsilon)$ where $\tau(\varepsilon) = \Theta(\varepsilon)$. Without loss of generality assume $t_0 \in [0, s_1)$ (otherwise we can re-numerate the points t_k, s_k).

Let $\delta_1 = \min\{\epsilon, b(\epsilon)\}$. From $a \in \mathcal{K}$ there exists $\delta_2 = \delta_2(\epsilon) > 0$ so if $s < \delta_2$ then $a(s) < \delta_1$. Let $\delta = \min(\epsilon, \delta_2)$. Choose the initial value $x_0 \in R^n$ such that $||x_0|| < \delta$ and let $x^*(t) = x(t; t_0, x_0),\ x^*(t) = (y^*(t), z^*(t)),\ t \geqslant t_0$ be a solution of the IVP for NIDE (2). We now prove that

$$||y^*(t)|| < \epsilon, \qquad t \geqslant t_0. \tag{8}$$

Assume inequality (8) is not true and let $t^* > t_0$ be such that

$$||y^*(t)|| < \epsilon \ \text{ for } \ t \in [t_0, t^*) \quad \text{and } ||y^*(t^*)|| = \epsilon.$$

Therefore, $y^*(t) \in S(\lambda)$ on $[t_0, t^*]$ and for any $r:\ 0 < r \leqslant ||y^*(t)|| \leqslant \varepsilon$ on $[t_0, t^*]$ and therefore for $t \in [t_0, t^*]$ the inequality $t \geqslant t_0 > \tau(\varepsilon) = \Theta(\varepsilon) \geqslant \Theta(r)$. Therefore, from condition 2 (i) of Theorem 1 the inequality ${}_{(1)}D_+V(t, x^*(t)) \leqslant 0$ is satisfied on $[t_0, t^*]$. Therefore, the condition 2 (i) of Lemma 1 is satisfied on $[t_0, t^*]$. Also, the condition 2 (ii) of Lemma 1 is satisfied on $[t_0, t^*]$ since $V(t, \phi_k(t, x^*(s_k - 0))) \leqslant V(s_k - 0, x^*(s_k - 0))$ holds, i.e. conditions of Lemma 1 are satisfied for $T = t^*$.

According to Lemma 1 applied to the solution $x^*(t)$ for $T = t^*$ and $\Delta = S(\lambda) \times R^l$ we get $V(t, x^*(t)) \leqslant V(t_0, x_0)$ on $[t_0, t^*]$. Then from condition 2 (iii) and the choice of the initial value x_0 we obtain $b(\varepsilon) = b(||y^*(t^*)||) \leqslant V(t^*, x^*(t^*)) \leqslant V(t_0, x_0) \leqslant a(||x_0||) < \delta_1 \leqslant b(\varepsilon)$. The contradiction proves (8) and therefore, the zero solution of NIDE (1) is uniformly eventually stable w.r.t. y. $\square$

Example 1. Consider the system of NIDE

$$\begin{aligned}
&y' = -y, \quad z' = z(t)y(t) for\ t \in (t_k, s_{k+1}],\ k = 0, 1, 2, \dots \\
&y(t) = \frac{1}{t} y(s_k - 0) \quad z(t) = \Psi_k(t, y(s_k - 0), z(s_k - 0)) \\
&\text{for } t \in (s_k, t_k],\ k = 1, 2, \dots, \\
&y(T_0) = y_0, \ \ z(T_0) = z_0.
\end{aligned} \tag{9}$$

where $\Psi_k:\ (s_k, t_{k+1}] \times R \to R, i = 1, 2, \dots,, s_1 = .25,\ s_{k+1} = s_k + 0.5$, $t_1 = .49,\ t_{k+1} = t_k + 0.5,\ k = 1, 2, \dots$. Denote $x = (y, z)$ and let $V(t, x) = y^2$. Then ${}_{(9)}D_+V(t, x) = -2y^2 \leqslant 0$ for $t \in (k, k+1),\ \ k = 0, 1, 2, \dots,\ \ x \in R^2$, i.e. condition 2 (i) of Theorem 2 is satisfied. For $y_0 = z_0 = 0$ the the scalar NIDE (9) has a zero solution.

Let $x = (y, z)$, $\phi_k = (\Phi_k, \Psi_k)$ and $\Phi_k(t, x) = \frac{1}{t} y$, $\quad k = 1, 2, 3, \ldots$.
Then, for $k = 1, 2$ and $t \in (s_1, t_1] = (0.25, 0.49] \cup (0.75, 0.99]$ we have $V(t, \phi_k(t, x)) = \Big(\Phi_k(t, x)\Big)^2 = \Big(\frac{1}{t} y\Big)^2 > y^2 = V(t, x)$, i.e. the condition 2 (ii) of Theorem 2 is not satisfied. But for $k > 2$ we have $V(t, \phi_k(t, x)) = \Big(\frac{1}{t} y\Big)^2 \leqslant y^2 = V(t, x)$, i.e. condition 2 (ii) is satisfied. Let $\Theta(r) = 1 + \frac{1}{r}$ is a decreasing function and for any $t \geqslant \Theta(r)$ for $0 < r \leqslant ||y||$ the condition 2 (ii) is satisfied. According to Theorem 2 the zero solution of the scalar NIDE (9) is eventually uniformly stable w.r. t. y (see Figure 1 and Figure 2). Note the zero solution is not stable for any impulsive functions Ψ_k.

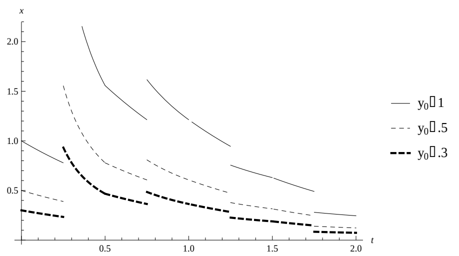

Figure 1. Example 1. Graph of the function $y(t)$ for various y_0 and $t_0 = 0$.

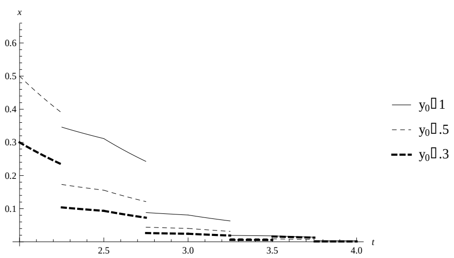

Figure 2. Example 1. Graph of the function $y(t)$ for various y_0 and $t_0 = 2$.

Example 2. Consider the system of NIDE which is slightly modified comparatively with (9):

$$
\begin{aligned}
& y' = -y\cos^2(z), \qquad z' = z(t)y(t) \quad \text{for } t \in (t_k, s_{k+1}],\ k = 0, 1, 2, \ldots \\
& y(t) = \frac{t}{t+1} y(s_k - 0)\sin(z(s_k - 0)) \qquad z(t) = \Psi_k(t, y(s_k - 0), z(s_k - 0)) \\
& \text{for } t \in (s_k, t_k],\ k = 1, 2, \ldots, \qquad\qquad (10) \\
& y(t_0) = y_0, \quad z(t_0) = z_0.
\end{aligned}
$$

where $\Psi_k : \ [s_k, t_k] \to R, k = 1, 2, \ldots,\ s_1 = .25,\ s_{k+1} = s_k + 0.5,$ $t_1 = .49,\ t_{k+1} = t_k + 0.5,\ k = 1, 2, \ldots$.

Denote $x = (y, z)$ and let $V(t, x) = y^2$. Then for any $x \in R^2 :\ y \in S(1)$ and $t \in [t_k, s_k],\ t \geqslant \Theta(r)$ for $0 < r \leqslant ||y|| \leqslant 1,\ \Theta(r) = 1 + \frac{1}{r}$, the inequality $\big(\frac{t}{t+1} y \sin(x)\big)^2 \leqslant y^2$ is satisfied. Therefore, condition 2 (ii) of

Theorem 2 is satisfied and the zero solution of the scalar NIDE (9) is uniformly eventually stable w.r.t. y.

Now we present some sufficient conditions for the eventual uniform asymptotic stability w.r.t. part of variables of the NIDE.

Theorem 3.4. *(Eventual uniform asymptotic stability w.r.t. part of variables). Let the following conditions be satisfied:*

1. *Conditions (H1) and (H2) are satisfied and $f(t,0) \equiv 0$ for $t \in [0,s_1] \bigcup \cup_{k=1}^{\infty}[t_k, s_{k+1}]$.*
2. *There exists a positive constant $M < \infty$ such that $\sum_{i=1}^{\infty}(t_i - s_i) \leqslant M$.*
3. *There exist a function $V \in \Lambda(R_+, S(\lambda) \times R^l)$ and a decreasing function $\Theta \in C(\ R_+, R_+)$ such that conditions 2 (ii) and 2 (iii) of Theorem 2 are satisfied and*

   (i) for any $t \in [0,s_1) \bigcup \cup_{k=1}^{\infty}(t_k, s_{k+1})$ and $x \in R^n :\ x = (y,z)$ such that $t \geqslant \Theta(r) :\ \ 0 < r \leqslant ||y|| \leqslant \lambda$ the inequality is satisfied

$$ {}_{(1)}D_+V(t,x) \leqslant -c(||x||) \tag{11}$$

where $c \in \mathcal{K}$.

Then the zero solution of NIDE (2) is eventually uniformly asymptotically stable w.r.t. y.

Proof. From Theorem 2 the zero solution of the NIDE (1) is eventually uniformly stable w.r.t. y. Therefore, for the number λ there exists $\alpha = \alpha(\lambda) \in (0,\lambda)$ and $\tau(\lambda)$ such that for any $\tilde{t}_0 \in [0,s_1) \bigcup \cup_{k=1}^{\infty}[t_k, s_{k+1})$ such that $\tilde{t}_0 \geqslant \tilde{\tau}(\lambda)$ and $\tilde{x}_0 \in R^n$ the inequality $||\tilde{x}_0|| < \alpha$ implies

$$||y(t;\tilde{t}_0,\tilde{x}_0)|| < \lambda \quad \text{for} \ \ t \geqslant \tilde{t}_0 \tag{12}$$

where $x(t;\tilde{t}_0,\tilde{x}_0) :\ \ x = (y,z)$ is any solution of the NIDE (2) (with initial data $(\tilde{t}_0,\tilde{x}_0)$).

Now we prove that the zero solution of NIDE (2) is eventually uniformly attractive w.r.t. y. Consider the constant $\beta \in (0,\alpha]$ such that $a(\beta) \leqslant b(\alpha)$. Let $\epsilon \in (0,\lambda]$ and $t_0 \in [0,s_1) \bigcup \cup_{k=1}^{\infty}[s_k, t_{k+1})$ be such that $t_0 \geqslant$

$\tau(\varepsilon) = \max\{\tilde{\tau}(\lambda), \Theta(\varepsilon)\}$. Without loss of generality assume $t_0 \in [0, s_1)$ (otherwise we can re-numerate the points t_k, s_k).

Let the point $x_0 \in R^n$, $||x_0|| < \beta$ and $x^*(t) = x(t; t_0, x_0) : \; x^*(t) = (y^*(t), z^*(t))$ be any solution of (2). Then $||x_0|| < \alpha$ and according to (12) the inequality

$$||y^*(t)|| < \lambda \quad \text{for} \quad t \geqslant t_0 \tag{13}$$

holds, i.e. the inclusion $y^*(t) \in S(\lambda)$ is satisfied on $[t_0, \infty)$.

Choose a constant $\gg=\gg(\epsilon) \in (0, \epsilon]$ such that $a(\gg) \leqslant b(\epsilon)$. Let $T > \frac{a(\alpha)}{c(\gg)} + M$ and m be a natural number such that $t_m < t_0 + T \leqslant s_{m+1}$. Note T depends only on ε but not on t_0. We now prove that

$$||x^*(t)|| < \epsilon \;\; \text{for} \;\; t \geqslant t_0 + T. \tag{14}$$

Assume

$$||x^*(t)|| \geqslant\gg \quad \text{for every } t \in [t_0, t_0 + T]. \tag{15}$$

For any $t \in [t_0, t_0 + T]$ the inequalities $t \geqslant t_0 \geqslant \tau(\varepsilon) \geqslant \Theta(\varepsilon) \geqslant \Theta(\lambda) \geqslant \Theta(r),\; 0 < r \leqslant ||y^*(t)|| < \lambda$ hold. Therefore, conditions 2 (ii) and 2 (iii) of Theorem 2 and condition 3 (i) of Theorem 3 are satisfied and thus the conditions 2 (i), 2 (ii), 2 (iii) of Lemma 2 are fulfilled.

From Lemma 2 (applied to the solution $x^*(t)$ for the interval $[t_0, t_0 + T]$ and $\Delta = S(\lambda) \times R^l$), and the choice of T we get

$$\begin{aligned}
&V(t_0 + T, x^*(t_0 + T)) \\
&\leqslant V(t_0, x_0) - \Big(\textstyle\sum_{i=0}^{m-1} \int_{t_i}^{s_{i+1}} c(||x^*(s)||)ds + \int_{t_m}^{t_0+T} c(||x^*(s)||)ds \Big) \\
&\leqslant a(||x_0||) - c(\gg)\Big(\textstyle\sum_{i=0}^{m-1} (s_{i+1} - t_i) + (T + t_0 - t_m) \Big) \\
&\leqslant a(\alpha) - c(\gg)\Big(-\textstyle\sum_{i=1}^{m} (t_i - s_i) + T \Big) \leqslant a(\alpha) - c(\gg)\Big(-M + T \Big) < 0.
\end{aligned}$$

The above contradiction proves there exists $t^* \in [t_0, t_0 + T]$ such that $||x^*(t^*)|| < \gamma$.

Consider the interval $[t^*, \infty)$. For any $t \in [t^*, \infty)$ the inequalities $t \geqslant t^* \geqslant t_0 \geqslant \tau(\varepsilon) \geqslant \Theta(\varepsilon) \geqslant \Theta(\lambda) \geqslant \Theta(r),\; 0 < r \leqslant ||y^*(t)|| < \lambda$ hold. Therefore, conditions 2 (ii) and 2 (iii) of Theorem 2 and condition 3 (i) of Theorem 3 are satisfied. From inequality (11) it follows that we have the inequality

${}_{(1)}D_+V(t, x^*(t)) \leqslant 0$ for $t \in \cup_{k=0}^{\infty}(t_k, s_{k+1}) \cap [t^*, \infty)$, i.e. condition 2 (i) of Lemma 1 with $\Delta = S(\lambda) \times R^l$ is satisfied. Therefore, according to Lemma 1 applied to the solution $x^*(t)$ for $\Delta = S(\lambda) \times R^l$ and $t \geqslant t^*$ the following inequality is satisfied:

$$V(t, x^*(t)) \leqslant V(t^*, x^*(t^*)). \tag{16}$$

Then for any $t \geqslant t^*$ applying (16), condition 2 (iii) of Theorem 2 and inequality (13) we get the inequalities

$$b(||y^*(t)||) \leqslant V(t, x^*(t)) \leqslant V(t^*, x^*(t^*)) \leqslant a(||x^*(t^*)||) < a(\gg) \leqslant b(\epsilon).$$

Therefore, inequality (14) holds for all $t \geqslant t^*$ (hence for $t \geqslant t_0 + T$). $\square$

Example 3. Consider the following system of NIDE

$$\begin{aligned}
&x' = (-2 + \tfrac{cos(z)}{1+t^2})x - \tfrac{z^2}{x},\\
&y' = (-2 + sin(x))y,\\
&z' = z \qquad\qquad \text{for } t \in (t_k, s_{k+1}],\ k = 0, 1, 2, \ldots\\
&x(t) = \Phi_k(t, x(s_k - 0), y(s_k - 0), z(s_k - 0))\\
&y(t) = \Psi_k(t, x(s_k - 0), y(s_k - 0), z(s_k - 0))\\
&z(t) = \Gamma_k(t, x(s_k - 0), y(s_k - 0), z(s_k - 0)), \quad \text{for } t \in (s_k, t_k],\\
&x(t_0) = x_0, \quad y(t_0) = y_0, \quad z(t_0) = z_0
\end{aligned} \tag{17}$$

where $s_k = 0.5k - \frac{1}{2^k}$ and $t_k = 0.5k + \frac{1}{2^k}$, $k = 1, 2, \ldots$, $\Phi_k(t, x, y, z) = \frac{1}{t}x, \Psi_k(t, x, y, z) = \cos(t)\sin(xz)y, \Gamma_k(t, x, y, z) = z$.

Then $\sum_{i=1}^{\infty}(t_i - s_i) = \sum_{i=1}^{\infty} \frac{1}{2^{k-1}} = 2$, i.e. condition 2 of Theorem 3 is satisfied for $M = 2$.

Denote $u = (x, y, z)$. Let $V(t, u) = x^2 + y^2$. Then

$$\begin{aligned}
{}_{(17)}D_+V(t, u) &= (-2 + \tfrac{cos(z)}{1+t^2})x^2 - z^2 + (-2 + sin(x))y^2\\
&\leqslant -(x^2 + y^2 + z^2) = -c(||u||).
\end{aligned} \tag{18}$$

where $c(s) = \sqrt{s} \in \mathcal{K}$, i.e. condition 3 (i) of Theorem 3 is satisfied.

Denote $\phi_k(t, x, y, z) = (\Phi_k(t, x, y, z), \Psi_k(t, x, y, z), \Gamma_k(t, x, y, z))$. Then for $k = 3, 4, \ldots$ we get

$V(t, \phi_k(t, x(s_k - 0), y(s_k - 0), z(s_k - 0)))$

$$= \Big(\Phi_k(t, x(s_k - 0), y(s_k - 0), z(s_k - 0))\Big)^2$$
$$+ \Big(\Psi_k(t, x(s_k - 0), y(s_k - 0), z(s_k - 0)))^2$$
$$= \Big(\frac{1}{t}\Big)^2 x^2(s_k - 0) + \cos^2(t)\sin^2(x(s_k - 0)z(s_k - 0))y^2(s_k - 0)$$
$$\text{for } t \in [s_k, t_k],\ x \in R, y \in S(\lambda) \tag{19}$$

i.e. condition 2 (ii) of Theorem 2 with $\Theta(r) = 1 + \frac{1}{r}$ is satisfied and therefore according to Theorem 3 the zero solution of the system of NIDE (17) is eventually uniformly asymptotically stable w.r.t. (x, y).

4. Partial Case

In the case when some of the equations in the system of non-instantaneous impulses are linear, then obtain very simple sufficient conditions for eventual stability in part of the variables.

Consider the system of NIDE of the type

$$\begin{aligned}
&y' = Ay \quad\ z' = G(t, y, z) \quad \text{for } t \in (t_k, s_{k+1}],\ k = 0, 1, 2, \dots \\
&y(t) = B_k(t)y(t_k - 0) \quad\ z(t) = \Psi_i(t, y(t), z(t), y(t_i - 0), z(t_i - 0)) \\
&\qquad \text{for } t \in (s_i, t_i],\ i = 1, 2, \dots, \\
&y(t_0) = y_0, \quad z(t_0) = z_0
\end{aligned} \tag{20}$$

where $G : \cup_{k=0}^{\infty}[t_k, s_{k+1}] \times R^n \to R^l$, $\Psi_i : [s_i, t_i] \times R^n \times R^n \to R^l$, $(i = 1, 2, 3, \dots)$, $A, B_k, k = 1, 2, \dots,$ are $m \times m$ dimensional matrices.

Theorem 2 is reduced to the following Theorem giving sufficient conditions for the NIDE (20).

Theorem 4.1. *(Eventual stability w.r.t. part of variables in linear case). Let the following conditions be satisfied:*

1. *The function $G \in C([0, s_1] \bigcup \cup_{k=1}^{\infty}[t_k, s_{k+1}] \times R^n, R^l), G(t, 0, 0) \equiv 0$, is such that for any initial point $(\tau, \tilde{x}_0) :\ t_p \leqslant \tau < s_{p+1},\ \tilde{x}_0 \in R^n$, the corresponding IVP for the system of ODE has a solution $x(t; \tau, \tilde{x}_0) \in C^1([\tau, s_{p+1}], R^n)$.*

2. *The functions $\Psi_k \in C([s_k, t_k] \times R^n \times R^n, R^l)$, and $\Psi_k(t, 0, 0) \equiv 0$ for $t \in [s_k, t_k]$, $k = 1, 2, \dots$.*

3. All eigenvalues of the matrix A have negative real parts.

4. There exists a natural number p such that for any $k \geqslant p$ the inequalities $|B_{k,i,j}(t)| \leqslant 1$ hold for $t \in [s_k, t_k],\ i,j = 1,2,\dots,m$, where $B_{k,i,j}(t)$ is the element of the matrix $B_k(t)$ on the i-th raw and the j-th column.

Then the zero solution of the NIDE (20) is uniformly eventually stable w.r.t. y.

The proof follows from Theorem 2 with $V(t,x) = \sum_{i=1}^{m} |y_i|$ where $x = (y,z), y \in R^m, z \in R^l$.

Remark 4.2. *Note that if in Theorem 4 $p = 1$ then the zero solution of the NIDE (20) is uniformly stable w.r.t. y.*

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