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Normal Equations for Singular Fuzzy Linear Systems

M. Nikuie*
Islamic Azad University-Tabriz Branch

M. K. Mirnia
Islamic Azad University-Tabriz Branch

Abstract. In this paper, normal equations for singular fuzzy linear systems is proposed. The principal application of the normal equations is to systems of equations that are inconsistent. The results are explained by solving some numerical examples.

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1. Introduction

The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh and Dubois and Prade. We refer the reader to [7] for more information on fuzzy number and fuzzy arithmetic. Friedman et al. ([9]) introduced a general model for solving a fuzzy linear system whose coefficient matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector. The linear system of equations $Ax = b$ where $A = [a_{ij}] \in R^{(n\times n)}$ is a crisp matrix and the right-hand side is a fuzzy vector are investigated ([2,3,4]). Normal equations play a major role in finding the least-square solution of an inconsistent linear system of equations. Let $Ax = b$ be an inconsistent singular linear system of

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*Corresponding author

equations, we obtain the normal equation $A^T Ax = A^T b$. Any x that minimizes the Euclidean norm of the residual $b - Ax$ is solution to the normal equations. Conversely, any solution to $A^T Ax = A^T b$ is a least-squares solution to the inconsistent singular linear system $Ax = b$ [8]. The linear system of equations $Ax = b$ where $A = [a_{ij}] \in R^{(n\times n)}$ is a crisp singular matrix and the right-hand side is a fuzzy vector is called a singular fuzzy linear system of equations. The purpose of this paper, is to propose, normal equations for singular fuzzy linear system. In Section 2, we recall some preliminaries. Some new results on singular fuzzy linear system of equations are given in Section 3. The proposed model for normal equations for singular fuzzy linear system is given in Section 4. Numerical examples are given in Section 5 followed by a suggestion and concluding remarks in Section 6.

2. Preliminaries and Basic Definitions

This section gives a brief summary of index of matrix, Drazin inverse, minimal solution, fuzzy numbers and fuzzy linear system. We refer the reader to [4,12,13] for more information.

Definition 2.1. ([13]) *Let $A \in C^{n\times n}$. We say the nonnegative integer number k to be the index of matrix A , if k is the smallest nonnegative integer number such that*

$$rank(A^{k+1}) = rank(A^k)$$

The index of matrix A , is denoted by $ind(A)$.

Definition 2.2. *Let $A \in C^{n\times n}$, with $ind(A) = k$. The matrix X of order n is the Drazin inverse of A ,denoted by A^D , if X satisfies the following conditions*

$$AX = XA, XAX = X, A^k XA = A^k \tag{1}$$

When $ind(A) = 1$, A^D is called the group inverse of A , and denoted by A_g .

Theorem 2.3. ([13]) *Let $A \in C^{n\times n}$, with $ind(A) = k$, $rank(A^k) = r$. We may assume that the Jordan normal form of A has the form as follows*

$$A = p \begin{pmatrix} D & 0 \\ 0 & N \end{pmatrix} P^{-1}$$

where P is a nonsingular matrix, D is a nonsingular matrix of order r, and N is a nilpotent matrix that $N^k = \bar{o}$. Then we can write the Drazin inverse of A in the form

$$A^D = P \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$$

When $ind(A) = 1$, it is obvious that $N = \bar{o}$.

Theorem 2.4. ([5]) *$A^D b$ is a solution of*

$$Ax = b, k = ind(A) \tag{2}$$

if and only if $b \in R(A^k)$, and $A^D b$ is an unique solution of (2) provided that $x \in R(A^k)$.

Definition 2.5. *The $m \times n$ linear system*

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_m \end{pmatrix} \tag{3}$$

where $A = (a_{ij})$, $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ is a crisp matrix, and the right-hand side is a fuzzy vector is called fuzzy linear system of equations.

Definition 2.6. ([9]) *A fuzzy number u in parametric form is a pair $(\bar{u}(r), \underline{u}(r))$ of functions $\bar{u}(r), \underline{u}(r), 0 \leqslant r \leqslant 1$, which satisfy the following requirements*
1. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0, 1]$.
2. $\bar{u}(r)$ is a bounded left continuous non-increasing function over $[0, 1]$.
3. $\underline{u}(r) \leqslant \bar{u}(r), 0 \leqslant r \leqslant 1$.

Definition 2.7. *The set of all these fuzzy numbers is denoted by E. For arbitrary fuzzy numbers $x = (\underline{x}(r), \bar{x}(r))$, $y = (\underline{y}(r), \bar{y}(r))$ and $k \in R$, we may define the addition and the scalar multiplication of fuzzy numbers by using the extension principle as [9]*
1. $x + y = (\underline{x}(r) + \underline{y}(r), \bar{x}(r) + \bar{y}(r))$,
2. $kx = \begin{cases} (k\underline{x}, k\bar{x}) & k \geqslant 0 \\ (k\bar{x}, k\underline{x}) & k < 0 \end{cases}$

Definition 2.8. *The fuzzy linear system*

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_n \end{pmatrix} \tag{4}$$

where $A = (a_{ij})$, $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$ is a crisp singular matrix, and the element $\tilde{b}_{ij}$ in the right-hand side matrix are fuzzy numbers is called singular fuzzy linear system. A fuzzy linear system (4) can be extended into a crisp linear system as follows

$$\begin{pmatrix} s_{1,1} & \cdots & s_{1,2n} \\ \vdots & \ddots & \vdots \\ s_{2n,1} & \cdots & s_{2n,2n} \end{pmatrix} \begin{pmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ \bar{x}_1 \\ \vdots \\ -\bar{x}_n \end{pmatrix} = \begin{pmatrix} \underline{b}_1 \\ \vdots \\ \underline{b}_n \\ -\bar{b}_1 \\ \vdots \\ \bar{b}_n \end{pmatrix}$$

where s_{ij} are determined as follows :

$$\begin{aligned} a_{ij} \geqslant 0 &\Rightarrow s_{ij} = a_{ij}, s_{i+n,j+n} = a_{ij} \\ a_{ij} < 0 &\Rightarrow s_{i,j+n} = -a_{ij}, s_{i+n,j} = -a_{ij} \end{aligned} \tag{5}$$

and any s_{ij} which is not determined by (5) is zero. Using matrix notation we get

$$SX = Y$$

The structure of $S = (s_{ij}), 1 \leqslant i \leqslant 2n$ and $1 \leqslant j \leqslant 2n$ implies that and that

$$S= \begin{pmatrix} B & C \\ C & B \end{pmatrix}$$

where B contains the positive entries of A and C contains the absolute value of the negative entries of A , i.e., $A = B - C$.

Definition 2.9. *Let $X = \{\underline{x}_i(r), -\bar{x}_i(r)), 1 \leqslant i \leqslant n\}$ denote a solution of $SX = Y$. The fuzzy number vector $U = \{\underline{u}_i(r), -\bar{u}_i(r)), 1 \leqslant i \leqslant n\}$ defined by*

$$\underline{u}_i(r) = min\{\underline{x}_i(r), \bar{x}_i(r), \underline{x}_i(1), \bar{x}_i(1)\}$$

$$\bar{u}_i(r) = max\{\underline{x}_i(r), \bar{x}_i(r), \underline{x}_i(1), \bar{x}_i(1)\}$$

is called a fuzzy solution of $SX = Y$. If $(\underline{x}_i(r), \bar{x}_i(r)), 1 \leqslant i \leqslant n)$, are all fuzzy numbers and $\underline{x}_i(r) = \underline{u}_i(r), \bar{x}_i(r) = \bar{u}_i(r), 1 \leqslant i \leqslant n$, then U is called a strong fuzzy solution. Otherwise, U is a weak fuzzy solution.

Definition 2.10. ([11]) *Let (3) be a singular fuzzy linear system of equations, (3) is consistent if $SX = Y$ be a consistent linear system i.e. $rank[S] = rank[S|Y]$.*

Definition 2.11. ([1, 8]) *Consider a system of equations written in matrix form as $Ax = b$ where A is $m \times n$, x is $n \times 1$, and b is $m \times 1$. The minimal solution of this problem is defined as follows:*

1. If the system is consistent and has a unique solution, x , then the minimal solution is defined to be x .

2. If the system is consistent and has a set of solutions, then the minimal solution is the element of this set having the least Euclidean norm.

3. If the system is inconsistent and has a unique least-squares solution, x, the minimal solution is defined to be x.

4. If the system is inconsistent and has set of least-squares solutions, then the minimal solution is the element of this set having the least Euclidean norm.

3. On "Singular Fuzzy Linear System of Equations"

The objective of this section, is to give the new properties of the index of matrix, Drazin inverse. This new results play major role in solving singular fuzzy linear system of equations.

Theorem 3.1. *If A, B be $n \times n$ matrices and $rank(A) \leqslant rank(B)$, then*

$$ind(A) \leqslant ind(B).$$

Proof. Let $\lambda_{1(A)}$ be zero eigenvalue of A and $\sigma(\lambda_{1(A)}) = m$ be the multiplicity of the eigenvalue $\lambda_{1(A)}$, then the maximum number of linearly independent eigenvectors associated with $\lambda_{1(A)}$ is

$$\rho_{\lambda_{1(A)}} = n - rank(A)$$

We assume that $\sigma(\lambda_{1(B)}) = m$. Since $rank(A) \leqslant rank(B)$ then $\rho_{\lambda_{1(B)}} \leqslant \rho_{\lambda_{1(A)}}$ thus by Definition 2.1 $ind(A) \leqslant ind(B)$. $\square$

Theorem 3.2. *Let A and B are both singular matrices, $ind(AB) \leqslant ind(A)$.*

Proof. Let $\lambda_{1(A)}$ be zero eigenvalue of A and $\sigma(\lambda_{1(A)}) = m$ be the multiplicity of the eigenvalue $\lambda_{1(A)}$, then the maximum number of linearly independent eigenvectors associated with $\lambda_{1(A)}$ is

$$\rho_{\lambda_{1(A)}} = n - rank(A)$$

Also if $\lambda_{1(AB)} = 0$ and $\sigma(\lambda_{1(AB)}) = m$ we can write, $\rho_{\lambda_{1(AB)}} = n - rank(AB)$. We have $\rho_{\lambda_{1(A)}} \leqslant \rho_{\lambda_{1(AB)}}$ since

$$rank(AB) \leqslant min\{rank(A), rank(B)\}$$

thus by Definition 2.1 $ind(AB) \leqslant ind(A)$. We can show that $ind(AB) \leqslant ind(B)$ similarly. $\square$

Theorem 3.3. *If A and B are both $n \times n$ singular matrices, then*

$$ind(\begin{pmatrix} A & B \\ B & A \end{pmatrix}) \geqslant 1.$$

Proof. According the Theorem3.2 let

$$\rho_{\lambda_{1(A)}} = n - rank(A)$$

associated with $\lambda_{1(A)} = 0$. We have

$$\underset{\lambda_1}{\rho} \begin{pmatrix} A & B \\ B & A \end{pmatrix}^{\leqslant \rho_{\lambda_{1(A)}}}$$

since $rank(A) \leqslant rank(\begin{pmatrix} A & B \\ B & A \end{pmatrix})$, thus by Definition 2.1

$$ind(A) \leqslant ind(\begin{pmatrix} A & B \\ B & A \end{pmatrix})$$

We can show that $ind(B) \leqslant ind(\begin{pmatrix} A & B \\ B & A \end{pmatrix})$ similarly. Therefore $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ is singular matrix. $\square$

Corollary 3.4. *Let* $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ *be a singular matrix.*

1. Since $(\begin{pmatrix} A & B \\ B & A \end{pmatrix}^T \begin{pmatrix} A & B \\ B & A \end{pmatrix})^T = \begin{pmatrix} A & B \\ B & A \end{pmatrix}^T \begin{pmatrix} A & B \\ B & A \end{pmatrix}$ *and by [6-PP.13] we have*

$$rank(\begin{pmatrix} A & B \\ B & A \end{pmatrix}^T \begin{pmatrix} A & B \\ B & A \end{pmatrix}) \leqslant rank\ (\begin{pmatrix} A & B \\ B & A \end{pmatrix})$$

Then, $(\begin{pmatrix} A & B \\ B & A \end{pmatrix}^T \begin{pmatrix} A & B \\ B & A \end{pmatrix})$ *is a singular and symmetric.*

2. We can show that

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix}^k \begin{pmatrix} A & B \\ B & A \end{pmatrix},\ ind(\begin{pmatrix} A & B \\ B & A \end{pmatrix}) = k$$

is a singular matrix, similarly.

4. Normal Equations for Singular Fuzzy Linear Systems

Consider the $n \times n$ fuzzy linear system of equations

$$A\tilde{x} = \tilde{b} \tag{6}$$

This system (6) is converted to the equivalent crisp system

$$SX = Y, \quad l = ind(S) \tag{7}$$

with coefficient matrix of dimension $2n \times 2n$.
1. If $ind(S) = 0$, the system (7) is consistent and has a unique solution. $x_l = S^{-1}Y$ is unique solution of (7)[8].
2. If (7) be consistent singular linear system, then $x^d = S^D Y$ [5].
3. If (7) be inconsistent. The system

$$S^l SX = S^l Y, \quad l = ind(S) \tag{8}$$

is called indicial equations[10]. If (7) be a consistent or inconsistent singular linear system, then the linear system (8) is consistent[12]. By Theorem 2.4, $x_D = (S^l S)^D S^l Y$ is a solution of (8).
4. If (7) be inconsistent. The system

$$S^T SX = S^T Y \tag{9}$$

is called normal equations. If (7) be a inconsistent singular linear system, then the linear system (9) is consistent ([8]). Therefore $x_T = (S^T S)^D S^T Y$ is a solution of the system (9).

5. Numerical Examples

In this section, two concrete examples of singular fuzzy linear system are presented.

Example 5.1. Consider the following singular fuzzy linear system of equations

$$\begin{cases} \widetilde{x}_1 - \widetilde{x}_2 & = (r, 2 - r) \\ 2\widetilde{x}_1 - 2\widetilde{x}_2 & = (2r, 4 - 2r) \end{cases}$$

The extended 4×4 system is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \\ -\overline{x}_1 \\ -\overline{x}_2 \end{pmatrix} = \begin{pmatrix} r \\ 2r \\ r-2 \\ 2r-4 \end{pmatrix} \tag{10}$$

$SX = Y$ is consistent. The index of matrix is equal to one. By Theorem 2.3 we have

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{pmatrix} = P^{-1} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P, P = \begin{pmatrix} -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{4}{3} \\ 0 & 0 & 2 & -1 \end{pmatrix}$$

Then, the Drazin inverse of S is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{pmatrix}^D = P^{-1} \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P = \begin{pmatrix} \frac{5}{9} & -\frac{4}{9} & -\frac{4}{9} & \frac{5}{9} \\ \frac{10}{9} & -\frac{8}{9} & -\frac{8}{9} & \frac{10}{9} \\ -\frac{4}{9} & \frac{5}{9} & \frac{5}{9} & -\frac{4}{9} \\ -\frac{8}{9} & \frac{10}{9} & \frac{10}{9} & -\frac{8}{9} \end{pmatrix}$$

Therefore

$$\begin{pmatrix} \frac{5}{9} & -\frac{4}{9} & -\frac{4}{9} & \frac{5}{9} \\ \frac{10}{9} & -\frac{8}{9} & -\frac{8}{9} & \frac{10}{9} \\ -\frac{4}{9} & \frac{5}{9} & \frac{5}{9} & -\frac{4}{9} \\ -\frac{8}{9} & \frac{10}{9} & \frac{10}{9} & -\frac{8}{9} \end{pmatrix} \begin{pmatrix} r \\ 2r \\ r-2 \\ 2r-4 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}r & -\frac{4}{3} \\ \frac{2}{3}r & -\frac{8}{3} \\ \frac{1}{3}r & +\frac{2}{3} \\ \frac{2}{3}r & +\frac{4}{3} \end{pmatrix}$$

is a solution of (10). Thus

$$\widetilde{x}_1 = (\tfrac{1}{3}r - \tfrac{4}{3}, \ -\tfrac{1}{3}r - \tfrac{2}{3})$$
$$\widetilde{x}_2 = (\tfrac{2}{3}r - \tfrac{8}{3}, \ -\tfrac{2}{3}r - \tfrac{4}{3})$$

$\widetilde{x}_1$, $\widetilde{x}_2$ are fuzzy numbers and by Definition2.9, Therefore is a strong fuzzy solution.

Example 5.2. Consider the following inconsistent singular fuzzy linear system of equations

$$\begin{cases} 2\widetilde{x}_1 & = (2+r, 3) \\ -\widetilde{x}_1 + \widetilde{x}_2 + \widetilde{x}_3 & = (-2, -1-r) \\ -\widetilde{x}_1 - \widetilde{x}_2 - \widetilde{x}_3 & = (2r+3, 7-2r) \end{cases}$$

The following extended 6×6 system is inconsistent

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \\ -\overline{x}_1 \\ -\overline{x}_2 \\ -\overline{x}_2 \end{pmatrix} = \begin{pmatrix} 2+r \\ -2 \\ 2r+3 \\ -3 \\ 1+r \\ 2r-7 \end{pmatrix}$$

The system $S^T S X = S^T Y$ is consistent. By Theorem 2.3 we have $S^T S =$

$$P^{-1} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} P, \;\; P = \begin{pmatrix} -\frac{1}{36} & \frac{1}{36} & \frac{1}{36} & -\frac{1}{36} & \frac{1}{36} & \frac{1}{36} \\ 0 & -\frac{1}{24} & -\frac{1}{24} & 0 & \frac{1}{24} & \frac{1}{24} \\ -\frac{1}{12} & 0 & 0 & \frac{1}{12} & 0 & 0 \\ \frac{1}{18} & \frac{1}{36} & \frac{1}{36} & \frac{1}{18} & \frac{1}{36} & \frac{1}{36} \\ 0 & -\frac{125}{24} & \frac{125}{24} & 0 & -\frac{149}{24} & \frac{149}{24} \\ 0 & -\frac{21}{8} & \frac{21}{8} & 0 & -\frac{25}{8} & \frac{25}{8} \end{pmatrix}$$

Then, the Drazin inverse of S is

$$\begin{pmatrix} 6 & 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 6 & 1 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 1 & 0 & 0 & 1 & 2 & 2 \end{pmatrix}^D = \begin{pmatrix} \frac{5}{24} & -\frac{1}{16} & -\frac{1}{16} & \frac{1}{24} & -\frac{1}{16} & -\frac{1}{16} \\ -\frac{1}{16} & \frac{5}{32} & \frac{5}{32} & -\frac{1}{16} & \frac{1}{32} & \frac{1}{32} \\ -\frac{1}{16} & \frac{5}{32} & \frac{5}{32} & -\frac{1}{16} & \frac{1}{32} & \frac{1}{32} \\ -\frac{1}{24} & \frac{1}{16} & \frac{1}{16} & -\frac{5}{24} & \frac{1}{16} & \frac{1}{16} \\ \frac{1}{16} & -\frac{1}{32} & -\frac{1}{32} & \frac{1}{16} & -\frac{5}{32} & -\frac{5}{32} \\ \frac{1}{16} & -\frac{1}{32} & -\frac{1}{32} & \frac{1}{16} & -\frac{5}{32} & -\frac{5}{32} \end{pmatrix}$$

Therefore

$$X_T = \begin{pmatrix} \frac{5}{24} & -\frac{1}{16} & -\frac{1}{16} & \frac{1}{24} & -\frac{1}{16} & -\frac{1}{16} \\ -\frac{1}{16} & \frac{5}{32} & \frac{5}{32} & -\frac{1}{16} & \frac{1}{32} & \frac{1}{32} \\ -\frac{1}{16} & \frac{5}{32} & \frac{5}{32} & -\frac{1}{16} & \frac{1}{32} & \frac{1}{32} \\ \frac{1}{24} & -\frac{1}{16} & -\frac{1}{16} & \frac{5}{24} & -\frac{1}{16} & -\frac{1}{16} \\ -\frac{1}{16} & \frac{1}{32} & \frac{1}{32} & -\frac{1}{16} & \frac{5}{32} & \frac{5}{32} \\ -\frac{1}{16} & \frac{1}{32} & \frac{1}{32} & -\frac{1}{16} & \frac{5}{32} & \frac{5}{32} \end{pmatrix} \begin{pmatrix} -2+5r \\ -9+2r \\ -9+2 \\ -5+2r \\ 4+3r \\ 4+3r \end{pmatrix} = \begin{pmatrix} \frac{1}{2}r \\ -\frac{17}{8}+\frac{3}{8}r \\ -\frac{17}{8}r+\frac{3}{8}r \\ -\frac{1}{2} \\ \frac{9}{8}+\frac{5}{8}r \\ \frac{9}{8}r+\frac{5}{8}r \end{pmatrix}$$

is a solution of $S^TSX = S^TY$. Thus

$$\widetilde{x}_1 = (\frac{1}{2}r, \frac{1}{2})$$
$$\widetilde{x}_2 = (\frac{3}{8}r - \frac{17}{8}, -\frac{5}{8}r - \frac{9}{8})$$
$$\widetilde{x}_2 = (\frac{3}{8}r - \frac{17}{8}, -\frac{5}{8}r - \frac{9}{8})$$

Therefore X_T is a fuzzy solution. $Ind(S) = 2$, the system $S^2SX = S^2Y$ is consistent.

$$S^3 = P^{-1} \begin{pmatrix} 8 & 1 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} P, \quad P = \begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{4} & 7 & \frac{1}{4} & \frac{1}{4} \\ 6 & 0 & 0 & 6 & 0 & 0 \\ \frac{3}{4} & 0 & 0 & \frac{7}{4} & 0 & 0 \\ 1 & \frac{21}{4} & \frac{1}{4} & -1 & \frac{13}{4} & -\frac{35}{4} \\ \frac{1}{6} & \frac{5}{6} & \frac{1}{6} & -\frac{1}{6} & \frac{13}{6} & -\frac{29}{6} \\ 0 & 1 & 0 & 0 & 1 & -2 \end{pmatrix}$$

Then

$$\begin{pmatrix} 8 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 2 & 10 & 2 & 2 \\ 6 & 2 & 2 & 6 & 2 & 2 \\ 0 & 0 & 0 & 8 & 0 & 0 \\ 10 & 2 & 2 & 2 & 2 & 2 \\ 6 & 2 & 2 & 6 & 2 & 2 \end{pmatrix}^D = \begin{pmatrix} \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\ -\frac{5}{32} & \frac{1}{32} & \frac{1}{32} & -\frac{1}{32} & \frac{1}{32} & \frac{1}{32} \\ -\frac{3}{32} & \frac{1}{32} & \frac{1}{32} & -\frac{3}{32} & \frac{1}{32} & \frac{1}{32} \\ 0 & 0 & 0 & \frac{1}{8} & 0 & 0 \\ -\frac{1}{32} & \frac{1}{32} & \frac{1}{32} & -\frac{5}{32} & \frac{1}{32} & \frac{1}{32} \\ -\frac{3}{32} & \frac{1}{32} & \frac{1}{32} & -\frac{3}{32} & \frac{1}{32} & \frac{1}{32} \end{pmatrix}$$

Therefore

$$X_D = \begin{pmatrix} \frac{1}{8} & 0 & 0 & 0 & 0 & 0 \\ -\frac{5}{32} & \frac{1}{32} & \frac{1}{32} & -\frac{1}{32} & \frac{1}{32} & \frac{1}{32} \\ -\frac{3}{32} & \frac{1}{32} & \frac{1}{32} & -\frac{3}{32} & \frac{1}{32} & \frac{1}{32} \\ 0 & 0 & 0 & \frac{1}{8} & 0 & 0 \\ -\frac{1}{32} & \frac{1}{32} & \frac{1}{32} & -\frac{5}{32} & \frac{1}{32} & \frac{1}{32} \\ -\frac{3}{32} & \frac{1}{32} & \frac{1}{32} & -\frac{3}{32} & \frac{1}{32} & \frac{1}{32} \end{pmatrix} \begin{pmatrix} 8+4r \\ -17+5r \\ -7+7r \\ -12 \\ 3+9r \\ -7+7r \end{pmatrix} = \begin{pmatrix} 1+\frac{1}{2}r \\ -\frac{7}{4}+\frac{1}{4}r \\ -\frac{1}{2}+\frac{1}{2}r \\ -\frac{3}{2} \\ \frac{3}{4}+\frac{3}{4}r \\ -\frac{1}{2}+\frac{1}{2}r \end{pmatrix}$$

is a solution of $S^2SX = S^2Y$. Thus

$$\widetilde{x}_1 = (1+\frac{1}{2}r, \frac{3}{2})$$

$$\widetilde{x}_2 = (-\frac{7}{4}+\frac{1}{4}r, -\frac{5}{8}r - \frac{9}{8})$$

$$\widetilde{x}_2 = (\frac{2}{3}r - \frac{8}{3}, -\frac{5}{8}r - \frac{9}{8})$$

6. Conclusions

In this paper, normal equations for singular fuzzy linear systems is given. Finding minimal solution for singular fuzzy linear system of equations by iterative methods is suggested for researchers.

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Muharram Nikuie
Young Researchers Club
Graduate Student
Islamic Azad University-Tabriz Branch
Tabriz, Iran
E-mail: nikoie_m@yahoo.com

Mir Kam Mirnia
Department of Computer engineering
Associate Professor
Islamic Azad University-Tabriz Branch
Tabriz, Iran
E-mail: mirnia-kam@tabrizu.ac.ir