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Closability of Module $\sigma-$Derivatoins

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Abstract. Let σ be a linear mapping from a dense subalgebra A of a Banach algebra B into B. In this note, we study the closability of a module $\sigma-$ derivation δ from A into a $B-$ bimodule M. Applying the notions of torsion-free modules and essential ideals, we present several results concerning the closability of such derivations. Also we investigate the closability of module $\sigma-$ derivations of the C^*- algebra B into a Hilbert $B-$ bimodule M.

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1. Introduction

Throughout the paper, A is a dense subalgebra of a Banach algebra B and M is a Banach $B-$ bimodule. We recall that a linear mapping $\delta : A \to M$ is a *(module) derivation* if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in A$. A derivation δ is said to be *inner* if there exists an element $u \in M$ such that $\delta(a) := ua - au$, for all $a \in A$. Recently, a number of

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analysts have studied various generalized notions of derivations in the context of Banach algebras. As an example, suppose that $\sigma : A \to B$ is a homomorphism. If for every $u \in M$, we take $\delta_u^\sigma : A \to M$ by $\delta_u^\sigma(a) := u\sigma(a) - \sigma(a)u$, then it is easily seen that $\delta_u^\sigma(ab) = \delta_u^\sigma(a)\sigma(b) + \sigma(a)\delta_u^\sigma(b)$ for all $a, b \in A$. Therefore considering the relation $\delta(ab) = \delta(a)b + a\delta(b)$ as an special case of $\delta(ab) = \delta(a)\sigma(b) + \sigma(a)\delta(b)$ for all $a, b \in A$, where $\sigma : A \to B$ is a linear mapping, leads the theory of derivations to be extensively developed.

The above consideration motivated the authors in [8,9] to generalize the notion of derivation as follows:

Let $\sigma : A \to B$ be a linear mapping. By a *(module) $\sigma-$ derivation* we mean a linear mapping $\delta : A \to M$ such that $\delta(ab) = \delta(a)\sigma(b) + \sigma(a)\delta(b)$ for all $a, b \in A$. In order to construct a $\sigma-$ derivation, suppose that u is an element of M satisfying

$$u\left(\sigma(ab) - \sigma(a)\sigma(b)\right) = \left(\sigma(ab) - \sigma(a)\sigma(b)\right)u.$$

Then the mapping δ_u^σ defined by $\delta_u^\sigma(a) := u\sigma(a) - \sigma(a)u$ is a module $\sigma-$ derivation which is called inner. Note that if σ is an endomorphism, then u can be any arbitrary unitary element of M. It is easy to see that if σ is bounded, then the module $\sigma-$ derivation δ_u^σ is bounded. The reader is referred to [5,8,9,10] for more details on $\sigma-$ derivations.

A linear mapping $\delta : A \to M$ is called closable if it has a closed linear extension. For a linear mapping $\delta : A \to M$, we let $S(\delta)$ denote the set

$$\{x \in M : \textit{there is a sequence } \{a_n\} \textit{ in } A \textit{ with } a_n \to 0 \textit{ and } \delta(a_n) \to x\}$$

and call it the separating space of δ. δ is closable iff $S(\delta) = \{0\}$[15]. It is obvious that if δ is continuous, then it is closable but the converse dose not hold in general. We refer the reader to [4,13,14] for more information on the concept of closability. In this note as a main result we show that if σ is a continuous surjective linear mapping and δ is a module $\sigma-$ derivation, then the separating space $S(\delta)$ is bimodule and applying this result we conclude the closability of a $\sigma-$ derivation δ under some restrictions on the codimensions of the sets $\{a \pm \delta(a) : a \in A\}$ which are called the *deficiency indices*.

Let $\delta_0 : A \to M$ be a linear mapping. Following [14], a module $\sigma-$ derivation δ is called *relative bounded with respect to* δ_0 (or briefly δ_0-

bounded) if there exist $\alpha, \beta > 0$ such that $\| \delta(a) \| \leqslant \alpha \| a \| + \beta \| \delta_0(a) \|$, for all $a \in A$. Among other facts we show that for a linear operator $\delta_0 : A \to M$ and a δ_0- bounded module $\sigma-$ derivation δ if there exists a core D for δ_0 such that the restriction of δ on D is closable, then δ is closable.

For an element a in a unital Banach algebra A, let $sp(a)$ be the set of all complex number λ such that $\lambda - a$ is not invertible in A and call it the *spectrum of a*. The *spectral radius* of a is defined by $\nu(a) := \sup\{| \lambda | : \lambda \in sp(a)\}$. An element a is called *quasi-nilpotent* if $\nu(a) = 0$. The set of all quasi-nilpotents is denoted by $Q(A)$. An algebra A is called *semi-simple* if $rad(A) = \{0\}$, where $rad(A)$ is defined to be the intersection of the maximal ideals in A, ([See 3]).

Let B be a C^*- algebra and M be a complex linear space which is a left $B-$ module and $\lambda(bx) = (\lambda b)x = b(\lambda x)$, where $\lambda \in \mathbb{C}$, $b \in B$ and $x \in M$. The space M is called a *left pre-Hilbert $B-$ module*, if there exists a $B-$ valued inner product $<,>: M \times M \to B$ such that for every $x, y, z \in M$, $\lambda \in \mathbb{C}$ and $b \in B$, satisfies the following conditions:

$(i) < x, x > \geqslant 0$
$(ii) < x, x >= 0$ if and only if $x = 0$
$(iii) < x + \lambda y, z >=< x, z > + \lambda < y, z >$
$(iv) < x, y >=< y, x >^*$
$(v) < ax, y >= a < x, y >$.

Similarly, we can define a right pre-Hilbert $B-$ module. The left (right) pre-Hilbert $B-$ module M is called *Hilbert $B-$ module* if it is a Banach space with respect to the norm $\| x \| := \| < x, x > \|^{\frac{1}{2}}$. The Hilbert module M is called *full* if the closed linear span $< M, M >$ of all elements of the form $< x, y > \ (x, y \in M)$ is equal to B. Let M be a right pre-Hilbert $B-$ module with the inner product $<,>_1$ and a left pre-Hilbert $B-$ module with the inner product $<,>_2$. Then M is a pre-Hilbert $B-$ bimodule if for every $x, y, z \in M$ and for each $a, b \in B$, the following conditins hold:

(i) $< x, y >_2 z = x < y, z >_1$
(ii) $< bx, bx >_1 \leqslant \| b \|^2 < x, x >_1$ and $< xa, xa >_2 \leqslant \| a \|^2 < x, x >_2$.

In [7] it is shown that if M is a pre-Hilbert $B-$ bimodule, then

$\parallel x \parallel := \parallel < x, x >_1 \parallel^{\frac{1}{2}} = \parallel < x, x >_2 \parallel^{\frac{1}{2}}$ defines a norm on M. We also investigate the closability of module $\sigma-$ derivations from a dense subalgebra A of a C^*- algebra B into Hilbert $B-$ bimodule M.

2. The Results

Theorem 2.1. *Let $\delta : A \to M$ be a bounded below module $\sigma-$ derivation such that $S(\delta) = R(\delta)$. Then $\delta = 0$. In particular, δ is closable.*

Proof. Let $x \in S(\delta)$. Then there exists a sequence $\{a_n\}$ in A such that $a_n \to 0$ and $\delta(a_n) \to x$. Since $x \in S(\delta) = R(\delta)$, so $\delta(a) = x$ for some $a \in A$. Also δ is bounded below hence there exists $C > 0$ such that $C \parallel a \parallel \leqslant \parallel \delta(a) \parallel$ for all $a \in A$. This implies that δ is an injection and δ^{-1} is bounded. Therefore $a_n \to \delta^{-1}(x) = a$. But $a_n \to 0$ thus $a = 0$ and $x = \delta(a) = 0$. □

Theorem 2.2. *Let M be a simple $B-$ bimodule in the sense that it has no non-trivial two-sided submodule, $\sigma : A \to B$ be a surjective continuous linear mapping and let $\delta : A \to M$ be a module $\sigma-$ derivation. Then either δ is closable or the range $R(\delta)$ of δ is dense in M.*

Proof. It is obvious that $S(\delta)$ is a closed subspace of M. We show that $S(\delta)$ is a two-sided submodule of M. Let $b \in B$ and $x \in S(\delta)$. Thus there is a sequence $\{a_n\}$ in A such that $a_n \to 0$ and $\delta(a_n) \to x$. Since σ is a surjection, so there exists $c \in A$ such that $\sigma(c) = b$. Hence $ca_n \to 0$ and by continuity of σ we have $\delta(ca_n) = \sigma(c)\delta(a_n) + \delta(c)\sigma(a_n) \to bx$. Thus $bx \in S(\delta)$. A similar argument shows that $xb \in S(\delta)$. By the hypothesis $S(\delta) = \{0\}$ or $S(\delta) = M$. Therefore δ is closable or the range of δ is dense in M. □

Since every simple Banach algebra B is itself a simple $A-$ bimodule, we have the two following results.

Corollary 2.3. *Let A be a dense subalgebra of a simple Banach algebra B, $\sigma : A \to B$ be a surjective continuous linear mapping and let $\delta : A \to B$ be a $\sigma-$ derivation. Then either δ is closable or both of the sets $\{a \pm \delta(a) : a \in A\}$ are dense in B.*

Proof. Following as stated in the proof of Theorem 2.2, one can observe that $S(\delta)$ is a two-sided ideal in B. If $S(\delta) = \{0\}$, then δ is closable. In the case that $S(\delta) = B$, then $R(\delta)$ is dense in B. Hence both of the sets $\{a \pm \delta(a) : a \in A\}$ are dense in B. □

Corollary 2.4. *Let A be a dense subalgebra of a simple Banach algebra B, $\sigma : A \to B$ be a surjective continuous linear mapping and let $\delta : A \to B$ be a $\sigma-$ derivation such that the set $\{a+\delta(a) : a \in A\}$ is closed. Then either δ is closable or the map from A into B which takes $a \mapsto a + \delta(a)$ is onto.*

Proof. Follows from the Corollary 2.3.
The proof of the following result is exactly similar to the method has been used in [13]. □

Theorem 2.5. *Let A be a dense subalgebra of a simple unital Banach algebra $(B, \| \,.\, \|)$, $\sigma : A \to B$ be a surjective continuous linear mapping and let $\delta : A \to B$ be a $\sigma-$ derivation. Suppose that $(A, | \,.\, |)$ is a Banach algebra for some norm $| \,.\, |$, defined on the domain A of δ such that $\delta : (A, | \,.\, |) \to (B, \| \,.\, \|)$ is continuous. If the deficiency indices of δ are finite and not equal, then $\delta : A \to B$ is closable.*

Remark 2.6. *Let A be a dense subalgebra of a simple C^*- algebra $(B, \| \,.\, \|)$, $\sigma : A \to B$ be a surjective continuous linear mapping and let $\delta, \delta_0 : A \to B$ be $\sigma-$ derivations such that δ_0 is closed and $\delta : (A, | \,.\, |) \to (B, \| \,.\, \|)$ is continuous, where the norm $| \,.\, |$ is defined by $| a | = \| a \| + \| \delta_0(a) \|$. Since $\| a \| \leqslant | a |$, it follows that $I(a)$ is continuous and therefore $I \pm \delta$ are continuous maps from $(A, | \,.\, |)$ into $(B, \| \,.\, \|)$. If one of the deficiency indices of δ is finite and non-zero, then one of the two sets $\{a \pm \delta(a) : a \in A\}$ is closed and not equal to B. Using Corollary 2.4, we conclude that δ is closable.*
Before we state the next theorem, we need the following useful lemma which can be found in [3].

Lemma 2.7. *Let A be a unital Banach algebra and I be an ideal in A with $I \subseteq Q(A)$. Then $I \subseteq rad(A)$. (See [3], Proposition 2.2.3, p 16). Using the concept of semi-simplicity and the above lemma, we have the* *following:*

Theorem 2.8. *Let A be a dense subalgebra of a unital semi-simple Banach algebra B, $\sigma : A \to B$ be a surjective continuous linear mapping and let $\delta : A \to B$ be a $\sigma-$ derivation. If $S(\delta)$ is contained in the set of quasi nilpotent elements of B, then δ is closable.*

Proof. The method has been used in the proof of Theorem 2.2 shows that $S(\delta)$ is a two-sided ideal in B and by our assumption $S(\delta) \subseteq Q(B)$. Thus $S(\delta)$ is contained in the radical of B. The semi-simplicity of B implies that $S(\delta) = \{0\}$. Hence δ is closable. □

Following the argument as stated in [14], we have the next two results.

Theorem 2.9. *Let $\delta, \delta_0 : A \to M$ be module $\sigma-$ derivations such that δ_0 is closable and $\delta-$ bounded. If δ is δ_0- bounded, then is closable.*

Proof. Let $x \in S(\delta)$. Thus there is a sequence $\{a_n\}$ in A such that $a_n \to 0$ and $\delta(a_n) \to x$. Since δ_0 is $\delta-$ bounded, so there exists a real number $\alpha > 0$ such that

$$\| \delta_0(a_n) - \delta_0(a_m) \| = \| \delta_0(a_n - a_m) \| \leqslant \alpha(\| a_n - a_m \| + \| \delta(a_n) - \delta(a_m) \|)$$

and

$$\alpha(\| a_n - a_m \| + \| \delta(a_n) - \delta(a_m) \|) \to 0 \qquad (as \quad m, n \to \infty).$$

Thus $\{\delta_0(a_n)\}$ is a Cauchy sequence in the Banach $B-$ module M and hence is convergent. Because of the closability of δ_0, we have $\delta_0(a_n) \to 0$. On the other hand since δ is δ_0- bounded, so there exists a real number $\beta > 0$ such that

$$\| \delta(a_n) \| \leqslant \beta(\| a_n \| + \| \delta_0(a_n) \|) \to 0 \qquad (as \quad n \to \infty).$$

Therefore $x = 0$ and hence δ is closable. □

Remark 2.10. *Let $\delta, \delta_0 : A \to M$ be module $\sigma-$ derivations such that δ_0 is closable and $\delta-$ bounded. Suppose that $\delta - \delta_0$ is $\delta-$ bounded. Hence* *there exists two positive numbers α, β such that*

$$\| (\delta - \delta_0)(a) \| \leqslant \alpha \| a \| + \beta \| \delta(a) \| .$$

Therefore

$$\| \delta(a) \| - \| \delta_0(a) \| \leqslant \alpha \| a \| + \beta \| \delta(a) \| .$$

An easy computation shows that if $1 - \beta > 0$, then δ is δ_0- bounded and by the above theorem, δ is closable.
Before we state the next theorem, we recall the following well-known definition.

Definition 2.11. *A subset D of domain $D(\delta_0)$ is called a core for δ_0, if δ_0 is the closure of its restriction on D.*

Theorem 2.12. *Let $\delta_0 : A \to M$ be a linear operator and $\delta : A \to M$ be a δ_0- bounded module $\sigma-$ derivation. If there exists a core D for δ_0 such that the restriction $\delta |_D : D \to M$ is closable, then δ is closable.*

Proof. First note that since δ is δ_0- bounded, so there exists a positive number β such that

$$\| \delta(a) \| \leqslant \beta(\| a \| + \| \delta_0(a) \|).$$

Let $x \in S(\delta)$. Thus there is a sequence $\{a_n\}$ in A such that $a_n \to 0$ and $\delta(a_n) \to x$. Let $n \in \mathbb{N}$. Since the subset D of A is a core for δ_0, so there exists a sequence $\{c_k^n\}$ in D such that $c_k^n \to a_n$ and $\delta_0(c_k^n) \to \delta_0(a_n)$. Hence for every fixed n there exist a positive integer N such that $\| c_N^n - a_n \| < \frac{1}{2n}$ and $\| \delta_0(c_N^n) - \delta_0(a_n) \| < \frac{1}{2n}$. Because of the δ_0- boundedness of δ we have

$$\| \delta(c_N^n) - \delta(a_n) \| \leqslant \beta(\| c_N^n - a_n \| + \| \delta_0(c_N^n - a_n) \|) < \frac{\beta}{n}.$$

Thus

$$\| \delta(c_N^n) - x \| \leqslant \| \delta(c_N^n) - \delta(a_n) \| + \| \delta(a_n) - x \| \to 0.$$

That is $\delta(c_N^n) \to x$. But $\delta\mid_D: D \to M$ is closable, therefore $x = 0$. $\square$
Before we state the next theorem, we recall the following well-known definition.

Definition 2.13. *Let x be in an $A-$ bimodule M. The annihilator $x^\perp$ of x is defined by $x^\perp := \{a \in A : ax = 0\}$. Then $A-$ bimodule M is called torsion-free if the torsion submodule $M_t := \{x \in M : x^\perp \neq \{0\}\}$ be zero. (i.e. for each $x \in M - \{0\}$, $x^\perp = \{0\}$.)*

Theorem 2.14. *Let I be a non-zero ideal in a dense subalgebra A of a Banach algebra B, $\sigma : A \to B$ be a surjective continuous linear mapping satisfying $\sigma(I) \neq \{0\}$. Suppose that $\delta : A \to M$ is a module $\sigma-$ derivation such that the restriction $\delta\mid_I: I \to M$ is closable. If $S(\delta)$ is a torsion-free module, then δ is closable.*

Proof. First note that the surjectivity of σ implies that $S(\delta)$ is a submodule of M. Let $x \in S(\delta)$. Then there exists a sequence $\{a_n\}$ in A such that $a_n \to 0$ and $\delta(a_n) \to x$. It is enough to show that $x^\perp \neq \{0\}$. For this, let a be a non-zero element in I such that $\sigma(a) \neq 0$. Then $aa_n \to 0$ and by continuity of σ we have $\delta(aa_n) \to \sigma(a)x$. But $aa_n \in I$ and by the assumption the restriction of δ on I is closable, so $\sigma(a)x = 0$. This shows that $x^\perp$ contains a non-zero element $\sigma(a)$. Now since $S(\delta)$ is torsion-free, hence $x = 0$. $\square$

Theorem 2.15. *Let A be a dense subalgebra of a Banach algebra B, M be a torsion-free $B-$ bimodule and let $\delta : A \to M$ be a non-zero continuous module $\sigma-$ derivation. Then σ is closable.*

Proof. Let $b \in S(\sigma)$. Then there exists a sequence $\{a_n\}$ in A such that $a_n \to 0$ and $\sigma(a_n) \to b$. Let x be a non-zero element in $R(\delta)$. So there exists a non-zero element $a \in A$ such that $\delta(a) = x$. Hence $a_na \to 0$ and by continuity of δ we have $\delta(a_na) \to b\delta(a) = bx$. Using the continuity of δ once more, we have $bx = 0$. This shows that $b \in x^\perp$ and since M is torsion-free, hence $b = 0$. $\square$

Definition 2.16. *An ideal I in an algebra B is called essential if its annihilator $I^\perp := \{b \in B : bI = \{0\}\}$ is zero.*

Replacing the module M by the Banach algebras B and using the concept of the "essential ideal", we have the following:

Theorem 2.17. *Let A be a dense subalgebra of a Banach algebra B, I be an essential ideal of B which is contained in A, $\sigma : A \to B$ be a continuous linear mapping such that $\{0\} \neq \sigma(I) \subseteq I$ and $\delta : A \to B$ be a $\sigma-$ derivation such that the restriction $\delta\mid_I : I \to B$ is closable, then δ is closable.*

Proof. Let $b \in S(\delta)$. Then there exists a sequence $\{a_n\}$ in A such that $a_n \to 0$ and $\delta(a_n) \to b$. Let a be a non-zero element in I such that $\sigma(a) \neq 0$. Hence $aa_n \to 0$ and by continuity of σ we have $\delta(aa_n) \to \sigma(a)b$. Because of the closability of the restriction $\delta\mid_I$ we have $\sigma(a)b = 0$. But I is an essential ideal of B so $b = 0$. □

Theorem 2.18. *Let M be a $B-$ bimodule with an approximate identity $\{e_i\}$, $\sigma : A \to B$ be a surjective continuous linear mapping and let $\delta : A \to M$ be a module $\sigma-$ derivation. If for every ideal (a) generated by $a \in A$, the restriction $\delta\mid_{(a)} : (a) \to M$ is closable, then δ is closable.*

Proof. Let $x \in S(\delta)$. Then there exists a sequence $\{a_n\}$ in A such that $a_n \to 0$ and $\delta(a_n) \to x$. Let $a \in A$. Since σ is surjective, there exists an element $c \in A$ such that $\sigma(c) = a$. Then $ca_n \to 0$ and by continuity of σ we have $\delta(ca_n) \to \sigma(c)x = ax$. But $ca_n \in (c)$ and by assumption the restriction of δ on (c) is closable, so $ax = 0$, for every $a \in A$. The density of A in B implies that $bx = 0$, for every $b \in B$. Since $e_i \in B$, then $e_i x = 0$. But $e_i x \to x$, hence $x = 0$. □

The following results concentrate on the closability of module $\sigma-$ derivations in Hilbert C^*- modules:

Let A be a dense subalgebra of a C^*- algebra B, M a Hilbert $B-$ module and let $\{e_i\}$ be an approximate identity for B. We have :

$$< x - e_i x, x - e_i x > = < x, x > - e_i < x, x > + e_i < x, x > e_i - < x, x > e_i$$

Hence $< x - e_i x, x - e_i x > \to 0$. Therefore $e_i x \to x$. So by the Theorem 2.18 we have the next corollary.

Corollary 2.19. *Let A be a dense subalgebra of a C^*- algebra B, M a Hilbert $B-$ module, $\sigma : A \to B$ be a surjective continuous linear mapping and let $\delta : A \to M$ be a module $\sigma-$ derivation. If for every ideal (a) generated by $a \in A$, the restriction $\delta |_{(a)}: (a) \to M$ is closable, then δ is closable.*

Corollary 2.20. *Let A be a dense subalgebra of a C^*- algebra B, $\sigma : A \to B$ be a surjective continuous linear mapping and let $\delta : A \to B$ be a module $\sigma-$ derivation. If for every ideal (a) generated by $a \in A$, the restriction $\delta |_{(a)}: (a) \to B$ is closable, then δ is closable.*
Before we state the next theorem, we need the following useful lemma which can be found in [11].

Lemma 2.21. *Let M be a full Hilbert $B-$ module and $b \in B$. If $bx = 0$, for every $x \in M$ then $b = 0$.*

Theorem 2.22. *Let A be a dense subalgebra of a C^*- algebra B, M be a full Hilbert $B-$ bimodule and let $\delta : A \to M$ be a surjective continuous module $\sigma-$ derivation. Then σ is closable.*

Proof. Let $b \in S(\sigma)$. Then there exists a sequence $\{a_n\}$ in A such that $a_n \to 0$ and $\sigma(a_n) \to b$. Let $x \in M$. Since δ is surjective, so there exists an element $a \in A$ such that $\delta(a) = x$. Hence $a_n a \to 0$ and by continuity of δ we have $\delta(a_n a) \to b\delta(a) = bx$. Using the continuity of δ once more, we have $bx = 0$, for all $x \in M$. But M is full and by the previous lemma,we have $b = 0$. $\square$

Before we state the next theorem, we need the following useful lemma which can be found in [1].

Lemma 2.23. *Let I be an ideal in a C^*- algebra B. The following conditions are mutually equivalent:*

(i) I is an essential ideal in B;

(ii) $\| c \| = \sup_{b \in I, \|b\| \leqslant 1} \| cb \|, \forall c \in B$;

(iii) $\| c \| = \sup_{b \in I, \|b\| \leqslant 1} \| bc \|, \forall c \in B$;

(iv) $\| c \| = \sup_{b \in I, \|b\| \leqslant 1} \| bcb^* \|, \forall c \in B$.

Theorem 2.24. *Let A be a dense subalgebra of a C^*- algebra B, I be an essential ideal of B which is contained in A, M be a Hilbert $B-$ bimodule and let $\sigma : A \to B$ be a continuous linear mapping such that $\sigma(I) = I$. If $\delta : A \to M$ is a module $\sigma-$ derivation such that the restriction $\delta |_I : I \to M$ is closable, then δ is closable.*

Proof. Let $x \in S(\delta)$. Then there exists a sequence $\{a_n\}$ in A such that $a_n \to 0$ and $\delta(a_n) \to x$. It is enough to show that $\| x \| = 0$. For this, let b be a non-zero element in I satisfying $\| b \| \leqslant 1$. Since $\sigma(I) = I$, so there exists a non-zero element $a \in I$ such that $\sigma(a) = b$. Then $aa_n \to 0$ and by continuity of σ we have $\delta(aa_n) \to \sigma(a)x = bx$. But $aa_n \in I$ and by assumption the restriction of δ on I is closable, so $bx = 0$. The fact that I is an essential ideal of B and the above lemma implies that

$$\| x \|^2 = \| < x, x > \| = \sup_{b \in I, \|b\| \leqslant 1} \| b < x, x > b* \| = \sup_{b \in I, \|b\| \leqslant 1} \| < bx, bx > \| = 0. \quad \square$$

The following is an immediate consequence of Theorem 2.24.

Corollary 2.25. *Let A be a dense subalgebra of a C^*- algebra B, I be a non-zero ideal of B which is contained in A, M be a Hilbert $B-$ bimodule such that $\| x \| := \sup_{b \in I, \|b\| \leqslant 1} \| < bx, bx > \|$ holds for all of $x \in M$ and let $\sigma : A \to B$ be a continuous linear mapping such that $\sigma(I) = I$. If $\delta : A \to M$ is a module $\sigma-$ derivation such that the restriction $\delta |_I$: $I \to M$ is closable, then δ is closable.*

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