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A New Three-step Iterative Method for Solving Nonlinear Equations

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Abstract. In this paper, a new three-step iterative method for finding a simple root of the nonlinear equation of $f(x) = 0$ will be introduced. This method is based on the two-step method of [C. Chun, Y. Ham, Some fourth-order modifications of Newton's method, Appl. Math. Comput. 197 (2008) 654-658]. The new method requires three evaluations of the function and two of its first-derivative. We will prove that the order of convergence of the new method and its efficiency index will respectively be 8, and 1.5157. Some numerical experiments are given to illustrate the performance of the three-step iterative method.

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Keywords and Phrases: Nonlinear equation, iterative method, three-step iterative method, convergence order, efficiency index.

1. Introduction

In recent years, attempts have been made to increase the order of convergence, and improve the efficiency index in iterative methods, examples of which can be found in [3–10]. Most of the mentioned papers have used

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the Newton's method to calculate the final approximation of the exact solution of the equation $f(x) = 0$. The calculation of the derivative in the third step is the fundamental drawback of these methods, which in turn decreases their efficiency index. However, the efficiency index of these methods can be increased by using approximation for calculating the derivatives. In this paper, a new three-step iterative method for solving nonlinear equations will be introduced. This method, using Steffensen's method has solved the problem of the calculation of derivative in the third step. This method also has an order of convergence of 8, and an efficiency index of 1.5157.

2. Development of Method and Convergence Analysis

In this section, we will first describe the ideas of the iterative method proposed by C. Chun and Y. Ham in [2], and then will give a new three-step iterative method and its convergence analysis.
To solve nonlinear equations, iterative methods such as Newton's method are usually used. Throughout this paper we will consider iterative methods to find a simple root, i.e., $f(\alpha) = 0$ and $f^{'}(\alpha) \neq 0$, of a nonlinear equation $f(x) = 0$ that uses f and $f^{'}$ but not the higher derivatives of f. Newton's method for the calculation of a simple root is probably the most widely used iterative methods defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1}$$

It is well known ([12]) that this method is quadratically convergent.
We will here consider Newton's method as the first step, and hence have:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{2}$$

Now, we will consider the following two-step methods:
double-Newton method with fourth-order convergence ([12]) given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(y_n)}{f'(y_n)}, \tag{3}$$

the fourth-order method [1] given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \left[1 + 2\frac{f(y_n)}{f(x_n)} + \frac{f(y_n)^2}{f(x_n)^2}\right]\frac{f(y_n)}{f'(x_n)}, \tag{4}$$

and King's fourth-order family of methods [7] given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)}\frac{f(y_n)}{f'(x_n)}, \tag{5}$$

where$\beta \in \Re$.
Approximately equaling the correction terms of the methods (3) and (4), we can obtain an approximation to $f^{'}(y_n)$

$$f^{'}(y_n) \approx \frac{f^{'}(x_n)f(x_n)^2}{f(x_n)^2 + 2f(x_n)f(y_n) + f(y_n)^2}. \tag{6}$$

Approximately equaling the correction terms of the methods (3) and (5), we can obtain an approximation to $f^{'}(y_n)$

$$f^{'}(y_n) \approx \frac{f^{'}(x_n)\left[f(x_n) + (\beta - 2)f(y_n)\right]}{f(x_n) + \beta f(y_n)}. \tag{7}$$

Now, we apply the respective approximation (6) and (7) to any other iterative method whose order is at least four and which also depend on $f^{'}(y_n)$, for example, the fifth-order method proposed in [5] given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^{'}(y_n) + 3f^{'}(x_n)}{5f^{'}(y_n) - f^{'}(x_n)}\frac{f(y_n)}{f'(x_n)}. \tag{8}$$

Per iteration this method requires two evaluations of the function and two evaluations of its first-derivative, so its efficiency index equals to $5^{1/4} \approx 1.495$ if we consider the definition of efficiency index [7] as $p^{1/m}$, where p is the order of the method and m is the number of functional evaluations per iteration required by the method. Using (7) in (8), we obtain the family of methods

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{2f(x_n) + (2\beta - 1)f(y_n)}{2f(x_n) + (2\beta - 5)f(y_n)}\frac{f(y_n)}{f'(x_n)}, \tag{9}$$

where$\beta \in \Re$. The family (9) includes, as particular cases, the following ones: For $\beta = 0$, we obtain a fourth-order method which is the second step of our three-step method:

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)} \frac{f(y_n)}{f'(x_n)}, \tag{10}$$

Now we will introduce Steffensen's method, by putting z_n in which, our new three-step method will be created. This method has the second order of convergence, and contrary to Newton's method, does not require the calculation of the function's derivative. In this method, $f^{'}(x_n)$, in the procedure of Newton's method will be substituted by the function

$$g(x_n) = \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}, \tag{11}$$

and one can obtain the iterative procedure of Steffensen as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)} = x_n - \frac{(f(x_n))^2}{f(x_n + f(x_n)) - f(x_n)} \qquad n = 0, 1, 2, \ldots \tag{12}$$

Now, we will introduce our new three-step method:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{f(x_n)}{f'(x_n)} - \frac{2f(x_n) - f(y_n)}{2f(x_n) - 5f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \frac{(f(z_n))^2}{f(z_n + f(z_n)) - f(z_n)} \qquad n = 0, 1, 2, \ldots \end{aligned} \tag{13}$$

For the methods defined by (13), we have

Theorem : *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \to R$ for an open interval I. If x_0 is sufficiently close to α, then the three-step iterative method* (13) *has eighth-order convergence and satisfies the following error equation :*

$$e_{n+1} = -2(c_2^2 c_3^2) e_n^8 + O(e_n^9), \tag{14}$$

where $e_n = x_n - \alpha$ for $n = 0, 1, 2, \ldots,$ and $c_n = \frac{f^{(n)}(\alpha)}{n! f'(\alpha)}$.

Proof. Since f is sufficiently differentiable, by expanding $f(x_n)$ and $f^{'}(x_n)$ about α, one obtains

$$\begin{aligned} f(x_n) &= f^{'}(\alpha)(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 \\ &+ c_6e_n^6 + c_7e_n^7 + c_8e_n^8 + O(e_n^9), \\ f^{'}(x_n) &= f^{'}(\alpha)(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 \\ &+ 7c_7e_n^6 + 8c_8e_n^7 + O(e_n^8)). \end{aligned} \quad (15)$$

where $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f^{'}(\alpha)}$ for $k = 2, 3, \ldots$ From (13), and (15), using the Maple software one obtains

$$\begin{aligned} y_n &= \alpha + c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 3c_4 - 4c_2^3)e_n^4 \\ &- (8c_2^4 - 20c_3c_2^2 + 6c_3^2 + 10c_4c_2 - 4c_5)e_n^5 + (-17c_4c_3 \\ &+ 28c_4c_2^2 - 13c_2c_5 + 5c_6 + 33c_2c_3^2 - 52c_3c_2^3 + 16c_2^5)e_n^6 \\ &+ (92c_3c_2c_4 - 22c_3c_5 + 18c_3^3 - 126c_3^2c_2^2 + 128c_3c_2^4 \\ &- 12c_4^2 - 72c_4c_2^3 + 36c_5c_2^2 + 6c_7 - 16c_2c_6 - 32c_2^6)e_n^7 \\ &+ (7c_8 + 118c_5c_2c_3 - 348c_4c_3c_2^2 - 19c_2c_7 + 64c_2^7 \\ &+ 64c_2c_4^2 - 31c_4c_5 + 75c_4c_3^2 + 176c_4c_2^4 - 92c_5c_2^3 - 27c_6c_3 \\ &+ 44c_6c_2^2 - 135c_2c_3^3 + 408c_3^2c_2^3 - 304c_3c_2^5)e_n^8 + O(e_n^9). \end{aligned} \quad (16)$$

Expanding $f(y_n)$ about α and using (16),

$$\begin{aligned} f(y_n) &= f^{'}(\alpha)(c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 - (7c_2c_3 - 3c_4 - 5c_2^3)e_n^4 \\ &+ (4c_5 - 12c_2^4 - 6c_3^2 - 10c_2c_4 + 24c_3c_2^2)e_n^5 + (-13c_2c_5 \\ &- 73c_3c_2^3 + 34c_4c_2^2 + 5c_6 - 17c_4c_3 + 37c_2c_3^2 + 28c_2^5)e_n^6 \\ &+ (44c_5c_2^2 + 104c_3c_2c_4 - 22c_3c_5 + 18c_3^3 + 6c_7 + 206c_3c_2^4 \\ &- 104c_4c_2^3 - 160c_3^2c_2^2 - 64c_2^6 - 12c_4^2 - 16c_2c_6)e_n^7 + (75c_4c_3^2 \\ &+ 134c_5c_2c_3 + 73c_2c_4^2 + 34c_6c_2^2 - 552c_3c_2^5 - 147c_2c_3^3 + 7c_8 \\ &- 19c_2c_7 + 297c_4c_2^4 - 134c_5c_2^3 + 582c_3^2c_2^3 + 144c_2^7 - 27c_6c_3 \\ &- 455c_4c_3c_2^2 - 31c_4c_5)e_n^8 + O(e_n^9)). \end{aligned} \quad (17)$$

From (13), one obtains

$$\begin{aligned} z_n &= \alpha - c_2c_3e_n^4 + ((3/2)c_2^4 - 2c_3^2 + 2c_3c_2^2 - 2c_2c_4)e_n^5 \\ &+ (8c_3c_2^3 + 6c_2c_3^2 - 7c_4c_3 - 3c_2c_5 - (35/4)c_2^5 + 3c_4c_2^2)e_n^6 \\ &+ (4c_5c_2^2 + (255/8)c_2^6 + 14c_4c_2^3 + 16c_3c_2c_4 - 10c_3c_5 \\ &+ 4c_3^3 + 16c_3^2c_2^2 - 4c_2c_6 - (135/2)c_3c_2^4 - 6c_4^2)e_n^7 + (19c_5c_2^3 \\ &- 5c_2c_7 + 14c_4c_3^2 + 20c_5c_2c_3 - 13c_6c_3 - (413/4)c_4c_2^4 + 61c_4c_3c_2^2 \\ &+ 13c_2c_3^3 - (1515/16)c_2^7 + 10c_2c_4^2 - 199c_3^2c_2^3 - 17c_4c_5. \\ &+ (573/2)c_3c_2^5 + 25c_6c_2^2)e_n^8 + O(e_n^9). \end{aligned} \quad (18)$$

Expanding $f(z_n)$ about α and using (18),

$$\begin{aligned} f(z_n) &= f^{'}(\alpha)(-c_2c_3e_n^4+((3/2)c_2^4+2c_3c_2^2-2c_3^2-2c_2c_4)e_n^5 \\ &+ (-(35/4)c_2^5-7c_4c_3-3c_2c_5+8c_3c_2^3+6c_2c_3^2+3c_4c_2^2)e_n^6 \\ &+ (4c-3^3+(255/8)c_2^6-6c_4^2+16c_3c_2c_4-(135/2)c_3c_2^4 \\ &- 10c_3c_5+14c_4c_2^3+16c_3^2c_2^2-4c_2c_6+4c_5c_2^2)e_n^7+(14c_4c_3^2 \\ &+ (573/2)c_3c_2^5-5c_2c_7-17c_4c_5+25c_6c_2^2-13c_6c_3 \\ &- (413/4)c_4c_2^4+c_2^2c_3^2-199c_3^2c_2^3-(1515/16)c_2^7+19c_5c_2^3 \\ &+ 13c_2c_3^3+10c_2c_4^2+61c_4c_3c_2^2+20c_5c_2c_3)e_n^8+O(e_n^9)). \end{aligned} \tag{19}$$

Expanding $f(z_n+f(z_n))$ about α and using (18) and (19),

$$\begin{aligned} f(z_n+f(z_n)) &= f^{'}(\alpha)(-2c_2c_3e_n^4+(3c_2^4-4c_2c_4+4c_3c_2^2-4c_3^2)e_n^5 \\ &+ (-14c_4c_3-6c_2c_5+16c_3c_2^3+6c_4c_2^2+12c_2c_3^2 \\ &- (35/2)c_2^5)e_n^6+(28c_4c_2^3+32c_3^2c_2^2+32c_3c_2c_4-135c_3c_2^4 \\ &- 20c_3c_5+8c_5c_2^2-8c_2c_6+8c_3^3+(255/4)c_2^6-12c_4^2)e_n^7 \\ &+ (26c_2c_3^3-26c_6c_3+50c_6c_2^2+40c_5c_2c_3+122c_4c_3c_2^2 \\ &+ 20c_2c_4^2+573c_3c_2^5+28c_4c_3^2-398c_3^2c_2^3+38c_5c_2^3 \\ &- 34c_4c_5-10c_2c_7-(1515/8)c_2^7+c_2^2c_3^2 \\ &- (413/2)c_4c_2^4)e_n^8+O(e_n^9)). \end{aligned} \tag{20}$$

Using (19),

$$\begin{aligned} (f(z_n))^2 &= (f^{'}(\alpha))^2((c_2^2c_3^2)e_n^8+(4c_2c_3^3-4c_2^3c_3^2+4c_4c_3c_2^2 \\ &- 3c_2^5c_3)e_n^9+(6c_2^2c_5c_3+(47/2)c_3c_2^6-6c_2^5c_4 \\ &- 14c_3c_2^3c_4+(9/4)c_2^8-18c_3^2c_2^4+22c_4c_3^2c_2-20c_3^3c_2^2 \\ &+ 4c_2^2c_4^2+4c_3^4)e_n^{10}+O(e_n^{11})). \end{aligned} \tag{21}$$

From (13), (20) and (21), one obtains

$$x_{n+1} = \alpha-(2c_2^2c_3^2)e_n^8+O(e_n^9), \tag{22}$$

which implies that $e_{n+1} = (-2c_2^2c_3^2)e_n^8+O(e_n^9)$.
Since in each iteration our new method requires three functional evaluations, and two functional evaluation of its derivative; and since it has an order of convergence of 8, its efficiency index will equal : $8^{1/5} \approx 1.5157$.

3. Numerical Experiments

We present some numerical results to illustrate the efficiency of the three-step iterative method proposed in this paper.

We compare (13) (MAA means: M:Matinfar, A:Asadpour, A:Aminzadeh) with the existing three-step methods in [13] which are respectively represented by XIA 1, XIA 2 and XIA 3.

XIA 1 is :

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)}[1/2 + \frac{5f(x_n)^2 + 8f(x_n)f(y_n) + 2f(y_n)^2}{5f(x_n)^2 - 12f(x_n)f(y_n)}(1/2 + \frac{f(z_n)}{f(y_n)})]. \end{aligned}$$

XIA 2 is :

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)}[\frac{5f(x_n)^2 - 2f(x_n)f(y_n) + f(y_n)^2}{5f(x_n)^2 - 12f(x_n)f(y_n)} + (1 + 4\frac{f(y_n)}{f(x_n)})(\frac{f(z_n)}{f(y_n)})]. \end{aligned}$$

XIA 3 is :

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \frac{f(x_n)}{f'(x_n)} - \frac{4f(x_n)^2 - 5f(x_n)f(y_n) - f(y_n)^2}{4f(x_n)^2 - 9f(x_n)f(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)}[1 + 4\frac{f(z_n)}{f(x_n)}][\frac{8f(y_n)}{4f(x_n) - 11f(y_n)} + 1 + \frac{f(z_n)}{f(y_n)}]. \end{aligned}$$

The iteration methods above will be stopped when the following stopping criteria is satisfied:

$$\Big|f(x_k)\Big| < \epsilon.$$

where ϵ represents tolerance. The test functions to be used are defined as follows:

$$\begin{aligned} f_1(x) &= x^3 + 4x^2 - 15, && x_* = 1.63198055660636, \\ f_2(x) &= xe^{x^2} - \sin^2(x) + 3\cos(x) + 5, && x_* = -1.207647827130919, \\ f_3(x) &= 10xe^{-x^2} - 1, && x_* = 1.67963061042845, \\ f_4(x) &= \sin^2(x) - x^2 + 1, && x_* = 1.4044916482153411, \\ f_5(x) &= x^5 + x^4 + 4x^2 - 15, && x_* \approx 1.347. \end{aligned}$$

where x_* is the exact root. The absolute values of the function ($|f(x_n)|$) and ($|x_n - x_*|$), are shown in Table 1.

Table 1: Comparison of iterative methods.

Function	Method	$\|x_k - x_*\|$	$f(x_k)$
$f_1(x), x_0 = 2$	MAA	2.489597032973023e-007	3.552713678800501e-015
	XIA 1	2.489597032973023e-007	3.552713678800501e-015
	XIA 2	2.489597035193469e-0074	3.552713678800501e-015
	XIA 3	2.489597035193469e-007	3.552713678800501e-015
$f_2(x), x_0 = -1$	MAA	0	2.664535259100376e-015
	XIA 1	6.661338147750939e-016	1.509903313490213e-014
	XIA 2	0	2.664535259100376e-015
	XIA 3	1.798561299892754e-014	3.632649736573512e-013
$f_3(x), x_0 = 1.5$	MAA	0	2.220446049250313e-016
	XIA 1	0	2.220446049250313e-016
	XIA 2	0	2.220446049250313e-016
	XIA 3	0	2.220446049250313e-016
$f_4(x), x_0 = 1.5$	MAA	0	3.330669073875470e-016
	XIA 1	0	3.330669073875470e-016
	XIA 2	2.220446049250313e-016	4.440892098500626e-016
	XIA 3	2.220446049250313e-016	4.440892098500626e-016
$f_5(x), x_0 = 1.2$	MAA	4.280989683049796e-004	1.776356839400251e-015
	XIA 1	4.280989683049796e-004	1.776356839400251e-015
	XIA 2	4.280989683049796e-004	1.776356839400251e-015
	XIA 3	4.280989683049796e-004	1.776356839400251e-015

All numerical tests have been written in Matlab. The numerical results imply that our three-step method has a good performance, and despite being simple and requiring a small number of calculations compared to other existing methods, delivers good numerical results.

4. Conclusions

In this paper, we proposed a new three-step iterative method for solving nonlinear equations by using Steffensen's method in the third step. We showed that the new three-step iterative method has eighth-order convergence. Numerical experiments also show that the numerical results of our new three-step method, in equal iterations, resemble the results of other existing three-step methods with eighth-order convergence. However, our new method relatively requires a smaller number of calculations and does not require the calculation of derivative in the third step.

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