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Generalized Composition Operators Between Weighted Dirichlet Type Spaces and Bloch Type Spaces

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Abstract. We characterize bounded and compact generalized composition operators between Bloch type spaces and weighted Dirichlet type spaces. Then, we show that these results can be employed to characterize bounded and compact Volterra type operators between these spaces. Finally, we give a connection between weighted and generalized composition operators to conclude some results about boundedness and compactness of them.

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1. Introduction

Let $\mathcal{H}(\mathbb{D})$ be the space of all analytic functions on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$. For $\alpha \in (0, \infty)$, the Bloch type space (or α-Bloch space) $\mathcal{B}^\alpha$ is the space of all $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

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The little Bloch type space $\mathcal{B}_0^\alpha$ consists of those functions $f \in \mathcal{B}^\alpha$ for which

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |f'(z)| = 0.$$

The space $\mathcal{B}^\alpha$ is a Banach space with the norm $\|f\| = |f(0)| + \|f\|_{\mathcal{B}^\alpha}$ and $\mathcal{B}_0^\alpha$ is a closed subspace of $\mathcal{B}^\alpha$. If $\alpha = 1$ or $\alpha \in (0,1)$, then $\mathcal{B}^\alpha$ coincides with the classical Bloch space $\mathcal{B}$ or the classical $(1-\alpha)$-Lipschitz class $\Lambda_{1-\alpha}$, respectively (see [5, Theorem 5.1] or [20]).
For $p \in (0,\infty)$ and $\beta > -1$, the weighted Bergman space $\mathcal{A}_\beta^p$ is the space of all $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{\mathcal{A}_\beta^p}^p = \int_{\mathbb{D}} |f(z)|^p \left(\log \frac{1}{|z|}\right)^\beta dA(z) < \infty,$$

where dA denotes normalized Lebesgue area measure on $\mathbb{D}$. Equivalently, $f \in \mathcal{A}_\beta^p$ if and only if

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\beta dA(z) < \infty.$$

It is well known that $\mathcal{A}_\beta^p$ is a Banach space for $p \geqslant 1$, and a Hilbert space for $p = 2$.
For $p \in (0,\infty)$ and $\beta > -1$, the weighted Dirichlet type space $\mathcal{D}_\beta^p$ is the space of all $f \in \mathcal{H}(\mathbb{D})$ satisfying

$$\|f\|_{\mathcal{D}_\beta^p}^p = \int_{\mathbb{D}} |f'(z)|^p \left(\log \frac{1}{|z|}\right)^\beta dA(z) < \infty.$$

We note that $f \in \mathcal{D}_\beta^p$ if and only if $f' \in \mathcal{A}_\beta^p$. It is well known that $f \in \mathcal{A}_\beta^p$ if and only if $f' \in \mathcal{A}_{p+\beta}^p$. Namely

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\beta dA(z) \simeq \int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{p+\beta} dA(z) + |f(0)|^p \qquad (1)$$

for $p \in (0,\infty)$, $\beta > -1$ and for any analytic function f in $\mathbb{D}$ which is a standard result. The inequality in one direction is a classical result due to Hardy and Littlewood (see [5, Theorem 5.6]), while the reverse inequality can easily be proved by the methods used in [5, Chapter 5].

Therefore one can say, $\mathcal{D}^p_{p+\beta} = \mathcal{A}^p_\beta$. For general background about weighted Bergman spaces $\mathcal{A}^p_\beta$ and Bloch type spaces $\mathcal{B}^\alpha$ we refer [7], [20] and the references therein.
In (1) and in the sequel of this paper, we use the notation $A \lesssim B$ to mean that there is a positive constant C such that $A \leqslant CB$, and $A \simeq B$ if $A \lesssim B \lesssim A$. Constants denoted by C, are positive and not necessarily the same in each occurrence.
Let φ be a non constant analytic self-map of $\mathbb{D}$ and $g \in \mathcal{H}(\mathbb{D})$. The *weighted composition operator* gC_φ is defined by $gC_\varphi(f) = g(f \circ \varphi)$ for $f \in \mathcal{H}(\mathbb{D})$. In the special case of $g = 1$, we get the *composition operator* $C_\varphi(f) = f \circ \varphi$. The *generalized composition operator* defined by

$$(C^g_\varphi f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi, \quad f \in \mathcal{H}(\mathbb{D}),$$

was introduced by Li and Stević in [8]. If $g = \varphi'$ then C^g_φ is essentially composition operator, since the difference $C^g_\varphi - C_\varphi$ is a constant. Therefore, gC_φ and C^g_φ are generalizations of the composition operators. The main subject in the study of (weighted) (generalized) composition operators is to describe operator theoretic properties of $(gC_\varphi)(C^g_\varphi)$ C_φ in terms of function theoretic properties of (g) φ. For general theory of composition operators, we refer the reader to [4] and [16]. Boundedness and compactness of composition operators C_φ on Bloch space $\mathcal{B}$ were described by Madigan and Matheson [10]. Recently, Ohno, Stroethoff and Zhao studied weighted composition operators between Bloch type spaces in [13]. Also composition operators from Bloch space $\mathcal{B}$ to the weighted Dirichlet space $\mathcal{D}^2_\beta$ were considered by Smith in [12]. Li and Stević investigated the boundedness and compactness of the generalized composition operators on Bloch type spaces and Zygmund spaces in [8]. The authors investigated boundedness and compactness of the generalized composition operators and the products of Volterra type operators and composition operators between $\mathcal{Q}_K$ spaces in [11] and between logarithmic Bloch type spaces and $\mathcal{Q}_K$ type spaces in [15].
In this paper we are interested in the problem of using function theoretic properties of analytic functions g and analytic self-maps φ on $\mathbb{D}$ to determine when generalized composition operator C^g_φ between the weighted

Dirichlet type spaces and Bloch type spaces is bounded or compact. As the following equations show, these results can be employed to characterize bounded and compact Volterra type operators between the above mentioned spaces where the Volterra type operators defined on $\mathcal{H}(\mathbb{D})$ as follows

$$V_g(f)(z) = \int_0^z f(\xi)g'(\xi)d\xi \quad \text{and} \quad U_g(f)(z) = \int_0^z f'(\xi)g(\xi)d\xi,$$

for symbol g in $\mathcal{H}(\mathbb{D})$. If $g(z) = z$ or $g(z) = \log\frac{1}{1-z}$, then V_g is the integration operator or the Cesàro operator, respectively. For a survey on the study of such operators see [17]. Note that

$$U_gC_\varphi(f) = \int_0^z (f\circ\varphi)'(\xi)g(\xi)d\xi = C_\varphi^{\varphi' g}(f),$$

so U_g is the generalized composition operator C_φ^g for $\varphi(z) = z$. Therefore, U_g is bounded (compact) whenever C_φ^g is bounded (compact) when $\varphi(z) = z$. As we know, on a general space of analytic functions, the differentiation operator D is typically unbounded. On the other hand, the composition operator C_φ is bounded on various spaces of analytic functions on $\mathbb{D}$ (see [5, 16]), though the product DC_φ is possibly still unbounded there. Here, we consider new operator V_gDC_φ, noting the equality

$$V_gDC_\varphi(f) = \int_0^z (f\circ\varphi)'(\xi)g'(\xi)d\xi = C_\varphi^{\varphi' g'}(f),$$

one can consider V_gDC_φ as a generalized composition operator, and apply the results obtained about generalized composition operators to characterize boundedness and compactness of V_gDC_φ. Finally, we give a connection between weighted composition operators gC_φ and generalized composition operators C_φ^g and conclude some results about boundedness and compactness of them.

In the sequel, we need the following estimates. For functions f in $\mathcal{B}^\alpha$, integrating the estimate $|f'(z)| \leqslant \frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^\alpha}$, we obtain

$$
\begin{aligned}
|f(z)-f(0)| = \left|\int_0^z f'(\zeta)d\zeta\right| &\leqslant \|f\|_{\mathcal{B}^\alpha}\int_0^{|z|}\frac{d\rho}{(1-\rho^2)^\alpha} \\
&\leqslant \frac{1}{(1-|z|^2)^\alpha}\|f\|_{\mathcal{B}^\alpha}. \qquad (2)
\end{aligned}
$$

To estimate functions in $\mathcal{D}^p_\beta$, we make use of the following lemma. For the proof see [6], [7], or the original source [19].

Lemma 1.1. *Let $0 < p < \infty$, $\beta > -1$ and $f \in \mathcal{A}^p_\beta$. Then*

$$
|f(z)|(1-|z|^2)^{\frac{2+\beta}{p}} \leqslant \left((1+\beta)\int_{\mathbb{D}}|f(z)|^p(1-|z|^2)^\beta dA(z)\right)^{\frac{1}{p}} \qquad (z\in\mathbb{D}),
$$

with equality if and only if f is a constant multiple of the function $f_a(z) = (-\sigma_a'(z))^{\frac{2+\beta}{p}}$.
From this Lemma, we have

$$
|f'(z)| \leqslant C\frac{\|f\|_{\mathcal{D}^p_\beta}}{(1-|z|^2)^{\frac{2+\beta}{p}}} \qquad (z\in\mathbb{D}).
$$

for $f \in \mathcal{D}^p_\beta$ and hence

$$
\begin{aligned}
|f(z)-f(0)| = \left|\int_0^z f'(\zeta)d\zeta\right| &\leqslant C\|f\|_{\mathcal{D}^p_\beta}\int_0^{|z|}\frac{d\rho}{(1-\rho^2)^{\frac{2+\beta}{p}}} \\
&\leqslant \frac{C}{(1-|z|^2)^{\frac{2+\beta}{p}}}\|f\|_{\mathcal{D}^p_\beta}. \qquad (3)
\end{aligned}
$$

Using the estimates (2) and (3), it follows from the proof of the Weak Convergence Theorem in [16] that:

Lemma 1.2. *Let $0 < \alpha, p < \infty$, $\beta > -1$, $g \in \mathcal{H}(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Let $X = \mathcal{B}^\alpha$ or $\mathcal{D}^p_\beta$; $Y = \mathcal{B}^\alpha$ or $\mathcal{D}^p_\beta$. Then $C^g_\varphi : X \longrightarrow Y$ is compact if and only if for any bounded sequence (f_n) in X which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, we have $\|C^g_\varphi f_n\|_Y \to 0$ as $n \to \infty$.*

2. The Boundedness and Compactness of

$$C_{\varphi}^{g}:\mathcal{B}^{\alpha}\longrightarrow\mathcal{D}_{\beta}^{p}$$

In this section, we characterize the boundedness and compactness of the generalized composition operators from Bloch type spaces $\mathcal{B}^{\alpha}$ to weighted Dirichlet type spaces $\mathcal{D}_{\beta}^{p}$. To do this, we need α-generalized hyperbolic $\mathcal{D}_{\beta}^{p}$ denoted by $D_{\alpha,\beta,p}^{h,g}$, defined as $\mathcal{D}_{\beta}^{p}$ but with the ordinary derivative $|\varphi'(z)|$ replaced by the α-generalized hyperbolic derivative $\frac{|g(z)|}{(1-|\varphi(z)|^2)^{\alpha}}$. Thus, for $0<\alpha,p<\infty$, $\beta>-1$ and $g\in\mathcal{H}(\mathbb{D})$, an analytic function φ on $\mathbb{D}$ belongs to $D_{\alpha,\beta,p}^{h,g}$ if and only if

$$\|\varphi\|_{D_{\alpha,\beta,p}^{h,g}}^{p}=\int_{\mathbb{D}}\frac{|g(z)|^{p}}{(1-|\varphi(z)|^{2})^{\alpha p}}\left(\log\frac{1}{|z|}\right)^{\beta}dA(z)<\infty.$$

Theorem 2.1. *Let $0<\alpha,p<\infty$, $\beta>-1$, $g\in\mathcal{H}(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(i) $\varphi\in D_{\alpha,\beta,p}^{h,g}$;

(ii) $C_{\varphi}^{g}:\mathcal{B}^{\alpha}\longrightarrow\mathcal{D}_{\beta}^{p}$ *is bounded;*

(iii) $C_{\varphi}^{g}:\mathcal{B}_{0}^{\alpha}\longrightarrow\mathcal{D}_{\beta}^{p}$ *is bounded.*

proof. (i)$\Longrightarrow$(ii). For $f\in\mathcal{B}^{\alpha}$, we have

$$|(C_{\varphi}^{g}f)'(z)|=|f'(\varphi(z))||g(z)|\leqslant\frac{|g(z)|}{(1-|\varphi(z)|^{2})^{\alpha}}\|f\|_{\mathcal{B}^{\alpha}},$$

and then

$$\|C_{\varphi}^{g}f\|_{\mathcal{D}_{\beta}^{p}}^{p}=\int_{\mathbb{D}}|f'(\varphi(z))|^{p}|g(z)|^{p}\left(\log\frac{1}{|z|}\right)^{\beta}dA(z)\leqslant\|f\|_{\mathcal{B}^{\alpha}}^{p}\|\varphi\|_{D_{\alpha,\beta,p}^{h,g}}^{p}.$$

Thus if $\varphi\in D_{\alpha,\beta,p}^{h,g}$, then $C_{\varphi}^{g}:\mathcal{B}^{\alpha}\longrightarrow\mathcal{D}_{\beta}^{p}$ is bounded.

(ii)$\Longrightarrow$(iii). It is trivial, since $\mathcal{B}_{0}^{\alpha}\subset\mathcal{B}^{\alpha}$.

(iii)$\Longrightarrow$(i). For $f \in \mathcal{B}^\alpha$ if we set $f_s(z) = f(sz)$ for $0 < s < 1$, then $f_s \in \mathcal{B}_0^\alpha$ and $\|f_s\|_{\mathcal{B}_0^\alpha} \leqslant \|f\|_{\mathcal{B}^\alpha}$. Using a well known result of Ramey and Ullrich [14], there are functions $f_1, f_2 \in \mathcal{B}^\alpha$ such that

$$(1-|z|^2)^{-\alpha} \lesssim |f_1'(z)| + |f_2'(z)| \qquad (z \in \mathbb{D}).$$

Then

$$\begin{aligned}\int_{\mathbb{D}} \tfrac{|sg(z)|^p}{(1-|s\varphi(z)|^2)^{\alpha p}} \left(\log \tfrac{1}{|z|}\right)^\beta \quad dA(z) &\lesssim \int_{\mathbb{D}} (|f_1'(s\varphi(z))|^p + |f_2'(s\varphi(z))|^p) \\ &\quad \times |sg(z)|^p \left(\log \tfrac{1}{|z|}\right)^\beta dA(z) \\ &= \|C_\varphi^g f_{1s}\|^p_{\mathcal{D}^p_\beta} + \|C_\varphi^g f_{2s}\|^p_{\mathcal{D}^p_\beta},\end{aligned}$$

holds for all $s \in (0,1)$. Hence, if $C_\varphi^g : \mathcal{B}_0^\alpha \longrightarrow \mathcal{D}_\beta^p$ is bounded, by an application of Fatou's lemma, this estimate implies that $\|\varphi\|^p_{D^{h,g}_{\alpha,\beta,p}} < \infty$ and $\varphi \in D^{h,g}_{\alpha,\beta,p}$. The proof is now complete. $\square$

Theorem 2.2. *Let $0 < \alpha, p < \infty$, $\beta > -1$, $g \in \mathcal{H}(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(i) $C_\varphi^g : \mathcal{B}^\alpha \longrightarrow \mathcal{D}_\beta^p$ *is compact;*

(ii) $C_\varphi^g : \mathcal{B}_0^\alpha \longrightarrow \mathcal{D}_\beta^p$ *is compact;*

(iii) $\lim_{t\to 1} \int_{|\varphi(z)|>t} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{\alpha p}} (\log \frac{1}{|z|})^\beta dA(z) = 0$ *and* $\varphi \in D^{h,g}_{\alpha,\beta,p}$.

proof. (iii)$\Longrightarrow$(i). From the boundedness of C_φ^g with $f(z) = z$, we have

$$L := \int_{\mathbb{D}} |g(z)|^p \left(\log \frac{1}{|z|}\right)^\beta dA(z) < \infty.$$

Let (f_n) be a sequence in the closed unit ball of $\mathcal{B}^\alpha$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. By hypothesis, for every $\varepsilon > 0$, there is $\delta \in (0,1)$ such that

$$\int_{|\varphi(z)|>\delta} \frac{|g(z)|^p}{(1-|\varphi(z)|^2)^{\alpha p}} \left(\log \frac{1}{|z|}\right)^\beta dA(z) < \varepsilon.$$

Let $K = \{w \in \mathbb{D} : |w| \leqslant \delta\}$. Then

$$\begin{aligned}\|C_\varphi^g f_n\|_{\mathcal{D}_\beta^p}^p &= \int_{\mathbb{D}} |f_n'(\varphi(z))|^p |g(z)|^p \left(\log\frac{1}{|z|}\right)^\beta dA(z)\\ &= \int_{|\varphi(z)|\leqslant\delta} |f_n'(\varphi(z))|^p |g(z)|^p \left(\log\frac{1}{|z|}\right)^\beta dA(z)\\ &+ \int_{\delta<|\varphi(z)|<1} |f_n'(\varphi(z))|^p |g(z)|^p \left(\log\frac{1}{|z|}\right)^\beta dA(z)\\ &\leqslant L\sup_{w\in K}|f_n'(w)|^p + \varepsilon\|f_n\|_{\mathcal{B}^\alpha}^p \leqslant L\sup_{w\in K}|f_n'(w)|^p + \varepsilon.\end{aligned}$$

By [3, IV,Theorem 2.1], the sequence (f_n') also converges to zero on compact subsets of $\mathbb{D}$ as $n\to\infty$. In particular, $\sup_{w\in K}|f_n'(w)|^p \to 0$ as $n\to\infty$. Hence $\|C_\varphi^g f_n\|_{\mathcal{D}_\beta^p}\to 0$ as $n\to\infty$. By Lemma 1.2, $C_\varphi^g : \mathcal{B}^\alpha \longrightarrow D_\beta^p$ is compact.

(i)$\Longrightarrow$(ii). It is trivial, since $B_0^\alpha \subset B^\alpha$.

(ii)$\Longrightarrow$(iii). Suppose $C_\varphi^g : \mathcal{B}_0^\alpha \longrightarrow \mathcal{D}_\beta^p$ is compact. Then by Theorem 2.1, $\varphi \in D_{\alpha,\beta,p}^{h,g}$. Next, since $(\frac{z^n}{n^{1-\alpha}})$ is norm bounded in $\mathcal{B}_0^\alpha$ and it converges to zero uniformly on compact subsets of $\mathbb{D}$, using Lemma 1.2, we have $\|C_\varphi^g(\frac{z^n}{n^{1-\alpha}})\|_{\mathcal{D}_\beta^p}\to 0$ as $n\to\infty$. Whence for every $\varepsilon>0$ there is an integer $N>1$ such that

$$n^{\alpha p}\int_{\mathbb{D}} |\varphi(z)|^{p(n-1)}|g(z)|^p \left(\log\frac{1}{|z|}\right)^\beta dA(z) < \varepsilon,$$

for all $n\geqslant N$. Thus for each $r\in(0,1)$

$$N^{\alpha p} r^{p(N-1)}\int_{|\varphi(z)|>r} |g(z)|^p \left(\log\frac{1}{|z|}\right)^\beta dA(z) < \varepsilon.$$

Taking $r \geqslant N^{\frac{-\alpha}{N-1}}$, we obtain

$$\int_{|\varphi(z)|>r} |g(z)|^p \left(\log\frac{1}{|z|}\right)^\beta dA(z) < \varepsilon. \tag{4}$$

On the other hand, for f in $\mathbb{B}_{\mathcal{B}_0^\alpha}$, the unit ball of $\mathcal{B}_0^\alpha$, if we let $f_t(z) = f(tz)$, then $f_t \to f$ uniformly on compact subsets of $\mathbb{D}$ as $t \to 1$. Since $C_\varphi^g : \mathcal{B}_0^\alpha \longrightarrow \mathcal{D}_\beta^p$ is compact, using Lemma 1.2, $\|C_\varphi^g(f_t - f)\|_{\mathcal{D}_\beta^p} \to 0$ as $t \to 1$. Thus for every $\varepsilon > 0$ there is a $t \in (0,1)$ such that

$$\int_{\mathbb{D}} |(C_\varphi^g f_t)'(z) - (C_\varphi^g f)'(z)|^p \left(\log \frac{1}{|z|}\right)^\beta dA(z) < \varepsilon.$$

Using this inequality along with (4), we have

$$\begin{aligned}\int_{|\varphi(z)|>r} &|(C_\varphi^g f)'(z)|^p (\log \frac{1}{|z|})^\beta dA(z) \\ &\leqslant C\varepsilon + C \int_{|\varphi(z)|>r} |(C_\varphi^g f_t)'(z)|^p (\log \frac{1}{|z|})^\beta dA(z) \\ &\leqslant C\varepsilon (1 + \sup_{z\in\mathbb{D}} |f_t'(z)|^p).\end{aligned}$$

Thus for every $f \in \mathbb{B}_{\mathcal{B}_0^\alpha}$ and every $\varepsilon > 0$, there exists a $\delta = \delta(f, \varepsilon)$ such that

$$\int_{|\varphi(z)|>r} |(C_\varphi^g f)'(z)|^p \left(\log \frac{1}{|z|}\right)^\beta dA(z) < \varepsilon,$$

for all $r \in [\delta, 1)$. Since C_φ^g is compact, thus $\{C_\varphi^g f : f \in \mathbb{B}_{\mathcal{B}_0^\alpha}\}$ is a relatively compact subset of $\mathcal{D}_\beta^p$. Hence there exist finitely many functions $f_1, \ldots, f_m \in \mathbb{B}_{\mathcal{B}_0^\alpha}$ such that for each $f \in \mathbb{B}_{\mathcal{B}_0^\alpha}$, $\|C_\varphi^g f - C_\varphi^g f_k\|_{\mathcal{D}_\beta^p} < \varepsilon$ for some $k \in \{1, \ldots, m\}$. Let $\delta = \max_{1\leqslant k\leqslant m} \delta(f_k, \varepsilon)$. Then for each $f \in \mathbb{B}_{\mathcal{B}_0^\alpha}$ we have

$$\int_{|\varphi(z)|>r} |(C_\varphi^g f)'(z)|^p \left(\log \frac{1}{|z|}\right)^\beta dA(z) < C\varepsilon,$$

if $r \in [\delta, 1)$. As mentioned in the previous theorem, there are two functions $f_1, f_2 \in \mathcal{B}^\alpha$ such that

$$(1 - |z|^2)^{-\alpha} \lesssim |f_1'(z)| + |f_2'(z)|.$$

If f_{1s}, f_{2s} defined as in the previous theorem, then we have

$$\begin{aligned}\varepsilon > C\int_{|\varphi(z)|>r}&\left[\frac{1}{\|f_{1s}\|^p_{\mathcal{B}_0^\alpha}}|(C^g_\varphi f_{1s})'(z)|^p+\frac{1}{\|f_{2s}\|^p_{\mathcal{B}_0^\alpha}}|(C^g_\varphi f_{2s})'(z)|^p\right]\left(\log\frac{1}{|z|}\right)^\beta dA(z)\\ \geqslant C\int_{|\varphi(z)|>r}&\left[|f_1'(s\varphi(z))|^p+|f_2'(s\varphi(z))|^p\right]|sg(z)|^p\left(\log\frac{1}{|z|}\right)^\beta dA(z)\\ \geqslant C\int_{|\varphi(z)|>r}&\frac{|sg(z)|^p}{|1-|s\varphi(z)|^2|^{\alpha p}}\left(\log\frac{1}{|z|}\right)^\beta dA(z),\end{aligned}$$

for all $r\in[\delta,1)$. Using Fatou's lemma, this estimate implies (iii). □

We recall that for $p\geqslant 1$, $\mathcal{D}^p_\beta$ is a Banach space. Using Lemma 4 in [8], one can say $C^g_\varphi:\mathcal{B}^\alpha_0\longrightarrow\mathcal{D}^p_\beta$ is weakly compact if and only if $C^g_\varphi:\mathcal{B}^\alpha_0\longrightarrow\mathcal{D}^p_\beta$ is compact.

In the case $p=2$ and $\alpha=1$, we obtain the following result. First we define $\sigma_a(z)=\frac{a-z}{1-\bar{a}z}$ for $a\in\mathbb{D}$, the conformal automorphism of $\mathbb{D}$ which exchanges the origin and the point a. Such a map is its own inverse and satisfies the fundamental identities

$$|\sigma_a'(z)|=\frac{1-|a|^2}{|1-\bar{a}z|^2}\quad\text{and}\quad 1-|\sigma_a(z)|^2=(1-|z|^2)|\sigma_a'(z)|.$$

Theorem 2.3. *Let $\beta>0$, $g\in\mathcal{H}(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. If $C^g_\varphi:\mathcal{B}\longrightarrow\mathcal{D}^2_\beta$ is compact, then*

$$\lim_{|\varphi(z)|\to 1}\frac{|g(z)|(1-|z|)^{\frac{\beta}{2}+1}}{1-|\varphi(z)|^2}=0.$$

The converse of this implication is true, provided that
$L:=\int_{|\varphi(z)|>r}\frac{dA(z)}{(1-|z|)^2}<\infty$, for $r\in(0,1)$.

proof. Let $f_a(z)=\frac{1-|a|^2}{\bar{a}}(2+\frac{1}{1-\bar{a}z})$ for any $a\in\mathbb{D}\setminus\{0\}$. Clearly

$$|f_a'(z)|=\frac{1-|a|^2}{|1-\bar{a}z|^2}=|\sigma_a'(z)|=\frac{1-|\sigma_a(z)|^2}{1-|z|^2}\leqslant\frac{1}{1-|z|^2}$$

which implies that $\sup_{z\in\mathbb{D}}|f_a'(z)|(1-|z|^2)\leqslant 1$. Also $f_a(0)=3\frac{1-|a|^2}{\bar{a}}$. So $\{f_a:\frac{1}{2}\leqslant|a|<1\}$ is a bounded family of functions in $\mathcal{B}$ and f_a converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. By compactness of C_φ^g and Lemma 1.2, we have $\|C_\varphi^g f_a\|_{\mathcal{D}_\beta^2} \to 0$ as $|a| \to 1$. Let $D(\lambda, r)$ denote the pseudohyperbolic disk $D(\lambda, r) = \{z \in \mathbb{D} : |\sigma_\lambda(z)| < r\}$. Observe that for any $\lambda \in \mathbb{D}$ and $0 < r < 1$, we have

$$\begin{aligned}
\|C_\varphi^g f_a\|_{\mathcal{D}_\beta^2}^2 &= \int_{\mathbb{D}} |f_a'(\varphi(z))|^2 |g(z)|^2 \left(\log \frac{1}{|z|}\right)^\beta dA(z) \\
&\geqslant \int_{D(\lambda,r)} |f_a'(\varphi(z))|^2 |g(z)|^2 \left(\log \frac{1}{|z|}\right)^\beta dA(z) \\
&= \int_{|w|<r} |f_a'(\varphi(\sigma_\lambda(w)))|^2 |g(\sigma_\lambda(w))|^2 |\sigma_\lambda'(w)|^2 \left(\log \frac{1}{|\sigma_\lambda(w)|}\right)^\beta dA(w) \\
&\geqslant C \int_{|w|<r} |((C_\varphi^g f_a) \circ \sigma_\lambda)'(w)|^2 (1 - |\sigma_\lambda(w)|^2)^\beta dA(w) \\
&\geqslant C \left(\frac{1-|\lambda|}{1+|\lambda|}\right)^\beta \int_{|w|<r} |((C_\varphi^g f_a) \circ \sigma_\lambda)'(w)|^2 dA(w) \\
&\geqslant C \left(\frac{1-|\lambda|}{1+|\lambda|}\right)^\beta |((C_\varphi^g f_a) \circ \sigma_\lambda)'(0)|^2 \\
&= C \left(\frac{1-|\lambda|}{1+|\lambda|}\right)^\beta \frac{(1-|a|^2)^2}{|1-\bar{a}\varphi(\lambda)|^4} |g(\lambda)|^2 (1-|\lambda|^2)^2 \\
&= C \frac{(1-|\lambda|)^{\beta+2}}{(1+|\lambda|)^{\beta-2}} \frac{(1-|a|^2)^2}{|1-\bar{a}\varphi(\lambda)|^4} |g(\lambda)|^2. \qquad (5)
\end{aligned}$$

Since $\|C_\varphi^g f_a\|_{\mathcal{D}_\beta^2} \to 0$ as $|a| \to 1$, then for given $\varepsilon > 0$ there is an r, $0 < r < 1$, such that $\|C_\varphi^g f_a\|_{\mathcal{D}_\beta^2} < \varepsilon$ whenever $|a| > r$. Hence for any point $z_0 \in \mathbb{D}$ with $|\varphi(z_0)| > r$ we have $\|C_\varphi^g f_{\varphi(z_0)}\|_{\mathcal{D}_\beta^2} < \varepsilon$. Hence, employing (5) with $\varphi(z_0)$ instead of a, we obtain

$$\frac{(1-|\lambda|)^{\frac{\beta}{2}+1}}{(1+|\lambda|)^{\frac{\beta}{2}-1}} \frac{1-|\varphi(z_0)|^2}{|1-\overline{\varphi(z_0)}\varphi(\lambda)|^2} |g(\lambda)| < \varepsilon.$$

Setting $\lambda = z_0$, we have

$$C \frac{|g(z_0)|(1-|z_0|)^{\frac{\beta}{2}+1}}{1-|\varphi(z_0)|^2} < \varepsilon.$$

Hence

$$\lim_{|\varphi(z)|\to 1}\frac{|g(z)|(1-|z|)^{\frac{\beta}{2}+1}}{1-|\varphi(z)|^2}=0.$$

To prove the converse, assume $L=\int_{|\varphi(z)|>r}\frac{dA(z)}{(1-|z|)^2}<\infty$ and

$$\lim_{|\varphi(z)|\to 1}\frac{|g(z)|(1-|z|)^{\frac{\beta}{2}+1}}{1-|\varphi(z)|^2}=0.$$

For compactness of $C_\varphi^g:\mathcal{B}\longrightarrow\mathcal{D}_\beta^2$, we show that the conditions in (iii) of Theorem 2.2 hold. By the definition of limit, for every $\varepsilon>0$ there exists $r\in(0,1)$ such that

$$\frac{|g(z)|(1-|z|)^{\frac{\beta}{2}}}{1-|\varphi(z)|^2}<\frac{\varepsilon^{\frac{1}{2}}}{1-|z|},$$

for every z with $|\varphi(z)|>r$. Integrating this inequality, we obtain

$$\begin{aligned}\int_{|\varphi(z)|>r}\frac{|g(z)|^2(\log\frac{1}{|z|})^\beta}{(1-|\varphi(z)|^2)^2}dA(z)&\leqslant C\int_{|\varphi(z)|>r}\frac{|g(z)|^2(1-|z|)^\beta}{(1-|\varphi(z)|^2)^2}dA(z)\\&\leqslant C\varepsilon\int_{|\varphi(z)|>r}\frac{dA(z)}{(1-|z|)^2}\\&=LC\varepsilon.\end{aligned}$$

On the other hand, we clearly have

$$\int_{|\varphi(z)|\leqslant r}\frac{|g(z)|^2(1-|z|)^\beta}{(1-|\varphi(z)|^2)^2}dA(z)<\infty.$$

Hence, $\varphi\in D_{1,\beta,2}^{h,g}$ and $\lim_{t\to 1}\int_{|\varphi(z)|>t}\frac{|g(z)|^2}{(1-|\varphi(z)|^2)^2}(\log\frac{1}{|z|})^\beta dA(z)=0$. Therefore, by Theorem 2.2, $C_\varphi^g:\mathcal{B}\longrightarrow\mathcal{D}_\beta^2$ is compact. □

3. The Boundedness and Compactness of $C_\varphi^g:\mathcal{D}_\beta^p\longrightarrow\mathcal{B}^\alpha$

In this section, we characterize the boundedness and compactness of the operator C_φ^g from weighted Dirichlet type spaces $\mathcal{D}_\beta^p$ to Bloch type spaces $\mathcal{B}^\alpha$.

Theorem 3.1. *Let $0 < \alpha, p < \infty$, $\beta > -1$, $g \in \mathcal{H}(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi^g : \mathcal{D}_\beta^p \longrightarrow \mathcal{B}^\alpha$ is bounded if and only if $L := \sup_{z\in\mathbb{D}} \frac{|g(z)|(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\frac{2+\beta}{p}}} < \infty$.*

proof. For the boundedness of C_φ^g, let $f \in \mathcal{D}_\beta^p$. Then $f' \in \mathcal{A}_\beta^p$ and from Lemma 1.1, we have

$$|f'(z)| \leqslant C\frac{\|f'\|_{\mathcal{A}_\beta^p}}{(1-|z|^2)^{\frac{2+\beta}{p}}} = C\frac{\|f\|_{\mathcal{D}_\beta^p}}{(1-|z|^2)^{\frac{2+\beta}{p}}} \quad (z \in \mathbb{D}). \tag{6}$$

From this inequality along with $C_\varphi^g f(0) = 0$, it follows that

$$\begin{aligned}
\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(C_\varphi^g f)'(z)| &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(\varphi(z)||g(z))| \\
&= \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha|g(z)|}{(1-|\varphi(z)|^2)^{\frac{\beta+2}{p}}}(1-|\varphi(z)|^2)^{\frac{\beta+2}{p}}|f'(\varphi(z))| \\
&\leqslant C\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha|g(z)|}{(1-|\varphi(z)|^2)^{\frac{\beta+2}{p}}}\|f\|_{\mathcal{D}_\beta^p} \\
&= CL\|f\|_{\mathcal{D}_\beta^p}.
\end{aligned}$$

Therefore, $C_\varphi^g : \mathcal{D}_\beta^p \longrightarrow \mathcal{B}^\alpha$ is bounded.

Conversely, consider the functions $f_a(z) = \int_0^z (\frac{1-|a|^2}{(1-\bar{a}\xi)^2})^{\frac{\beta+2}{p}} d\xi$, for $a \in \mathbb{D}$. Then by Lemma 1.1, $f_a \in \mathcal{D}_\beta^p$ and $\|f_a\|_{\mathcal{D}_\beta^p} \simeq 1$ with constant depending only on β and p. Fix $z_0 \in \mathbb{D}$ and $a = \varphi(z_0)$, the boundedness of $C_\varphi^g : \mathcal{D}_\beta^p \longrightarrow \mathcal{B}^\alpha$ implies that $\|C_\varphi^g f_a\|_{\mathcal{B}^\alpha} \leqslant C\|f_a\|_{\mathcal{D}_\beta^p} \leqslant C$ for all a, since $C_\varphi^g f_a(0) = 0$. Therefore,

$$\begin{aligned} C \geqslant \|C_\varphi^g f_a\|_{\mathcal{B}^\alpha} &= \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f_a'(\varphi(z)||g(z)| \\ &= \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha \left|\frac{1-|a|^2}{(1-\bar{a}\varphi(z))^2}\right|^{\frac{\beta+2}{p}} |g(z)| \\ &\geqslant \frac{(1-|z_0|^2)^\alpha |g(z_0)|}{(1-|\varphi(z_0)|^2)^{\frac{\beta+2}{p}}}. \end{aligned}$$

Hence, $\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha |g(z)|}{(1-|\varphi(z)|^2)^{\frac{\beta+2}{p}}} < \infty$. □

Theorem 3.2. *Let $0 < \alpha, p < \infty$, $\beta > -1$, $g \in \mathcal{H}(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi^g : \mathcal{D}_\beta^p \longrightarrow \mathcal{B}^\alpha$ is compact if and only if $C_\varphi^g : \mathcal{D}_\beta^p \longrightarrow \mathcal{B}^\alpha$ is bounded and $\lim_{|\varphi(z)|\to 1} \frac{|g(z)|(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\frac{\beta+2}{p}}} = 0$.*

proof. Let $C_\varphi^g : \mathcal{D}_\beta^p \longrightarrow \mathcal{B}^\alpha$ be compact. Note that f_a, defined as in the previous theorem, converges to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1$. Let (z_n) be a sequence in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Using the test function $f_n(z) = f_{\varphi(z_n)}(z)$, we get

$$\begin{aligned} \|C_\varphi^g f_n\|_{\mathcal{B}^\alpha} &= \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |(C_\varphi^g f_n)'(z)| \\ &\geqslant (1-|z_n|^2)^\alpha (\frac{1}{1-|\varphi(z_n)|^2})^{\frac{\beta+2}{p}} |g(z_n)|, \end{aligned}$$

from which the result follows.
Conversely, from the boundedness of C_φ^g with $f(z) = z$, we have

$$L := \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |g(z)| < \infty.$$

Suppose (f_n) is a sequence in unit ball of $\mathcal{D}_\beta^p$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. By definition of limit, for every $\varepsilon > 0$ there is $\delta \in (0,1)$ such that $\sup_{\{z:|\varphi(z)|>\delta\}} \frac{|g(z)|(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\frac{\beta+2}{p}}} < \varepsilon$. Then using (6), we get

$$
\begin{aligned}
\|C_\varphi^g f_n\|_{\mathcal{B}^\alpha} &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(C_\varphi^g f_n)'(z)| \\
&= \sup_{|\varphi(z)|\leqslant\delta}(1-|z|^2)^\alpha|f_n'(\varphi(z))||g(z)| \\
&+ \sup_{\delta<|\varphi(z)|<1}(1-|z|^2)^\alpha|f_n'(\varphi(z))||g(z)| \\
&\leqslant L\sup_{|w|\leqslant\delta}|f_n'(w)| + C\|f_n\|_{\mathcal{D}_\beta^p}\sup_{|\varphi(z)|>\delta}\frac{|g(z)|(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{\frac{\beta+2}{p}}} \\
&\leqslant L\sup_{|w|\leqslant\delta}|f_n'(w)| + C\varepsilon\|f_n\|_{\mathcal{D}_\beta^p} \\
&\leqslant L\sup_{|w|\leqslant\delta}|f_n'(w)| + C\varepsilon.
\end{aligned}
$$

By [3, IV, Theorem 2.1], the sequence (f_n') also converges to zero on compact subsets of $\mathbb{D}$ as $n\to\infty$. In particular $\sup_{|w|\leqslant\delta}|f_n'(w)|\to 0$ as $n\to\infty$. It follows that $\|C_\varphi^g f_n\|_{\mathcal{B}^\alpha}\to 0$ as $n\to\infty$. By Lemma 1.2, $C_\varphi^g : \mathcal{D}_\beta^p \longrightarrow \mathcal{B}^\alpha$ is compact. $\square$

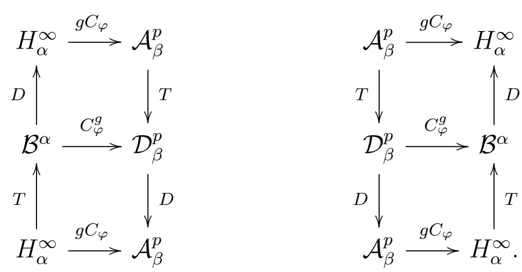
Finally, we give a connection between two operators gC_φ and C_φ^g.

To do this, for $\alpha>0$ we consider the standard weighted Banach spaces of analytic functions defined as follows:

$$H_\alpha^\infty = \{f\in\mathcal{H}(\mathbb{D}) : \|f\|_{H_\alpha^\infty} = \sup(1-|z|)^\alpha|f(z)|<\infty\},$$

$$H_\alpha^0 = \{f\in H_\alpha^\infty : \lim_{|z|\to 1}(1-|z|)^\alpha|f(z)|=0\}.$$

For more details about spaces of this type we refer to [1, 2, 9, 12] and the references therein. Then, we consider the differentiation operator D and the integration operator T from $\mathcal{H}(\mathbb{D})$ into $\mathcal{H}(\mathbb{D})$ defined by $(Df)(z)=f'(z)$ and $(Tf)(z)=\int_0^z f(\xi)d\xi$, respectively. Now, consider the following diagrams



Clearly, by the definitions of the spaces H_α^∞, $\mathcal{B}^\alpha$, $\mathcal{A}_\beta^p$ and $\mathcal{D}_\beta^p$, the operators D and T between these spaces as mentioned in the above diagrams are bounded for $\alpha, p > 0$ and $\beta > -1$. Also these diagrams commute, that is,

$$T \circ gC_\varphi \circ D = C_\varphi^g \quad \text{and} \quad D \circ C_\varphi^g \circ T = gC_\varphi,$$

which imply that, weighted composition operators $gC_\varphi : \mathcal{A}_\beta^p \longrightarrow H_\alpha^\infty$ and $gC_\varphi : H_\alpha^\infty \longrightarrow \mathcal{A}_\beta^p$ are bounded (compact) if and only if the generalized composition operators $C_\varphi^g : \mathcal{D}_\beta^p \longrightarrow \mathcal{B}^\alpha$ and $C_\varphi^g : \mathcal{B}^\alpha \longrightarrow \mathcal{D}_\beta^p$ are bounded (compact), respectively.

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