Journal of Mathematical Extension
Vol. 6, No. 1, (2012), 1-9

Enlargements of Monotone Operators Determined by Representing Functions

M. Rezaei
University of Isfahan

Abstract. In this paper, we study a new enlargement of subdifferential for any proper lower semicontinuous function. We know that ε-subdifferential of any proper lower semicontinuous function is an enlargement of its subfifferential and any point from the graph of ε-subdifferential can be approximated by a point from the graph of subfifferential. This nice property, apart from its theoretical importance, gives also the possibility to use the enlargement of subdifferentials in finding approximate solutions of inclusions determined by subdifferentials. We define a new enlargement and observe, in the case subdifferentials, the relation between this new enlargement and the ε- subdifferential.

AMS Subject Classification: 49J40; 90C25; 90C30.
Keywords and Phrases: Maximal monotone operators, ε-subdifferential, lower semicontinuous function, Fitzpatrick function, enlargement of an operator.

1. Introduction

Let X be real Banach space with continuous dual X^* with pairing between them denoted by $\langle . \, , \, . \rangle$. As usual, we will use the same symbol $\|.\|$ for the norms in X and X^*, w and w^* will stand for the weak and weak star topology in X and X^* respectively. For given operator $T : X \rightrightarrows X^*$, its graph will be denoted by

$$Gr(T) := \{(x, x^*) \in X \times X^* : x^* \in T(x)\},$$

Received: November 2011; Final Revision: April 2012

and the sets

$$Dom(T) := \{x \in X : T(x) \neq \emptyset\}$$

and

$$R(T) := \cup\{T(x) : x \in Dom(T)\}$$

will stand for the domain and the range of T respectively. The inverse operator of T, $T^{-1} : X^* \rightrightarrows X$ is defined by

$$T^{-1}(x^*) := \{x \in X : x^* \in T(x)\}, \qquad x^* \in X^*,$$

and evidently has the range of T as domain and the same graph as T. The operator T is said to be monotone if

$$\langle y - x, y^* - x^* \rangle \geqslant 0 \qquad \forall (x, x^*), (y, y^*) \in Gr(T).$$

Observe that the property of T being monotone is a property of the graph of T, hence, if T is monotone, the same is true for T^{-1}.
A monotone operator $T : X \rightrightarrows X^*$, is called maximal if its graph can not be properly extended to a graph of another monotone mapping between X and X^*. In an equivalent way, T is maximal, if the condition $\langle y - x, y^* - x^* \rangle \geqslant 0$ for every Gr(T), implies that $(x, x^*) \in Gr(T)$.
A well-known example of a maximal monotone operator is the subdifferential of a proper lower semicontinuous convex function $f : X \to \mathbb{R} \cup \{+\infty\}$. Recall that proper f means that the set $domf := \{x \in X : f(x) < +\infty\}$ (which is the effective domain of f) is nonempty. For any $\varepsilon \geqslant 0$, if $x \in domf$, we define the ε-subdifferential of f by:

$$\partial_\varepsilon f(x) := \{x^* \in X^* : f(y) - f(x) \geqslant \langle y - x, x^* \rangle - \varepsilon, \quad \forall y \in X\},$$

if $x \in domf$, and $\partial_\varepsilon f(x) := \emptyset$, if $x \notin domf$. For every $\varepsilon > 0$, $\partial_\varepsilon f$ is always non-empty valued at the points of $domf$. In other words, for every $\varepsilon > 0$ one has $Dom(\partial_\varepsilon f) = domf$. For $\varepsilon = 0$, $\partial_0 f$ is subdifferential ∂f of f (the latter could be empty at some points of dom f). According to the well-known result of Rockafellar ([16]), ∂f is a maximal monotone operator.

2. Enlargements

As it was seen from the definition of ε-subdifferential, for any proper lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$ and any $\varepsilon > 0$, the ε-subdifferential $\partial_\varepsilon f$ is an enlargement of the subdifferential ∂f, i.e. $\partial f(x) \subset \partial_\varepsilon f(x)$ for any $x \in X$. On the other hand, this enlargement is not far enough from the initial operator. The well-known Brøndsted-Rockafellar theorem ([1]) asserts that any point from the graph of $\partial_\varepsilon f$ can be approximated (depending ε) with a point from the graph ∂f. This nice property, apart from its theoretical importance, give also the possibility to use the enlargement of subdifferentials in finding approximate solutions of inclusions determined by subdifferentials.
Motivated by the above, the search of possible enlargements of an arbitrary (maximal) monotone operators has been done during in recent years. Different attempts for such notions, inspired by various properties of variational inequalities, could be found in [8],[15] and [19]. Recently the following notion of enlargement has been paid a lot of attention. Given a monotone $T : X \rightrightarrows X^*$, $\varepsilon > 0$ and for every $(y, y^*) \in Gr(T)$, let

$$T^\varepsilon(x) := \{x^* \in X^* : \langle y - x, y^* - x^* \rangle \geqslant -\varepsilon., \quad (1)$$

This definition was given in [9] but the notion was not studied. Independently, this concept has been studied, first in finite dimensions in [2] with applications to approximate solutions of variational inequalities, and then in Hilbert space (most of the proofs work straitforward in the reflexive case) in [3], with applications to finding a zero of a maximal monotone operator. An approach with families of enlargements was further investigated in [18].
In the case of subdifferentials, i.e. $T = \partial f$ for a convex proper (lower semicontinuous) function, one easily sees that $\partial_\varepsilon f \subset T^\varepsilon$ for every $\varepsilon > 0$. Let us mention that this inclusion is, in general, strict and a simple example is $\frac{1}{2}\|x\|^2$. But the enlargement T^ε satisfies a Brøndsted -Rockafellar type theorem in reflexive Banach spaces. The result was proved by Torralba[19] by making use of the Minty-Rockafellar subjectivity theorem (see also [3]) for Hilbert space setting. But, as mentioned above, the proof works for the case of a reflexive space because it uses the same

surjectively theorem. An extension outside the reflexive case for a class of maximal monotone operators is given in [14].
Here we wish to introduce and investigate another enlargement, which is generated by representation function associated with given monotone operator proposed by Fitzpatrick in [6]. The idea is to try from one side to enlarge enough the given operator in order to able to treat it numerically. However on the other side to have a smaller one which will allow to approximate always the enlargement by couples from the graph of the initial operator.
Let us consider the Cartesian product $X \times X^*$ equipped with a product topology determined by the norm topology in X and the w^*-topology in X^*. In this case the dual of $X \times X^*$ can be identified with $X^* \times X$ and hence, for the couples $\langle (x, x^*), (y^*, y)\rangle = \langle x, y^*\rangle + \langle y, x^*\rangle$. First, for given a (proper)convex function $g : X \times X^* \to \mathbb{R} \cup \{+\infty\}$, let us define the following operator $T_g : X \rightrightarrows X^*$ by

$$T_g(x) := \{x^* : (x^*, x) \in \partial g(x, x^*)\}, \qquad x \in X.$$

The so-defined operator T_g is monotone ([6], Proposition 2.2).
Further, for a given monotone operator $T : X \rightrightarrows X^*$, let us define the following function $L_T : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ by

$$L_T(x, x^*) := \sup\{\langle y, x^*\rangle + \langle x-y, y^*\rangle : (y, y^*) \in Gr(T)\}, \quad (x, x^*) \in X \times X^*. \tag{2}$$

The following theorem summarizes some of the most important properties of L_T proved by Fitzpatrick in [6]:

Theorem 2.1. ([6]) *Let $T : X \rightrightarrows X^*$ be monotone operator with $Dom(T) \neq \emptyset$. Then*

(a) The function L_T is a proper convex $\|.\| \times w^*$-lower semicontinuous function;

(b) For any $x \in X$ one has $T(x) \subseteq T_{L_T}(x)$. If T is maximal monotone $T = T_{L_T}$;

(c) If T is maximal monotone, then $L_T(x, x^*) \geqslant \langle x, x^*\rangle$ for every $(x, x^*) \in X \times X^*$ and $L_T(x, x^*) = \langle x, x^*\rangle$ if and only if $(x, x^*) \in Gr(T)$. Moreover, L_T is the minimal convex function on $X \times X^*$ with these two properties.

The above representation of a given monotone operator by subdifferentials of convex functions in $X \times X^*$ is the transformation of the representation of the monotone operators by subdifferentials of saddle functions provided by Krauss[7]. The Fitzpatrick approach was also studied in [10].
Now, let us use the usual ε-subdifferetials of the function L_T in order to define an enlargement of a given monotone operator $T : X \rightrightarrows X^*$ for which we will always assume that $Dom(T) \neq \emptyset$. For $\varepsilon > 0$ let

$$T_\varepsilon(x) := \{x^* \in X^* : (x^*, x) \in \partial_\varepsilon L_T(x, x^*)\}, \qquad x \in X. \quad (3)$$

Because of Theorem 2.1, this operator needs an enlargement of T, i.e. $T(x) \subset T_\varepsilon(x)$ for any $x \in X$. Moreover, it can be easily by verified that $T_\varepsilon(x)$ is convex and since $L_T(x, .)$ is lower semicontinuous for the w^*-topology, then for any ε and $x \in X$ the image $T_\varepsilon(x)$ is w^*-closed in X^*. First, we know that the new enlargement is contained in the one form (1).

Proposition 2.2. *Let $T : X \rightrightarrows X^*$ be maximal monotone. Then $T_\varepsilon \subset T^\varepsilon$.*

Proof. Let $\varepsilon > 0$ and $x^* \in T_\varepsilon(x)$ for $x \in X$. Then, by definition, for every $(u, u^*) \in X \times X^*$, we have

$$L_T(u, u^*) - L_T(x, x^*) \geqslant \langle (u - x, u^* - x^*), (x^*, x)\rangle - \varepsilon.$$

Since $L_T(x, x^*) \geqslant \langle x, x^*\rangle$ and for $(u, u^*) \in Gr(T)$ one has $L_T(u, u^*) = \langle u, u^*\rangle$, for every $(u, u^*) \in Gr(T)$, the latter inequality gives

$$\langle (u - x, u^* - x^*), (x^*, x)\rangle \geqslant -\varepsilon,$$

which, according to (2.1) $x^* \in T^\varepsilon(x)$. The proof is completed. □

Further, we wish to investigate the particular case of subdifferentials. First we observe the following simple estimation.

Lemma 2.3. *Let $T : X \rightrightarrows X^*$ be monotone, $\varepsilon > 0$ and $x^* \in T^\varepsilon(x)$. Then*

$$L_T(x, x^*) \leqslant \langle x, x^*\rangle + \varepsilon.$$

Proof. The proof comes directly from the definitions. Since $x^* \in T^\varepsilon(x)$ for every $(y, y^*) \in Gr(T)$, we have

$$\langle y - x, y^* - x^* \rangle \geqslant -\varepsilon$$

which for every $(y, y^*) \in Gr(T)$, gives

$$\langle x, x^* \rangle + \varepsilon \geqslant \langle y, x^* \rangle + \langle x - y, y^* \rangle.$$

Now the desired estimation follows from the definition of L_T in (2). □

Observe that, due to the previous proposition, this estimation holds also for the new enlargement T_ε.
Now, let us see, in the case of subdifferentials, what is the relation between the new enlargement and the ε-subdifferential. Given a proper lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, by f^* we denoted the Fenchel-Moreou conjugate of f, i.e. the function $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$, defined by

$$f^*(x^*) := \sup\{\langle x, x^* \rangle - f(x) : x \in X\}, \qquad x^* \in X^*$$

Observe that f^* is lower semicontinuous with respect to the w^*- topology in X^*. Hence, if we put $g(x, x^*) = f(x) + f^*(x^*), (x, x^*) \in X \times X^*$, we obtain a convex proper and $\|.\| \times w^*$-lower semicontinuous function in $X \times X^*$. It is easily verified that for this function we have $T_g = \partial f$ (Example 2.3 from [6]). Moreover, since $g(x, x^*) \geqslant \langle x, x^* \rangle$ for any $(x, x^*) \in X \times X^*$ and $g(x, x^*) = \langle x, x^* \rangle$ exactly when $x^* \in \partial f(x)$, then by Theorem 2.1 for $T = \partial f = T_g$, for every $(x, x^*) \in X \times X^*$ we get

$$g(x, x^*) = f(x) + f^*(x^*) \geqslant L_T(x, x^*).$$

Theorem 2.4. *Let $T = \partial f$ for some proper convex and lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$. Then for any $\varepsilon > 0$ and $x \in dom f$ we have $T_\varepsilon(x) \subset \partial_\varepsilon f(x)$.*

Proof. Take $x \in dom f$ and $x^* \in T_\varepsilon(x)$ for some $\varepsilon > 0$. By definition this means for every $(u, u^*) \in X \times X^*$

$$L_T(u, u^*) - L_T(x, x^*) \geqslant \langle (u - x, u^* - x^*), (x^*, x) \rangle - \varepsilon.$$

As above, using $L_T(x,x^*) \geqslant \langle x,x^*\rangle$, for every $(u,u^*) \in X \times X^*$

$$L_T(u,u^*) - \langle x,u^*\rangle \geqslant \langle u-x,x^*\rangle - \varepsilon.$$

Let us take an arbitrary $u \in X$ and $\delta > 0$. Take a $u^*_\delta \in \partial_\delta f(x)$. By the last inequality we have

$$L_T(u,u^*_\delta) - \langle x,u^*_\delta\rangle \geqslant \langle u-x,x^*\rangle - \varepsilon,$$

from $f^*(x^*) = \sup\{\langle x,x^*\rangle - f(x) : x \in X\}$, by the Theorem 2.1 (c) we have $L \geqslant f + f^*$ and

$$f(u) + f^*(u^*_\delta) - \langle x,u^*_\delta\rangle \geqslant \langle u-x,x^*\rangle - \varepsilon.$$

Since $u^*_\delta \in \partial_\delta f(x)$, we know that

$$f(u) - f^*(u^*_\delta) \leqslant \langle x,u^*_\delta\rangle + \delta,$$

which together with the previous inequality give

$$f(u) - f(x) + \delta \geqslant \langle u-x,x^*\rangle - \varepsilon.$$

Passing to the limit for δ we get

$$f(u) - f(x) \geqslant \langle u-x,x^*\rangle - \varepsilon,$$

and since u was arbitrary, we conclude that $x^* \in \partial_\varepsilon f(x)$. $\square$

Acknowledgment: In this research, the author was partially supported by the center of Excellence for Mathematics, University of Isfahan, Isfahan, Iran.

References

[1] A. Brøndsted and R. T. Rockafellar, On the subdifferentiability of convex functions, *Proc. Amer. Soc.*, 16 (1965), 605-611.

[2] R. S. Burachik, A. N. Iusem, and B. F, Svaiter, Enlargements of maximal monotone opertors with applications to varitional inequalities, *Set-valued Anal.*, 5 (1997), 159-180.

[3] R. S. Burachik, C. A. Sagastizabal, and B. F. Svaiter, ε-Enlargement of maximal monotone operators: Theory and Application, in Nonsmooth , Piecewise Smooth, Semismooth and Smoothing Methods, M. Fukushima and L. Qi (eds), *Kluwer Academic Publishers*, (1998), 25-43.

[4] R. S. Burachik and A. N. Iusem, On non-enlargeable and fully-enlargeable maximal monotone operators, *Journal of Convex Analysis*, 13 (2006), 603-622.

[5] R. S. Burachik, A. N. Iusem, And J. G. Melo, Duality and exact penalization for general augmented Lagrangians, *Journal of Optimization Theory and Applications*, 147 (2010), 125-140.

[6] S. Fitzpatrick, Representing monotone operators by convex functions, *Proc. of Workshop on Functional Analysis and Optimization*, Canberra, Australia, 1988, Center Math. Anal., Austral. Nat. Uni., 20 (1988), 59-65.

[7] E. Krauss, A representation of maximal monotone operators by saddle functions, *Rev. Roumiane math. Pures Appl.*, 30 (1985), 823-837.

[8] R. Lucchetti and F. Patrone, A characterization of Tykhonov well-posedness for minimum problems with applications to variational inequalities, *Number. Function. Anal. Optim.,* 4 (1981), 461-476.

[9] J. E. Martinez-Legas and M. Thera, ε-subdifferentials in terms of subdifferentials, *Set-valued Anal.*, 4 (1996), 327-332.

[10] J. E. Martinez-Legaz and M. Thera, A convex representation of maximal monotone operators, *J. Nonlinear and Convex Analysis,* 2 (2001), 243-247.

[11] R. R. Phelps, *Convex functions, monotone operators and differentiability,* Lec. Notes in Math. Vol. 1364, Springer-Verlag, 1993.

[12] M. Rocco and J. E. Martnez-Legaz, On surjectivity results for maximal monotone operators of type (D), *Convex Anal.,* (to appear)

[13] J. P. Revalski, Variational inequalites with unique solution, in Math. and Education in Math., *Proc. og the 14-th Spring Conf. of the Union of the Bulgarian Mathematicians,* (1985), 534-541.

[14] J. P. Revalski and M. Thera, Enlargements and sums of monotone operators, *Nonlinear Analysis,* 48 (2002), 505-519.

[15] R. T. Rockafellar, Local boundeness of nonlinear monotone operators, *Michigan Math.,* 16 (1969), 397-407.

[16] R. T. Rockafellar, On the maximal monotonicity of subdifferntial mappings, *Pacific J. Math.,* 33 (1970), 209-216.

[17] S. Simons, Minimax and Monotonicity, *Lec. Notes in Math.,* Vol. 1693, Springer-Verlag, Berlin, Heidelberg, 1998.

[18] B. F. Svaiter, A family of enlargements of maximal monotone operators, *Set-valued Anal.,* 8 (2000), 311-328.

[19] D. Torralba, *Convergence é pgraphique et changements déchelle en analyse varionelle et optimization*, Thèse de Université de Montpellier II, 1996.

[20] L. Vesely, Local uniform boundedness principle for families of ε-monotone operators, *Nonlinear Anal., TMA,* 24 (1994), 1299-1304.

Mahboubeh Rezaei
Department of Mathematics
Assistant Professor of Mathematics
University of Isfahan
P.O. Box 81745-163
Isfahan, Iran
E-mail: mrezaie@sci.ui.ac.ir