Homotopy Analysis Method Based on Optimal Value of the Convergence Control Parameter for Solving Semi-Differential Equations

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Abstract. In this paper, homotopy analysis method is directly extended to investigate nth order semi-differential equations and to derive their numerical solutions which is introduced by replacing some integer-order space derivatives by fractional derivatives. The fractional derivatives are described in the Caputo sense. So the homotopy analysis method for differential equations of integer-order is directly extended to derive explicit and numerical solutions of the fractional differential equations. An optimal value of the convergence control parameter is given through the square residual error. Comparison is made between Homotopy perturbation method, collocation spline method, and the present method.

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1. Introduction

Fractional differential equations have gained importance and popularity during the past three decades or so, mainly due to its demonstrated

applications in numerous seemingly diverse fields of science and engineering. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives, and the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow. The differential equations with fractional order have recently proved to be valuable tools to the modeling of many physical phenomena ([3, 20]). This is because of the fact that the realistic modeling of a physical phenomenon does not depend only on the instant time, but also on the history of the previous time which can also be successfully achieved by using fractional calculus. Most nonlinear fractional equations do not have exact analytic solutions, so approximation and numerical techniques must be used. The Adomain decomposition method (ADM) ([16]), the homotopy perturbation method (HPM) ([18]), the variational iteration method (VIM) ([19]) and other methods have been used to provide analytical approximation to linear and nonlinear problems. However, the convergence region of the corresponding results is rather small. In 1992, Liao employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely Homotopy Analysis Method (HAM), ([9-13]). This method has been successfully applied to solve many types of nonlinear problems in science and engineering, such as the viscous flows of non-Newtonian fluids ([7]), the KdV-type equations ([1]), finance problems ([28]), Falkner-Skan boundary layer flows ([17]), electrohydrodynamic flow ([14]), systems of fractional algebraic-differential equations ([26, 27]) and so on. The HAM contains a certain auxiliary parameter h which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution.

The HAM offers certain advantages over routine numerical methods. Numerical methods use discretization which gives rise to rounding off errors causing loss of accuracy, and requires large computer memory and time. This computational method yields analytical solutions and has certain advantages over standard numerical methods. The HAM method is better since it does not involve discretization of the variables and hence is free from rounding off errors and does not require large computer memory or time.

The aim of this paper is to use homotopy analysis method for solving semi-differential equations. An optimal value of the convergence control parameter is defined through the square residual error concept. The paper has been organized as follows. In Section 2, a brief review of the theory of fractional calculus will be given to fix notation and provide a convenient reference. In Section 3, we give analysis of the HAM. In Section 4, we extend the application of the HAM to construct numerical solution for the fractional semi-differential equation. Conclusions are presented in Section 5.

2.   Description on the Fractional Calculus

Definition 2.1. *A real function $f(t), t > 0$ is said to be in the space $C_\mu, \mu \in \mathrm{R}$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$ where $f_1 \in (0, \infty)$, and it is said to be in the space C_n^μ l if and only if $h(n) \in C_\mu, n \in N$. Clearly $C_\mu \subset C_\nu$ if $\nu \leqslant \mu$ [3].*

Definition 2.2. *The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geqslant 0$, of a function $f \in C_\mu, \mu \geqslant -1$, is defined as*

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} f(t) dt, \qquad \alpha > 0, x > 0.) \quad (1)$$

$$J^0 f(x) = f(x). \tag{2}$$

$\Gamma(\alpha)$ is the well-known Gamma function. Some of the properties of the operator J^α, which we will need here, are as follows [21]:
For $f \in C_\mu, \mu \geqslant -1, \alpha, \beta \geqslant 0$ and $\gamma \geqslant -1$

$$J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x),$$
$$J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x),$$
$$J^\alpha t^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\alpha + \gamma + 1)} t^{\alpha+\gamma}.$$

Definition 2.3. *For the concept of fractional derivative, there exist many mathematical definitions [24, 15, 23, 3, 21]. In this paper, the two most commonly used definitions: the Caputo derivative and its reverse*

operator Riemann-Liouville integral are adopted. That is because Caputo fractional derivative [3] allows the traditional assumption of initial and boundary conditions. The Caputo fractional derivative is defined as

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,t)}{\partial t^n} d\tau, & n-1 < \alpha < n, \\ \frac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n \in N. \end{cases}$$

$$(3)$$

Here, we also need two basic properties about them:

$$D^\alpha J^\alpha f(x) = f(x), \tag{4}$$

$$J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^\infty f^{(k)}(0^+) \frac{x^k}{k!}, \qquad x > 0. \tag{5}$$

Definition 2.4. *The MittagLeffler function $E_\alpha(z)$ with $a > 0$ is defined by the following series representation, valid in the whole complex plane [21]:*

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\alpha n + 1)}, \qquad \alpha > 0, z \in \mathrm{C} \tag{6}$$

Definition 2.5. *A fractional differential equation of the form [22]:*

$$[D^{\frac{n}{2}} + c_1 D^{\frac{n-1}{2}} + \cdots + c_n D^0] = f(t), \tag{7}$$

is called a semi-differential equation of nth order, where $c_1, \cdots, c_n \in \mathbb{R}$ and $f(t)$ is a given function from I into $\mathbb{R}$, I is the interval $[0, T]$.

In (7) D^α denotes the fractional differential operator of order $\alpha \notin N$ in the sense of Caputo, and is given by (3). The theory for the derivatives of fractional order was developed in the 19th century. Recently, fractional derivatives have proved to be tools in the modeling of many physical phenomena (see, [4, 5, 6]). We mention the important example: the Bagley-Torvik equation,

$$[D^2 + c_1 D^{\frac{3}{2}} + \cdots + c_4 D^0]y(t) = f(t), \quad y(0) = a_1, y'(0) = a_2, \tag{8}$$

which arises, for example, in the modeling of the motion of a rigid plate immersed in a Newtonian fluid. Studying the numerical solution of (7) has been increased in the last two decades. A survey of some numerical methods is given by Podlubny [21]. Blank [2] proposed the collocation spline method and also Rawashdeh [22] applied the collocation spline method to solve semi-differential equations. In this paper we will apply Homotopy analysis method for solving semi-differential equations of nth order.

3. Basic Idea of HAM

To describe the basic ideas of the HAM, we consider the following fractional initial value problem:

$$N_\alpha[y(t)] = 0, \quad t \geqslant 0, \tag{9}$$

where N_α is a nonlinear differential operator that may involves fractional derivatives, t denotes an independent operator and $y(t)$ is an unknown function. The highest order derivative is n, subject to the initial conditions

$$y^{(k)}(0) = c_k, \quad k = 0, 1, \cdots, n - 1. \tag{10}$$

By means of generalizing the traditional homotopy method, the so-called zeroth-order deformation equation can be defined as

$$(1 - q)D_*^\beta[\phi(t; q) - u_0(t)] = q\, hH(t)N_\alpha[\phi(t; q)], \tag{11}$$

where $q \in [0, 1]$ is the embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, and the auxiliary linear operator L can be considered as $L = D_*^\beta$, $\beta > 0$. $y_0(t)$ is initial guess of $y(t)$, $y(t)$ is an unknown functions, respectively. It is important, that one has great freedom to choose auxiliary things in HAM.
Obviously, when $q = 0$ and $q = 1$, it holds

$$\phi(t; 0) = y_0(t), \quad \phi(t; 1) = y(t),$$

respectively. Thus, as q increases from 0 to 1, the solution $y(t; q)$ varies from the initial guess $y_0(t)$ to the solution $y(t)$. Expanding $y(t; q)$ in Taylor series with respect to q, we have

$$\phi(t; q) = y_0(t) + \sum_{m=1}^{+\infty} y_m(t)q^m, \tag{12}$$

where

$$y_m(t) = \frac{1}{m!} \frac{\partial^m \phi(t; q)}{\partial q^m}|_{q=0}, \tag{13}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series (12) converges at $q = 1$, then we have

$$y(t) = y_0(t) + \sum_{m=1}^{+\infty} y_m(t), \tag{14}$$

which must be one of solutions of original nonlinear equation, as proved by Liao [11]. As $h = -1$ and $H(t) = 1$, Eq. (11) becomes

$$(1 - q)D_*^{\beta}[\phi(t; q) - y_0(t)] + q \, N_\alpha[\phi(t; q)] = 0, \tag{15}$$

which is used mostly in the homotopy perturbation method [8], where as the solution obtained directly, without using Taylor series . According to the definition (13), the governing equation can be deduced from the zero-order deformation equation (11).
Define the vector

$$\vec{y}_n = \{y_0(t), y_1(t), \dots, y_n(t)\}.$$

Differentiating equation (11) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$D_*^{\beta}[y_m(t) - \chi_m y_{m-1}(t)] = h \, H(t)R_m(\vec{y}_{m-1}), \tag{16}$$

where

$$R_m(\vec{y}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N_\alpha[\phi(t;q)]}{\partial q^{m-1}}\Big|_{q=0}, \tag{17}$$

and

$$\chi_m = \left\{ \begin{array}{ll} 0, & m \leqslant 1, \\ 1, & m > 1. \end{array} \right.$$

Applying the Riemann-Liouville integral operator J^β on both side of Eq. (16), we have

$$y_m(t) = \chi_m y_{m-1}(t) + h J^\beta [H(t) R_m(\vec{y}_{m-1}(t))], \tag{18}$$

In this way, it is easy to obtain $y_m(t)$ for $m \geqslant 1$, at mth-order, we have

$$y(t) = \sum_{m=0}^{M} y_m(t). \tag{19}$$

When $M \to \infty$, we get an accurate approximation of the original Eq. (9).

Remark 1. *In 2007, Yabushita et al. ([25]) applied the HAM to solve two coupled nonlinear ODEs, and suggested the so-called optimization method to find out two optimal convergence-control parameters by means of the minimum of the square residual error integrated in the whole region having physical meanings. Their approach is based on the square residual error*

$$\Delta(h) = \int_\Omega \left(N \left[\sum_{k=0}^{M} y_k(t) \right] \right)^2 d\Omega, \tag{20}$$

of a nonlinear equation $N[y(t)] = 0$, where $\sum_{k=0}^{M} y_k(t)$ gives the Mth-order HAM approximation. Obviously, $\Delta(h) \to 0$ (as $M \to +\infty$) corresponds to a convergent series solution. For given order M of approximation, the optimal value of h is given by a nonlinear algebraic equation

$$\frac{d\Delta(h)}{dh} = 0.$$

We use exact square residual error (20) integrated in the whole region of interest Ω, at the order of approximation M.

4.   Application

In this section we apply homotopy analysis method for solving Bagley-Torvik equation and the initial value problem, which are semi-differential equations of order 4, ([22]).

Example 3.1.   Consider the Bagley-Torvik equation

$$[D^2 + D^{\frac{3}{2}} + D^0]y(t) = 2 + 4\sqrt{\frac{t}{\pi}} + t^2, \quad y(0) = y'(0) = 0. \qquad (21)$$

The exact solutions of Eq. (12) is

$$y(t) = t^2, \qquad (22)$$

For application of homotopy analysis method, it in convenient to choose

$$y_0(t) = 0, \qquad (23)$$

as the initial approximate of Eq. (26). By taking $\beta = 2$, we choose the linear operator

$$L[\phi(t;q)] = \frac{d^2\phi(t;q)}{dt^2}, \qquad (24)$$

with the property $L(c_1 t + c_2) = 0$ where c_1 and c_2 are constants of integrations. Furthermore, we define nonlinear operators as

$$N_\alpha[\phi(t;q)] = \frac{d^2\phi(t;q)}{dt^2} + D^{\frac{3}{2}}\phi(t;q) + \phi(t;q) - 2 - 4\sqrt{2}\frac{t}{\pi} - t^2,$$

We construct the zeroth-order and the mth-order deformation equations where

$$R_m(\vec{y}_{m-1}) = \frac{d^2 y_{m-1}}{dt^2} + D^{\frac{3}{2}} y_{m-1} + y_{m-1} + (1 - \chi_m)(-2 - 4\sqrt{2}\frac{t}{\pi} - t^2),$$

We now successively obtain the solution to each high order deformation equations:

$$y_m(t) = \chi_m y_{m-1}(t) + L^{-1}\left[R_m(\vec{y}_{m-1})\right], m \geqslant 1, \tag{25}$$

We now successively obtain

$$y_1(t) = -ht^2 - \frac{16ht^{\frac{5}{2}}}{15\pi^{\frac{1}{2}}} - \frac{1}{12}ht^4,$$

$$y_2(t) = -ht^2 - \frac{16ht^{\frac{5}{2}}}{15\pi^{\frac{1}{2}}} - \frac{1}{12}ht^4 - h^2t^2 - \frac{1}{6}h^2t^4 - \frac{1}{360}h^2t^6$$

$$- \frac{5081767996463981}{132996926495784960}h^2t^{\frac{9}{2}}, -\frac{64h^2t^{\frac{9}{2}}}{945\pi^{\frac{1}{2}}} - \frac{5081767996463981}{8444249301319680}h^2t^{\frac{5}{2}}$$

$$- \frac{16h^2t^{\frac{5}{2}}}{15\pi^{\frac{1}{2}}} - \frac{5081767996463981}{27021597764222976}\frac{h^2t^{\frac{5}{2}}\pi^{\frac{1}{2}}}{\frac{1}{t}^{\frac{1}{2}}}$$

$$\vdots$$

In this case, for given order of approximation n, the optimal value of h is given by the minimum of Δ_n, corresponding to a nonlinear algebraic equation

$$\frac{\partial \Delta_m}{\partial h} = 0.$$

Thus, the optimal value of h is determined by the minimum of Δ_8, corresponding to the nonlinear algebraic equation $\frac{d\Delta_8}{dh} = 0$. According to Table 1, Δ_8 has its minimum value at -0.74689.

Table 1.

Optimal value of h at different order of approximation.

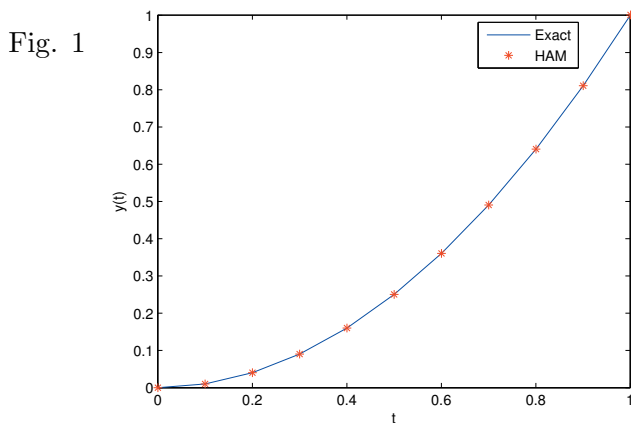| m	Optimal value of h	Minimum value of Δ_m
3	-0.6113	9.0419e-004
4	-0.6765	1.3410e-005
5	-0.7178	2.2031e-007
6	-0.7464	3.9224e-009
7	-0.74661	1.1865e-011
8	-0.74689	1.1395e-012

Fig. 1

Fig. 1. Comparison between the 8-term HAM solution and the exact solution.

Table 2.

The comparison of the results of the HAM($h = -0.74689$, $h = -1$) and exact solution

t	HAM ($h = -0.74689$)	$\left|t^2 - y_{10}\right|$	HPM	$\left|t^2 - y_{10}\right|$
0.1	0.01	2.3265e-013	0.01	4.0215e-011
0.2	0.04	1.4385e-011	0.04	5.2739e-009
0.3	0.09	6.1890e-011	0.09	9.2959e-008
0.4	0.16	2.2736e-011	0.16	7.2230e-007
0.5	0.25	1.3680e-010	0.25	3.5881e-006
0.6	0.36	3.5678e-011	0.35999	1.3445e-005
0.7	0.49	2.6188e-010	0.48996	4.1497e-005
0.8	0.64	4.3416e-010	0.63989	1.1118e-004
0.9	0.81	1.0816e-010	0.80973	2.6745e-004
1.0	1.00	7.9195e-010	0.99941	5.9110e-004

Example 3.2.   Consider the Bagley-Torvik equation

$$[D^2 - 2D + D^{\frac{1}{2}} + D^0]y(t) = 6t - 6t^2 + \frac{16}{5\sqrt{\pi}}t^{\frac{5}{2}} + t^3, \quad y(0) = y'(0) = 0. \quad (26)$$

The exact solutions of Eq. (12) is

$$y(t) = t^3, \tag{27}$$

For application of homotopy analysis method, it in convenient to choose

$$y_0(t) = 0, \tag{28}$$

as the initial approximate of Eq. (26). By taking $\beta = 2$, we choose the linear operator

$$L[\phi(t;q)] = \frac{d^2\phi(t;q)}{dt^2},\tag{29}$$

with the property $L(c_1 t + c_2) = 0$ where c_1 and c_2 are constants of integrations. Furthermore, we define nonlinear operators as

$$N_\alpha[\phi(t;q)] = \frac{d^2\phi(t;q)}{dt^2} - 2\frac{d\phi(t;q)}{dt} + D^{\frac{1}{2}}\phi(t;q) + \phi(t;q) - 6t + 6t^2 - \frac{16}{5\sqrt{\pi}}t^{\frac{5}{2}} - t^3,$$

We construct the zeroth-order and the mth-order deformation equations where

$$R_m(\vec{y}_{m-1}) = \frac{d^2 y_{m-1}}{dt^2} - 2\frac{dy_{m-1}}{dt} + D^{\frac{1}{2}}y_{m-1} + y_{m-1} - 6t + 6t^2$$
$$+(1 - \chi_m)(-\frac{16}{5\sqrt{\pi}}t^{\frac{5}{2}} - t^3),$$

We now successively obtain the solution to each high order deformation equations:

$$y_m(t) = \chi_m y_{m-1}(t) + L^{-1}\left[R_m(\vec{y}_{m-1})\right], m \geqslant 1,\tag{30}$$

We now successively obtain

$$y_1(t) = -ht^2 - \frac{16ht^{\frac{5}{2}}}{15\pi^{\frac{1}{2}}} - \frac{1}{12}ht^4,$$

$$y_2(t) = -ht^2 - \frac{16ht^{\frac{5}{2}}}{15\pi^{\frac{1}{2}}} - \frac{1}{12}ht^4 - h^2 t^2 - \frac{1}{6}h^2 t^4 - \frac{1}{360}h^2 t^6$$
$$- \frac{5081767996463981}{132996926495784960}h^2 t^{\frac{9}{2}}, -\frac{64h^2 t^{\frac{9}{2}}}{945\pi^{\frac{1}{2}}} - \frac{5081767996463981}{8444249301319680}h^2 t^{\frac{5}{2}}$$
$$- \frac{16h^2 t^{\frac{5}{2}}}{15\pi^{\frac{1}{2}}} - \frac{5081767996463981}{27021597764222976}\frac{h^2 t^{\frac{5}{2}}\pi^{\frac{1}{2}}}{\frac{1}{t}^{\frac{1}{2}}}$$

$$\vdots$$

In this case, for given order of approximation n, the optimal value of h is given by the minimum of Δ_n, corresponding to a nonlinear algebraic equation

$$\frac{\partial \Delta_m}{\partial h} = 0.$$

Thus the optimal value of h is determined by the minimum of Δ_8, corresponding to the nonlinear algebraic equation $\frac{d\Delta_8}{dh} = 0$. According to Table 3, Δ_8 has its minimum value at $h = -1.1014$. Fig. 2 discussed the numerical comparison between the 8th-order HAM with the exact solution. The absolute error of the 8th-order HAM and analytic solution with $h = -1.1014, h = -1$ as shown in Table 4.

Table 3.

Optimal value of h at different order of approximation.

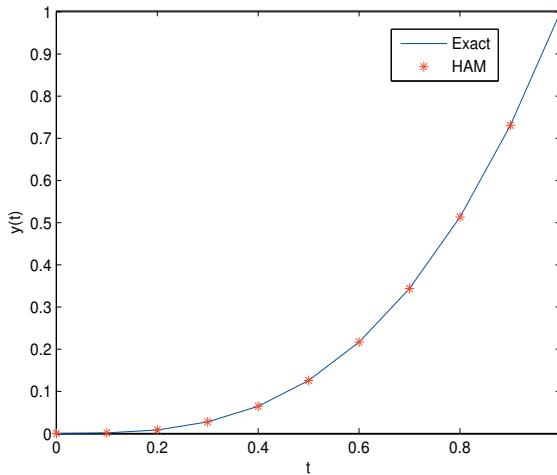| m	Optimal value of h	Minimum value of Δ_m
3	-1.2580	2.3474e-004
4	-1.1814	5.0780e-006
5	-1.2349	4.0954e-007
6	-1.1761	3.2435e-009
7	-1.1406	2.5118e-011
8	-1.1014	2.0748e-012

Fig.

Fig. 2. Comparison between the 8-term HAM solution and the exact solution.

Table 4.

The comparison of the results of the HAM($h = -1.1014$, $h = -1$) and exact solution

t	HAM ($h = -1.1014$)	$\left|t^3 - y_8\right|$	HPM	$\left|t^3 - y_8\right|$
0.1	0.001	3.5997e-013	0.001	2.5435e-016
0.2	0.008	1.5021e-012	0.008	4.2573e-013
0.3	0.027	9.6606e-012	0.027	3.1145e-011
0.4	0.064	2.9515e-011	0.064	6.3416e-010
0.5	0.125	4.1907e-011	0.125	6.4131e-009
0.6	0.216	3.3163e-010	0.216	4.1671e-008
0.7	0.343	5.0078e-010	0.343	1.9955e-007
0.8	0.512	2.0880e-009	0.512	7.6424e-007
0.9	0.729	8.0036e-009	0.729	2.4674e-006
1.0	1.0	4.4094e-009	0.99999	6.9604e-006

Remark 2. *Now We will compare numerical solution semi-differential equations of order 4, by collocation spline method based on Lagrange interpolation as Showed Rawashdeh in [22]. It is clear that the main disadvantage of the collocation spline method is its complex and difficult procedure. Also Rawashdeh in [22] reported the computed absolute error (error between exact and approximate value) with $N = 100$ (N is the division number of the given interval) for Examples 3.1 and 3.2, see Table 5. In the studies by Rawashdeh, much time was spent and boring operations were done by collocation spline method based on Lagrange interpolation to get approximate solutions.*

Table 5.

Numerical solution of $y(t)$ in Examples 3.1 and 3.2.

$t_n(T = 5)$	$h = \frac{T}{N}$	Absolute error of Example 3.1	Absolute error of Example 3.2
0.1	100	0.2e-010	0.18e-010
1	100	0.44e-007	0.534e-009
2.5	100	0.25e-005	0.25e-006
4	100	0.15e-006	0.88e-005
5	100	0.22e-004	0.39e-004

5.   Conclusion

In this paper, we have successfully developed an optimal homotopy-analysis approach for solving Semi-Differential Equations of nth Order. An optimal value of the convergence control parameter h has also been given by means of the exact square residual error integrated in the whole region of interest Ω. We show that HAM provides accurate numerical solution for Semi-Differential Equations of nth Order in comparison with the homotopy perturbation method. The results show that HAM is a powerful mathematical tool and HPM is a special case of HAM.

Matlab has been used for computations in this paper.

References

[1] S. Abbasbandy, Soliton solutions for the 5th-order KdV equation with the homotopy analysis method, *Nonlinear Dyn.,* 51 (2008), 83-87.

[2] L. Blank, Numerical treatment of differential equations of fractional order, *Tech. Rep. Numerical Analysis Report,* 287, The University of Manchester, Manchester, UK., 1996.

[3] M. Caputo, Linear models of dissipation whose Q is almost frequency independent, Part II. *J. Roy Austral Soc.,* 13 (1967), 529-539.

[4] K. Diethelm and N. J. Ford, Numerical solution of the Bagley-Torvik equation, *BIT.,* 42-3 (2002), 490-507.

[5] L. Gaul, P. Klein, and S. Kemple, Damping and description involving fractional operators, *Mechanical Systems and Signal Processing,* 5(2) (1991), 81-88.

[6] W. G. Glockle and T. F. Nonnenmacher, A fractional calculus approach to self-similar protein dynamics, *Biophysical Journal,* 68(1) (1995), 46-53.

[7] T. Hayat, M. Khan, and M. Ayub, On non-linear flows with slip boundary condition, *ZAMP.,* 56 (2005), 1012-1029.

[8] J. H. He, Homotopy perturbation technique, *Comput. Meth. Appl. Eng.,* 178 (1999), 257-262.

[9] S. J. Liao, *The proposed homotopy analysis technique for the solution of nonlinear problems,* Ph.D. Thesis, 1992, Shanghai Jiao Tong University.

[10] S. J. Liao, Beyond perturbation: *introduction to the homotopy analysis method,* 2003, CRC Press, Boca Raton: Chapman & Hall.

[11] S. J. Liao, On the homotopy anaylsis method for nonlinear problems, *Appl. Math. Comput.,* 147 (2004), 499-513.

[12] S. J. Liao, Comparison between the homotopy analysis method and homotopy perturbation method, *Appl. Math. Comput.,* 169 (2005), 1186-1194.

[13] S. J. Liao, Homotopy Analysis Method: A New Analytical Technique for Nonlinear Problems, *Journal of Commun Nonlinear Sci Numer Simul.,* 2-2 (1997), 95-100.

[14] A. Mastroberardino, Homotopy analysis method applied to electrohydrodynamic flow, *Commun Nonlinear Sci Numer Simulat,* 16 (2011), 2730-2736.

[15] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations,* Wiley, New York, 1993.

[16] S. Momani and N. T. Shawagfeh, Decomposition method for solving fractional Riccati differential equations, *Appl. Math. Comput.,* 182 (2006), 1083-1092.

[17] S. S. Motsa and P. Sibanda, An efficient numerical method for solving Falkner-Skan boundary layer flows, *International Journal for Numerical Methods in Fluids,* (2011), doi: 10.1002/fld.2570.

[18] Z. Odibat and S. Momani, Modified homotopy perturbation method: Application to quadratic Riccati differential equation of fractional order, Chaos, *Solitons and Fractals,* 36(1) (2008), 167-174.

[19] Z. Odibat and S. Momani, Application of variation iteration method to nonlinear differential equations of fractional order, *Int. J . Nonlin. Sci. Numer. Simulat.,* 1(7) (2006), 15-27.

[20] K. B. Oldham and J. Spanier, *The Fractional calculus,* New York: Academic Press, 1974.

[21] I. Podlubny, *Fractional Differential Equations,* Academic Press, San Diego, 1999.

[22] E. A. Rawashdeh, Numerical solution of semi-differential equations by collocation method, *Applied Mathematics and Computation,* 174(2) (2006), 869-876.

[23] S. G. Samko, A. A. Kilbas, and O.I. Marichev, Fractional Integrals and Derivatives: *Theory and Applications, Gordon and Breach,* Yverdon, 1993.

[24] B. J. West, M. Bolognab, and P. Grigolini, *Physics of Fractal Operators,* Springer, New York, 2003.

[25] K. Yabushita, M. Yamashita, and K. Tsuboi. An analytic solution of projectile motion with the quadratic resistance law using the homotopy analysis method, *J Phys A.,* 40 (2007), 8403-8416.

[26] M. Zurigat, S. Momani, and A. Alawneh, Analytical approximate solutions of systems of fractional algebraic-differential equations by homotopy analysis method, *Computers and Mathematics with Applications,* 59 (2010), 1227-1235.

[27] M. Zurigat, S. Momani, and H. Xu, A reliable algorithm of homotopy analysis method for solving nonlinear fractional differential equations, *Applied Mathematical Modelling,* 34 (2010), 593-600.

[28] S. P. Zhu, An exact and explicit solution for the valuation of American put options, *Quantitative Finance.,* 6 (2006), 229-242.

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