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# On the Epsilon Hypercyclicity of a Pair of Operators

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**Abstract.** In this paper we prove that if a pair of operators is $\epsilon$-hypercyclic for all $\epsilon > 0$, then it is topologically transitive.

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## 1. Introduction

From now on, let $T_1$, $T_2$ be commutative bounded linear operators on an infinite dimensional Banach space $X$ and consider the pair $T = (T_1, T_2)$.

**Definition 1.1.** *Put $\mathcal{F} = \{{T_1}^m {T_2}^n : m, n \geqslant 0\}$. For $x \in X$, the orbit of $x$ under $T$ is the set $Orb(T, x) = \{Sx : S \in \mathcal{F}\}$. The vector $x$ is called a hypercyclic vector for the pair $T$ if $Orb(T, x)$ is dense in $X$.*

**Definition 1.2.** *We say that the pair $T = (T_1, T_2)$ is topologically transitive if for every nonempty open subsets $U$ and $V$ of $X$ there exists $S \in \mathcal{F}$ such that $S(U) \cap V \neq \emptyset$.*

**Definition 1.3.** *Let $\epsilon \in (0, 1)$ and $x \in X$. If for every non-zero vector $y \in X$, there exist integers $m$, $n$ such that*

$$\| T_1^m T_2^n x - y \| < \epsilon \| y \|,$$

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*then the vector $x$ is called $\epsilon$-hypercyclic for the pair $T = (T_1, T_2)$. A pair of operators is $\epsilon$-hypercyclic if it admits an $\epsilon$-hypercyclic vector.*
For some sources on these topics see [1–18].

## 2. Main Results

In this section we prove that if a pair is $\epsilon$-hypercyclic for all $\epsilon > 0$, then it is topologically transitive and consequently it is hypercyclic. We will extend Theorem 1.4 in [1] for a pair of operators and we will use the idea of it's proof. We will denote $\mathbb{N} \cup \{0\}$ by $\mathbb{N}_0$.

**Theorem 2.1.** *Let $X$ be a separable infinite dimensional Banach space and $T = (T_1, T_2)$ be the pair of operators $T_1$ and $T_2$. If for every $\epsilon > 0$, $T$ is $\epsilon$-hypercyclic, then $T$ is topologically transitive.*

**Proof.** Suppose that $U$ and $V$ are nonempty open subsets of $X$. Let $u \in U$ and $v \in V$ be two nonzero vectors, and consider

$$0 < \delta < min\{\| u \|, \| v \|\}$$

small enough such that $B(u, \delta) \subset U$ and $B(v, \delta) \subset V$. Choose

$$\epsilon < \delta/(6max\{\| u \|, \| v \|\}),$$

and let $x \in X$ be an $\epsilon$-hypercyclic vector for $T$. This implies that there exist nonnegative integers $m_0$ and $n_0$ such that

$$\| T_1^{m_0} T_2^{n_0} x - u \| < \epsilon \| u \| < \delta.$$

Hence

$$T_1^{m_0} T_2^{n_0} x \in B(u, \delta) \subset U.$$

We want to show that

$$V \cap \{{T_1}^m {T_2}^n x : m, n \in \mathbb{N}_0\}$$

contain infinitely many elements. Suppose on the contrary that it contains only the elements ${T_1}^{n_i^{(1)}}{T_2}^{n_i^{(2)}}x$ for $i = 1, ..., k$. As we saw earlier, for each $v' \in B(v, \frac{2\delta}{3})$ there exist integers $m(v')$ and $n(v')$ which satisfies

$$\| T_1^{m(v')}T_2^{n(v')}x - v' \| \leqslant \epsilon \| v' \| \leqslant 2\epsilon \| v \| < \frac{\delta}{3}.$$

Hence

$${T_1}^{m(v')}{T_2}^{n(v')}x \in \{{T_1}^{n_i^{(1)}}{T_2}^{n_i^{(2)}}x : i = 1, ..., k\},$$

because

$$\| T_1^{m(v')}T_2^{n(v')}x - v \| \leqslant \| T_1^{m(v')}T_2^{n(v')}x - v' \| + \| v' - v \| < \delta.$$

Therefore

$$B(v, \frac{2\delta}{3}) \subset \bigcup_{i=1}^{k} B({T_1}^{n_i^{(1)}}{T_2}^{n_i^{(2)}}x, \frac{\delta}{3}),$$

that is a contradiction since $X$ is infinite dimensional. Thus indeed the set

$$V \cap \{{T_1}^m{T_2}^n x : m, n \in \mathbb{N}_0\}$$

contain infinitely many elements and so the set

$$B(v, \delta) \cap \{{T_1}^m{T_2}^n x : m, n \in \mathbb{N}_0\}$$

has infinite elements. In particular, there exist $m$, $n \in \mathbb{N}_0$ satisfying $m \geqslant m_0$ and $n \geqslant n_0$ such that $T_1^m T_2^n x \in V$. Thus we get

$$T_1^{m-m_0}T_2^{n-n_0}T_1^{m_0}T_2^{n_0}x = T_1^m T_2^n x,$$

which belongs to

$$T_1^{m-m_0}T_2^{n-n_0}(U) \cap V.$$

This completes the proof. □

In the proof of the following lemma, we use a method of the proof of Theorem 1.2 in [5] to extend the results for tuples . We will use HC($T$) for the collection of hypercyclic vectors for the pair of operator $T$.

**Lemma 2.2.** *Let $X$ be a separable infinite dimensional Banach space and $T = (T_1, T_2)$ be the pair of operators $T_1$ and $T_2$. Then $T$ is topologically transitive if and only if HC($T$) is dense in $X$.*

**Proof.** Fix an enumeration $\{B_n : n \in \mathbb{N}\}$ of the open balls in $X$ with rational radii, and centers in a countable dense subset of $X$. By the continuity of the operators $T_1$ and $T_2$, the sets

$$G_k = \bigcup \{T_1^{-m} T_2^{-n}(B_k) : m, n \in \mathbb{N}_0\}$$

are open. Clearly HC($T$) is equal to

$$\bigcap \{G_k : k \in \mathbb{N}\}.$$

Now let $T$ be topologically transitive and let $W$ be an arbitrary nonempty open set in $X$. Then for all $k \in \mathbb{N}$, there exist $m(k)$ and $n(k)$ in $\mathbb{N}$ such that

$$T_1^{m(k)} T_2^{n(k)} W \cap B_k \neq \emptyset$$

which implies that $W \cap G_k \neq \emptyset$ for all $k$. Thus each $G_k$ is dense in $X$ and so by the Bair Category Theorem HC($T$) is also dense in $X$. Conversely, if HC($T$) is dense in $X$, then each set $G_k$ is so. This implies clearly that $T$ is topologically transitive.

**Corollary 2.3.** *Let $X$ be a separable infinite dimensional Banach space and $T = (T_1, T_2)$ be the pair of operators $T_1$ and $T_2$. If for every $\epsilon > 0$, $T$ is $\epsilon$-hypercyclic, then $T$ is hypercyclic.*

**Proof.** If for every $\epsilon > 0$, $T$ is $\epsilon$-hypercyclic then by Theorem 2.1, $T$ is topologically transitive. Now, by the Lemma 2.2, HC($T$) is dense in $X$ and this implies clearly that the pair $T$ is hypercyclic. □

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