Some Common Fixed Point of Two Families of Weakly Compatible Self-Maps on Quasi-Metric Spaces

M. Avar, K. Jahedi and M. J. Mehdipour

Abstract. In this paper, we find the conditions guaranteeing the existence of a unique common fixed point of two families of weakly compatible self-maps on quasi-metric spaces.

Keywords: common fixed point, weakly compatible, quasi metric.

1 Introduction

Through out this paper, ρ denotes a quasi-metric on a nonempty set X; that is, a real valued function ρ on $X \times X$ such that for every $x, y, z \in X$,

(i) $\rho(x, y) \geq 0$;

(ii) $x = y$ if and only if $\rho(x, y) = \rho(y, x) = 0$;

(iii) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

A sequence $\{x_n\}$ in a quasi-metric space (X, ρ) is called $\rho-$*convergence* at a point $x \in X$ if for every $\varepsilon > 0$ there is an integer n_0 such that $n \geq n_0$ implies that $\rho(x, x_n) < \varepsilon$. It is said to be $\rho-$*Cauchy* if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\rho(x_n, x_m) < \varepsilon$ if $n_0 \leq n \leq m$. A quasi-metric space (X, ρ) is called $\rho-$*complete* if every $\rho-$Cauchy sequence in X is $\rho-$convergent. A point $x_0 \in X$ is called a *limit point* of set $E \subseteq X$ if there exists a sequence $\{x_n\}$ in E such that

$$\lim_{n\to\infty} \rho(x_0, x_n) = 0.$$

We denote by E' the set of all limit points of E in X, and set

$$\overline{E} = E \cup E'.$$

A self-mapping A on a quasi-metric space (X, ρ) is called $\rho-$*continuous* at $x_0 \in X$ if

$$\lim_{n\to\infty} \rho(A(x_0), A(x_n)) = \lim_{n\to\infty} \rho(A(x_n), A(x_0)) = 0,$$

when for any sequence $\{x_n\}$ in X

$$\lim_{n\to\infty} \rho(x_0, x_n) = \lim_{n\to\infty} \rho(x_n, x_0) = 0.$$

Also, self-mappings A and S of a quasi-metric space (X, ρ) is said to be $\rho-$*compatible* if

$$\lim_{n\to\infty} \rho(SAx_n, ASx_n) = \lim_{n\to\infty} \rho(ASx_n, SAx_n) = 0,$$

whenever $\{x_n\}$is a sequence in X such that

$$\lim_{n\to\infty}\rho(x_0, Ax_n) = \lim_{n\to\infty}\rho(x_0, Sx_n) = 0$$

for some $x_0 \in X$. In particular, the pair (A, S) is said to be *weakly compatible* if $Ax = Sx$ for some $x \in X$, then $ASx = SAx$.

Schellekens [18] introduced the concept of quasi-metric spaces as a generalization of the concept of metric spaces. Quasi-metric spaces have some applications in the study of computer science; for example see [7, 9, 17] for the applications of this theory to the asymptotic complexity analysis of Divide and Conquer algorithms. Some other authors extended the fixed point theorems in metric spaces to quasi-metric spaces [4, 6, 11, 16, 10, 15]. For instance, Hick [10] proved if there exists $0 \le \gamma < 1$ such that

$$\rho(Ax, Ay) \le \gamma \max\{\rho(x, y), \rho(x, Ax), \rho(y, Ay), 1/2[\rho(x, Ay) + \rho(y, Ax)]\},$$

then A has a fixed point. He also proved a fixed point theorem for self-mappings A of a $\rho-$complete quasi-metric (X, ρ) which satisfying the following condition.

$$\rho(y, Ay) \le \phi(y) - \phi(Ay),$$

where ϕ is a positive function on X. Ciric [4] generalized this result by proving the following common fixed point theorem.

Theorem 1.1 *Suppose $A, S : X \to X$ and $\phi : X \to [0, \infty)$, where X is a complete quasi-metric space. Let there is $x_0 \in X$ such that*

$$\rho(y, Ay) + \rho(Ay, SAy) \le \phi(y) - \phi(SAy)$$

for all $y \in \{x_0, Ax_0, SAx_0, A(SA)x_0, ..., (SA)^n x_0, A(SA)^n x_0, ...\}$. If $G_1(x) = \rho(x, Ax)$ and $G_2(x) = \rho(x, Sx)$ are $(S, A)-$orbitally weak lower semi-continuous relative to x_0, then $Ap = p = Sp$ for some $p \in X$.

Jungck [12] and Jungck and Rhoades [13] introduced the notions of compatible and weakly compatible mappings on metric spaces. These notions are a generalization of the notion of commuting self-mappings. Using concepts of compatible and weakly compatible mappings on metric spaces, Singh and Jain [19] proved the following result.

Theorem 1.2 *Let P_i and Q_j be self-mappings of a complete metric space $(\mathcal{X}, d)$ for $i = 1, ..., 4$ and $j = 0, 1$. If*

(i) $Q_0(\mathcal{X}) \subseteq P_1P_3(\mathcal{X}), Q_1(\mathcal{X}) \subseteq P_2P_4(\mathcal{X})$.

(ii) $P_2P_4 = P_4P_2, P_1P_3 = P_3P_1, Q_0P_4 = P_4Q_0, Q_1P_3 = P_3Q_1$.

(iii) *for every $x, y \in \mathcal{X}$ and for some $0 < \gamma < 1$*

$$\begin{aligned} d(Q_0x, Q_1y) \le\ & \gamma \max\{d(Q_0x, P_2P_4x), d(Q_1y, P_1P_3y), d(P_2P_4x, P_1P_3y), \\ & 1/2[d(Q_0x, P_1P_3y) + d(Q_1y, P_2P_4x)]\}. \end{aligned} \quad (1)$$

(iv) *the pair (Q_0, P_2P_4) is compatible and the pair (Q_1, P_1P_3) is weakly compatible.*

(v) *either P_2P_4 or Q_0 is continuous.*

Then P_i and Q_j have a unique common fixed point for $i = 1, ..., 4$ and $j = 0, 1$.

Ciric et al. [5] obtained an extension of Theorem 1.2. In fact, they proved the theorem for a countable family of compatible self-mappings of a complete metric space by replacing relation (1) by

$$\begin{aligned} d(Q_0x, Q_1y) \;\leq\; & \max\{\varphi(d(Q_0x, \pi_{i=1}^n P_{2i}x)), \varphi(d(Q_1y, \pi_{i=1}^n P_{2i-1}y)), \\ & \varphi(d(\pi_{i=1}^n P_{2i}x, \pi_{i=1}^n P_{2i-1}y)), \varphi(1/2[d(Q_0x, \pi_{i=1}^n P_{2i-1}y)) \\ + \; & \varphi(d(Q_1y, \pi_{i=1}^n P_{2i}x)])\}, \end{aligned} \tag{2}$$

where $\pi_{i=\ell}^m P_i = P_\ell P_{\ell+1}...P_m$ and φ is an element of Φ, the set of continuous non-decreasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t > 0$.

In this paper, we investigate the question and prove an analogue of Ciric et al. [5] for quasi-metric spaces.

2 Main Results

We commence this section with the main result of the paper.

Theorem 2.1 *Let $S_1, S_2, \ldots, S_{2n}, A_0$ and A_1 be self-mappings of a $\rho-$complete quasi-metric space (X, d) such that*

(i) $A_0(X) \subseteq \pi_{i=1}^n S_{2i-1}(X)$ and $A_1(X) \subseteq \pi_{i=1}^n S_{2i}(X)$;

(ii) $\pi_{i=1}^\ell S_{2i}\pi_{i=\ell+1}^n S_{2i} = \pi_{i=\ell+1}^n S_{2i}\pi_{i=1}^\ell S_{2i}$ *for* $\ell = 1, ..., n-1$;

(iii) $A_0(\pi_{i=\ell}^n S_{2i}) = (\pi_{i=\ell}^n S_{2i})A_0$ *for* $\ell = 2, ..., n$;

(iv) $\pi_{i=1}^\ell S_{2i-1}\pi_{i=\ell+1}^n S_{2i-1} = \pi_{i=\ell+1}^n S_{2i-1}\pi_{i=1}^\ell S_{2i-1}$ *for* $\ell = 1, ..., n-1$;

(v) $A_1(\pi_{i=\ell}^n S_{2i-1}) = (\pi_{i=\ell}^n S_{2i-1})A_1$ *for* $\ell = 2, ..., n$;

(vi) $\pi_{i=1}^n S_{2i}$ *or* A_0 *is* $\rho-$*continuous;*

(vii) *the pair* $(A_0, \pi_{i=1}^n S_{2i})$ *is* $\rho-$*compatible and pair* $(A_1, \pi_{i=1}^n S_{2i-1})$ *is weakly compatible;*

(viii) *there exists* $\varphi \in \Phi$ *such that for every* $u, v \in X$, $x \in \overline{\pi_{i=1}^n S_{2i-1}(X)}$ *and* $y \in \overline{\pi_{i=1}^n S_{2i}(X)}$,

$$\begin{aligned} \rho(A_0u, y) + \rho(A_1v, x) \;\leq\; & \max\{\varphi(\rho(x, A_0u)), \varphi(\rho(y, A_1v)), \varphi(\rho(x, y)), \\ & \varphi(1/2[\rho(x, \pi_{i=1}^n S_{2i-1}v)) + \rho(y, \pi_{i=1}^n S_{2i}u)])\}. \end{aligned} \tag{3}$$

Then $S_1, S_2, \ldots, S_{2n}, A_0, A_1$ have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Choose $x_1, x_2 \in X$ such that

$$A_0x_0 = \pi_{i=1}^n S_{2i-1}x_1 := y_0 \quad \text{and} \quad A_1x_1 = \pi_{i=1}^n S_{2i}x_2 := y_1.$$

For any $k \in \mathbb{N}$, set

$$A_0x_{2k} = \pi_{i=1}^n S_{2i-1}x_{2k+1} := y_{2k} \quad \text{and} \quad A_1x_{2k+1} = \pi_{i=1}^n S_{2i}x_{2k+2} := y_{2k+1}.$$

From properties of φ and condition (viii) we see that

$$\begin{aligned}
\rho(y_{2k}, y_{2k+1}) &+ \rho(y_{2k+1}, y_{2k}) \\
&\leq \max\{\varphi(\rho(A_0x_{2k}, A_0x_{2k})), \varphi(\rho(A_1x_{2k+1}, A_1x_{2k+1})), \\
&\quad \varphi(\rho(A_0x_{2k}, A_1x_{2k+1})), \varphi(1/2[\rho(A_0x_{2k}, \pi_{i=1}^n S_{2i-1}x_{2k+1}) \\
&+ \rho(A_1x_{2k+1}, \pi_{i=1}^n S_{2i}x_{2k})])\} \\
&= \max\{\varphi(\rho(y_{2k}, y_{2k+1})), \varphi(1/2[\rho(y_{2k}, y_{2k}) + \rho(y_{2k+1}, y_{2k-1})])\} \\
&\leq \max\{\varphi(\rho(y_{2k}, y_{2k+1})), \varphi(1/2[\rho(y_{2k+1}, y_{2k}) + \rho(y_{2k}, y_{2k-1})])\} \\
&\leq \varphi(\max\{\rho(y_{2k}, y_{2k+1}), \rho(y_{2k+1}, y_{2k}), \rho(y_{2k}, y_{2k-1})\}) \\
&\leq \varphi(\rho(y_{2k}, y_{2k-1})).
\end{aligned}$$

This shows that

$$\rho(y_{2k+1}, y_{2k}) \leq \varphi(\rho(y_{2k}, y_{2k-1})) \leq \rho(y_{2k}, y_{2k-1}) \tag{4}$$

and

$$\rho(y_{2k}, y_{2k+1}) \leq \rho(y_{2k}, y_{2k-1}). \tag{5}$$

A similar argument shows that

$$\rho(y_{2k+2}, y_{2k+1}) \leq \varphi(\rho(y_{2k+1}, y_{2k})) \leq \rho(y_{2k+1}, y_{2k}) \tag{6}$$

and

$$\rho(y_{2k+1}, y_{2k+2}) \leq \rho(y_{2k+1}, y_{2k}). \tag{7}$$

By relation (4)–(7), we have

$$0 \leq \rho(y_{n+1}, y_n) \leq \varphi(\rho(y_n, y_{n-1})) \leq \rho(y_n, y_{n-1}) \tag{8}$$

and

$$0 \leq \rho(y_n, y_{n+1}) \leq \rho(y_n, y_{n-1}) \tag{9}$$

for all $n \in \mathbb{N}$. Hence $\{\rho(y_{n+1}, y_n)\}$ is a non-increasing sequence. Thus there exists $\alpha \geq 0$ such that $\lim_{n\to\infty} \rho(y_{n+1}, y_n) = \alpha$. This together with (8) and continuity of ϕ shows that

$$\alpha = \lim_{n\to\infty} \varphi(\rho(y_{n+1}, y_n)) = \varphi(\alpha).$$

So $\alpha = 0$. Thus

$$\lim_{n\to\infty} \rho(y_{n+1}, y_n) = \lim_{n\to\infty} \rho(y_n, y_{n-1}) = 0.$$

From (9) we see that

$$\lim_{n\to\infty} \rho(y_n, y_{n+1}) = 0.$$

Let ε and δ be positive numbers with $\delta < (\varepsilon - \varphi(\varepsilon))/3$. By

$$\lim_{n\to\infty} \rho(y_n, y_{n+1}) = \lim_{n\to\infty} \rho(y_{n+1}, y_n) = 0,$$

choose $N \in \mathbb{N}$ such that $\rho(y_n, y_{n+1}) < \delta$ and $\rho(y_{n+1}, y_n) < \delta$ for all $n \geq N$. If $k, q \in \mathbb{N}$, then by (viii) we have

$$\begin{aligned}
\rho(y_{2q+1}, y_{2k+1}) &\leq \rho(A_1 x_{2q+1}, A_0 x_{2k+2}) + \rho(A_0 x_{2k+2}, y_{2k+1}) \\
&\leq \max\{\varphi(\rho(A_0 x_{2k+2}, A_0 x_{2k+2})), \varphi(\rho(y_{2k+1}, A_1 x_{2q+1})), \\
&\quad \varphi(\rho(A_0 x_{2k+2}, y_{2k+1})), \varphi(1/2[\rho(A_0 x_{2k+2}, \pi_{i=1}^{n} S_{2i-1} x_{2q+1}) \\
&+ \rho(y_{2k+1}, \pi_{i=1}^{n} S_{2i} x_{2k+2})])\} \\
&= \max\{\varphi(\rho(y_{2k+1}, y_{2q+1})), \varphi(\rho(y_{2k+2}, y_{2k+1})), \\
&\quad \varphi(1/2[\rho(y_{2k+2}, y_{2q}) + \rho(y_{2k+1}, y_{2k+1})])\} \qquad (10) \\
&\leq \max\{\varphi(\rho(y_{2k+1}, y_{2q+1})), \varphi(\rho(y_{2k+2}, y_{2k+1})), \\
&\quad \varphi(\rho(y_{2k+2}, y_{2k+1}) + \rho(y_{2k+1}, y_{2q+1}) + \rho(y_{2q+1}, y_{2q})\} \\
&\leq \varphi(\rho(y_{2k+2}, y_{2k+1}) + \rho(y_{2k+1}, y_{2q+1}) + \rho(y_{2q+1}, y_{2q})) \\
&\leq 2\delta + \rho(y_{2k+1}, y_{2q+1}).
\end{aligned}$$

From properties of φ and (viii) with $x = y_{2k}, y = A_1 x_{2q+1}, u = x_{2k}$ and $v = x_{2q+1}$, we infer that

$$\begin{aligned}
\rho(y_{2k}, y_{2q+1}) &\leq \rho(A_0 x_{2k}, A_1 x_{2q+1}) + \rho(A_1 x_{2k+1}, y_{2k}) \\
&\leq \max\{\varphi(\rho(y_{2k}, A_0 x_{2k}), \varphi(\rho(A_1 x_{2q+1}, A_1 x_{2k+1})), \\
&\quad \varphi(\rho(y_{2k}, A_1 x_{2q+1})), \varphi(1/2[\rho(y_{2k}, \pi_{i=1}^{n} S_{2i-1} x_{2k+1}) \\
&+ \rho(A_1 x_{2q+1}, \pi_{i=1}^{n} S_{2i} x_{2k})])\} \\
&= \max\{\varphi(\rho(y_{2q+1}, y_{2k+1})), \varphi(\rho(y_{2k}, y_{2q+1})), \\
&\quad \varphi(1/2[\rho(y_{2k}, y_{2k}) + \rho(y_{2q+1}, y_{2k-1})])\} \\
&\leq \varphi(t_{n,m}),
\end{aligned}$$

where

$$t_{n,m} = \max\{\rho(y_{2q+1}, y_{2k+1}), 1/2(\rho(y_{2q+1}, y_{2k-1})\}.$$

In view of (10), we conclude that

$$\begin{aligned}
t_{n,m} &\leq \max\{\rho(y_{2q+1}, y_{2k+1}), \\
&\quad \max\{\rho(y_{2q+1}, y_{2k+1}), \rho(y_{2k+1}, y_{2k-1})\}\} \\
&= \max\{\rho(y_{2q+1}, y_{2k+1}), \rho(y_{2k+1}, y_{2k-1})\} \\
&\leq \max\{2\delta + \rho(y_{2k+1}, y_{2q+1}), 2\delta\} \\
&= 2\delta + \rho(y_{2k+1}, y_{2q+1}).
\end{aligned}$$

Now, we prove that if

$$\rho(y_n, y_m) < \varphi(\varepsilon) + (\varepsilon - \varphi(\varepsilon))/3 + 2\delta \qquad (11)$$

for any $m \geq n \geq N$, then $t_{n,m} < \varepsilon + 6\delta$. For this end, we consider the following cases.

Case 1. Let $n = 2r$ and $m = 2s$ for some $r, s \in \mathbb{N}$. Then

$$\begin{aligned}\rho(y_{2r+1}, y_{2s+1}) &\leq \rho(y_{2r+1}, y_{2r}) + \rho(y_{2r}, y_{2s}) + \rho(y_{2s}, y_{2s+1}) \\ &\leq \varphi(\varepsilon) + (\varepsilon - \varphi(\varepsilon))/3 + 4\delta.\end{aligned}$$

Hence $t_{n,m} < \varepsilon + 6\delta$.

Case 2. Let $n = 2r$ and $m = 2s + 1$ for some $r, s \in \mathbb{N}$. Then

$$\begin{aligned}\rho(y_{2r+1}, y_{2s+1}) &\leq \rho(y_{2r+1}, y_{2r}) + \rho(y_{2r}, y_{2s+1}) \\ &\leq \varphi(\varepsilon) + (\varepsilon - \varphi(\varepsilon))/3 + 3\delta.\end{aligned}$$

So $t_{n,m} < \varepsilon + 6\delta$.

Case 3. Let $n = 2r + 1$ and $m = 2s$ for some $r, s \in \mathbb{N}$. Then

$$\begin{aligned}\rho(y_{2r+1}, y_{2s+1}) &\leq \rho(y_{2r+1}, y_{2s}) + \rho(y_{2s}, y_{2s+1}) \\ &\leq \varphi(\varepsilon) + (\varepsilon - \varphi(\varepsilon))/3 + 3\delta.\end{aligned}$$

Thus $t_{n,m} < \varepsilon + 6\delta$.

Case 4. Let $n = 2r + 1$ and $m = 2s + 1$ for some $r, s \in \mathbb{N}$. According to (11), we get $t_{n,m} < \varepsilon + 6\delta$.

By a similar argument as given in [5], we can show that the sequence $\{y_n\}$ is $\rho-$Cauchy. Hence from the $\rho-$completeness of X, it follows that there exists $z \in X$ such that $\lim_{n\to\infty} \rho(z, y_n) = 0$. Hence

$$\begin{aligned}\lim_{k\to\infty} \rho(z, A_1x_{2k+1}) &= \lim_{k\to\infty} \rho(z, \pi_{i=1}^n S_{2i-1}x_{2k+1}) \\ &= \lim_{k\to\infty} \rho(z, A_0x_{2k}) = \lim_{k\to\infty} \rho(z, \pi_{i=1}^n S_{2i}x_{2k}) = 0 \qquad (12)\end{aligned}$$

and so

$$\begin{aligned}\lim_{k\to\infty} (\rho(A_0x_{2k}, z) &+ \rho(A_1x_{2k+1}, z)) \\ &\leq \lim_{k\to\infty} (\max\{\varphi(\rho(z, A_0x_{2k})), \varphi(\rho(z, A_1x_{2k+1})), \varphi(\rho(z, z)), \\ &\quad \varphi(1/2[\rho(z, \pi_{i=1}^n S_{2i-1}x_{2k+1}) + \rho(z, \pi_{i=1}^n S_{2i}x_{2k})])\}) \\ &= 0.\end{aligned}$$

Thus

$$\begin{aligned}\lim_{k\to\infty} \rho(A_1x_{2k+1}, z) &= \lim_{k\to\infty} \rho(\pi_{i=1}^n S_{2i-1}x_{2k+1}, z) \\ &= \lim_{k\to\infty} \rho(A_0x_{2k}, z) = \lim_{k\to\infty} \rho(\pi_{i=1}^n S_{2i}x_{2k}, z) \qquad (13) \\ &= 0.\end{aligned}$$

Now, we consider the following cases.

Case I. Let $\pi_{i=1}^{n} S_{2i}$ is $\rho-$continuous. From (12) and (13) we see that

$$\lim_{k\to\infty} \rho(\pi_{i=1}^{n} S_{2i}\pi_{i=1}^{n} S_{2i}x_{2k}, \pi_{i=1}^{n} S_{2i}z) = \lim_{k\to\infty} \rho(\pi_{i=1}^{n} S_{2i}z, \pi_{i=1}^{n} S_{2i}\pi_{i=1}^{n} S_{2i}x_{2k}) = 0$$

and

$$\lim_{k\to\infty} \rho(\pi_{i=1}^{n} S_{2i}A_0x_{2k}, \pi_{i=1}^{n} S_{2i}z) = \lim_{k\to\infty} \rho(\pi_{i=1}^{n} S_{2i}z, \pi_{i=1}^{n} S_{2i}A_0x_{2k}) = 0.$$

Since $(A_0, \pi_{i=1}^{n} S_{2i})$ is $\rho-$compatible, we have

$$\lim_{k\to\infty} \rho(A_0\pi_{i=1}^{n} S_{2i}x_{2k}, \pi_{i=1}^{n} S_{2i}z) = \lim_{k\to\infty} \rho(\pi_{i=1}^{n} S_{2i}z, A_0\pi_{i=1}^{n} S_{2i}x_{2k}) = 0.$$

Step 1. From (viii) with $u = \pi_{i=1}^{n} S_{2i}x_{2k}, v = x_{2k+1}, x = A_0\pi_{i=1}^{n} S_{2i}x_{2k}$ and $y = A_1x_{2k+1}$, we have

$$\begin{aligned}
\rho(A_0\pi_{i=1}^{n} S_{2i}x_{2k}, A_1x_{2k+1}) &+ \rho(A_1x_{2k+1}, A_0\pi_{i=1}^{n} S_{2i}x_{2k}) \\
&\leq \max\{\varphi(\rho(A_0\pi_{i=1}^{n} S_{2i}x_{2k}, A_0\pi_{i=1}^{n} S_{2i}x_{2k})), \\
&\quad \varphi(\rho(A_1x_{2k+1}, A_1x_{2k+1})), \\
&\quad \varphi(\rho(A_0\pi_{i=1}^{n} S_{2i}x_{2k}, A_1x_{2k+1})), \\
&\quad \varphi(1/2[\rho(A_0\pi_{i=1}^{n} S_{2i}x_{2k}, \pi_{i=1}^{n} S_{2i-1}x_{2k+1}) \\
&+ \rho(A_1x_{2k+1}, \pi_{i=1}^{n} S_{2i}\pi_{i=1}^{n} S_{2i}x_{2k})])\}.
\end{aligned}$$

Letting $k \to \infty$, we see that

$$\begin{aligned}
\rho(\pi_{i=1}^{n} S_{2i}z, z) &+ \rho(z, \pi_{i=1}^{n} S_{2i}z) \\
&\leq \max\{\varphi(\rho(\pi_{i=1}^{n} S_{2i}z, z)), \\
&\quad \varphi(1/2[\rho(\pi_{i=1}^{n} S_{2i}z, z) + \rho(z, \pi_{i=1}^{n} S_{2i}z)])\} \\
&\leq \varphi(\max\{\rho(z, \pi_{i=1}^{n} S_{2i}z), \rho(\pi_{i=1}^{n} S_{2i}z, z)\}).
\end{aligned}$$

If

$$\max\{\rho(z, \pi_{i=1}^{n} S_{2i}z), \rho(\pi_{i=1}^{n} S_{2i}z, z)\} = \rho(z, \pi_{i=1}^{n} S_{2i}z), \tag{14}$$

then

$$\rho(z, \pi_{i=1}^{n} S_{2i}z) \leq \rho(\pi_{i=1}^{n} S_{2i}z, z) + \rho(z, \pi_{i=1}^{n} S_{2i}z) \leq \varphi(\rho(z, \pi_{i=1}^{n} S_{2i}z)).$$

So $\rho(z, \pi_{i=1}^{n} S_{2i}z) = 0$. By (14), we have

$$0 \leq \rho(\pi_{i=1}^{n} S_{2i}z, z) \leq \rho(z, \pi_{i=1}^{n} S_{2i}z) = 0.$$

It follows that

$$\rho(\pi_{i=1}^{n} S_{2i}z, z) = \rho(z, \pi_{i=1}^{n} S_{2i}z) = 0.$$

Thus $\pi_{i=1}^{n} S_{2i}z = z$. Similarly, if

$$\max\{\rho(z, \pi_{i=1}^{n} S_{2i}z), \rho(\pi_{i=1}^{n} S_{2i}z, z)\} = \rho(\pi_{i=1}^{n} S_{2i}z, z),$$

then $\pi_{i=1}^{n} S_{2i} z = z$.

Step 2. Put $u = z$, $v = x_{2k+1}$, $x = A_0 z$ and $y = A_1 x_{2k+1}$ in condition (viii). Then

$$\begin{aligned}\rho(A_0 z, A_1 x_{2k+1}) &+ \rho(A_1 x_{2k+1}, A_0 z)\\ &\leq \max\{\varphi(\rho(A_0 z, A_0 z)), \varphi(\rho(A_1 x_{2k+1}, A_1 x_{2k+1})),\\ &\quad \varphi(\rho(A_0 z, A_1 x_{2k+1})), \varphi(1/2[\rho(A_0 z, \pi_{i=1}^{n} S_{2i-1} x_{2k+1})\\ &+ \rho(A_1 x_{2k+1}, \pi_{i=1}^{n} S_{2i} z)])\}.\end{aligned}$$

Letting $k \to \infty$, we obtain

$$\begin{aligned}\rho(A_0 z, z) &+ \rho(z, A_0 z)\\ &\leq \max\{\varphi(\rho(A_0 z, z)), \varphi(1/2[\rho(A_0 z, z) + \rho(z, \pi_{i=1}^{n} S_{2i} z)])\}.\end{aligned}$$

Since $\pi_{i=1}^{n} S_{2i} z = z$ and φ is non-decreasing, it follows that

$$\rho(A_0 z, z) + \rho(z, A_0 z) \leq \varphi(\rho(A_0 z, z)). \tag{15}$$

This implies that $\rho(A_0 z, z) = 0$. From (15) and the fact that $\varphi(0) = 0$ we see that $\rho(z, A_0 z) = 0$. Therefore,

$$A_0 z = \pi_{i=1}^{n} S_{2i} z = z.$$

Step 3. From (viii) with $u = \pi_{i=2}^{n} S_{2i} z, v = x_{2k+1}, x = A_0 \pi_{i=2}^{n} S_{2i} z$ and $y = A_1 x_{2k+1}$, we see that

$$\begin{aligned}\rho(A_0 \pi_{i=2}^{n} S_{2i} z, A_1 x_{2k+1}) &+ \rho(A_1 x_{2k+1}, A_0 \pi_{i=2}^{n} S_{2i} z)\\ &\leq \max\{\varphi(\rho(A_0 \pi_{i=2}^{n} S_{2i} z, A_0 \pi_{i=2}^{n} S_{2i} z)),\\ &\quad \varphi(\rho(A_1 x_{2k+1}, A_1 x_{2k+1})),\\ &\quad \varphi(\rho(A_0 \pi_{i=2}^{n} S_{2i} z, A_1 x_{2k+1})),\\ &\quad \varphi(1/2[\rho(A_0 \pi_{i=2}^{n} S_{2i} z, \pi_{i=1}^{n} S_{2i-1} x_{2k+1})\\ &+ \rho(A_1 x_{2k+1}, \pi_{i=1}^{n} S_{2i} \pi_{i=2}^{n} S_{2i} z)])\}.\end{aligned}$$

Since $A_0 z = z$, by letting $k \to \infty$, we get

$$\begin{aligned}\rho(\pi_{i=2}^{n} S_{2i} z, z) &+ \rho(z, \pi_{i=2}^{n} S_{2i} z)\\ &\leq \max\{\varphi(\rho(\pi_{i=2}^{n} S_{2i} z, z)), \varphi(1/2[\rho(\pi_{i=2}^{n} S_{2i} z, z)\\ &+ \rho(z, \pi_{i=2}^{n} S_{2i} z)])\}\\ &\leq \varphi(\max\{\rho(\pi_{i=2}^{n} S_{2i} z, z), \rho(z, \pi_{i=2}^{n} S_{2i} z)\}).\end{aligned}$$

This shows that $\pi_{i=2}^{n} S_{2i} z = z$. Thus $S_2(\pi_{i=2}^{n} S_{2i} z) = S_2 z$ and so $S_2 z = \pi_{i=1}^{n} S_{2i} z = z$. Continuing this procedure, we obtain $A_0 z = S_{2i} z = z$ for $i = 1, ..., n$.

Step 4. By condition (i), there exists $v \in X$ such that

$$z = A_0 z = \pi_{i=1}^{n} S_{2i-1} v.$$

Putting $u = x_{2k}$, $x = A_0x_{2k}$ and $y = A_1v$ in condition (viii), we have

$$\begin{aligned} \rho(A_0x_{2k}, A_1v) &+ \rho(A_1v, A_0x_{2k}) \\ &\leq \max\{\varphi(\rho(A_0x_{2k}, A_0x_{2k})), \varphi(\rho(A_1v, A_1v)), \\ &\quad \varphi(\rho(A_0x_{2k}, A_1v)), \varphi(1/2[\rho(A_0x_{2k}, \pi_{i=1}^n S_{2i-1}v) \\ &+ \rho(A_1v, \pi_{i=1}^n S_{2i}x_{2k})])\}. \end{aligned}$$

Letting $k \to \infty$, we find

$$\begin{aligned} \rho(z, A_1v) &+ \rho(A_1v, z) \\ &\leq \max\{\varphi(\rho(z, A_1v)), \varphi(1/2[\rho(z, \pi_{i=1}^n S_{2i-1}v) + \rho(A_1v, z)])\} \\ &= \max\{\varphi(\rho(z, A_1v)), \varphi(1/2[\rho(z, z) + \rho(A_1v, z)])\} \\ &\leq \varphi(\max\{\rho(z, A_1v), \rho(A_1v, z)\}). \end{aligned}$$

Hence $A_1v = z$ and therefore

$$\pi_{i=1}^n S_{2i-1}v = A_1v = z.$$

As $(A_1, \pi_{i=1}^n S_{2i-1})$ is weakly compatible, we have

$$\pi_{i=1}^n S_{2i-1}A_1v = A_1\pi_{i=1}^n S_{2i-1}v.$$

Thus $\pi_{i=1}^n S_{2i-1}z = A_1z$.

Step 5. Putting $u = x_{2k}$, $v = z$, $x = A_0x_{2k}$ and $y = A_1z$ in condition (viii), we have

$$\begin{aligned} \rho(A_0x_{2k}, A_1z) &+ \rho(A_1z, A_0x_{2k}) \\ &\leq \max\{\varphi(\rho(A_0x_{2k}, A_0x_{2k})), \varphi(\rho(A_1z, A_1z)), \\ &\quad \varphi(\rho(A_0x_{2k}, A_1z)), \varphi(1/2[\rho(A_0x_{2k}, \pi_{i=1}^n S_{2i-1}z) \\ &+ \rho(A_1z, \pi_{i=1}^n S_{2i}x_{2k})])\}. \end{aligned}$$

Letting $k \to \infty$, we get

$$\begin{aligned} \rho(z, A_1z) &+ \rho(A_1z, z) \\ &\leq \max\{\varphi(\rho(z, A_1z)), \varphi(1/2[\rho(z, A_1z) + \rho(A_1z, z)])\} \\ &\leq \varphi(\max\{\rho(z, A_1z), \rho(A_1z, z)\}). \end{aligned}$$

So $\pi_{i=1}^n S_{2i-1}z = A_1z = z$.

Step 6. Putting $u = x_{2k}, v = \pi_{i=2}^n S_{2i-1}z, x = A_0x_{2k}$ and $y = A_1\pi_{i=2}^n S_{2i-1}z$ in condition (viii), we have

$$\begin{aligned} \rho(A_0x_{2k}, A_1\pi_{i=2}^n S_{2i-1}z) &+ \rho(A_1\pi_{i=2}^n S_{2i-1}z, A_0x_{2k}) \\ &\leq \max\{\varphi(\rho(A_0x_{2k}, A_0x_{2k})), \varphi(\rho(A_1\pi_{i=2}^n S_{2i-1}z, \\ &\quad A_1\pi_{i=2}^n S_{2i-1}z)), \varphi(\rho(A_0x_{2k}, A_1\pi_{i=2}^n S_{2i-1}z)), \\ &\quad \varphi(1/2[\rho(A_0x_{2k}, \pi_{i=1}^n S_{2i-1}\pi_{i=2}^n S_{2i-1}z) \\ &+ \rho(A_1\pi_{i=2}^n S_{2i-1}z, \pi_{i=1}^n S_{2i}x_{2k})])\}. \end{aligned}$$

Letting $k \to \infty$ shows that

$$\begin{aligned}\rho(z, \pi_{i=2}^n S_{2i-1}z) &+ \rho(\pi_{i=2}^n S_{2i-1}z, z) \leq \max\{\varphi(\rho(z, \pi_{i=2}^n S_{2i-1}z)),\\ &\varphi(1/2[\rho(z, \pi_{i=2}^n S_{2i-1}z) + \rho(\pi_{i=2}^n S_{2i-1}z, z)])\}\\ &\leq \varphi(\max\{\rho(z, \pi_{i=2}^n S_{2i-1}z), \rho(\pi_{i=2}^n S_{2i-1}z, z)\}).\end{aligned}$$

So $\pi_{i=2}^n S_{2i-1}z = z$ and hence $p_3 z = z$. Continuing this procedure, we have $A_1 z = S_{2i-1}z$ for $i = 1, ..., n$. Thus $A_0 z = A_1 z = S_i z = z$ for $i = 1, ..., 2n$. That is, z is a common fixed point of $A_0, A_1, S_1, S_2, ..., S_{2n}$.

Case II. Let A_0 be $\rho-$continuous. By (12) and (13),

$$\lim_{k\to\infty} \rho({A_0}^2 x_{2k}, A_0 z) = \lim_{k\to\infty} \rho(A_0 z, {A_0}^2 x_{2k}) = 0.$$

Since $(A_0, \pi_{i=1}^n S_{2i})$ is $\rho-$compatible, we have

$$\lim_{k\to\infty} \rho(\pi_{i=1}^n S_{2i} A_0 x_{2k}, A_0 z) = \lim_{k\to\infty} \rho(A_0 z, \pi_{i=1}^n S_{2i} A_0 x_{2k}) = 0.$$

Step 7. Putting $u = A_0 x_{2k}, v = x_{2k+1}, x = A_0^2 x_{2k}$ and $y = A_1 x_{2k+1}$ in condition (viii), we have

$$\begin{aligned}\rho(A_0^2 x_{2k}, A_1 x_{2k+1}) &+ \rho(A_1 x_{2k+1}, A_0^2 x_{2k})\\ &\leq \max\{\varphi(\rho(A_0^2 x_{2k}, A_0^2 x_{2k})), \varphi(\rho(A_1 x_{2k+1}, A_1 x_{2k+1})),\\ &\quad \varphi(\rho(A_0^2 x_{2k}, A_1 x_{2k+1})), \varphi(1/2[\rho(A_0^2 x_{2k}, \pi_{i=1}^n S_{2i-1} x_{2k+1})\\ &+ \rho(A_1 x_{2k+1}, \pi_{i=1}^n S_{2i} A_0 x_{2k})])\}.\end{aligned}$$

Letting $k \to \infty$ gives that

$$\begin{aligned}\rho(A_0 z, z) &+ \rho(z, A_0 z)\\ &\leq \max\{\varphi(\rho(A_0 z, z)), \varphi(1/2[\rho(A_0 z, z) + \rho(z, A_0 z)])\}\\ &\leq \varphi(\max\{\rho(A_0 z, z), \rho(z, A_0 z)\}).\end{aligned}$$

Hence $A_0 z = z$ and therefore, for every $i = 1, ..., n$

$$A_1 z = S_{2i-1} z = z,$$

by the same argument that was used in Steps 4–6.

Step 8. By condition (i), there exists $u \in X$ such that

$$z = A_1 z = \pi_{i=1}^n S_{2i} u.$$

Putting $v = x_{2k+1}$, $x = A_0 u$ and $y = A_1 x_{2k+1}$ in condition (viii), we have

$$\begin{aligned}\rho(A_0 u, A_1 x_{2k+1}) &+ \rho(A_1 x_{2k+1}, A_0 u)\\ &\leq \max\{\varphi(\rho(A_0 u, A_0 u)), \varphi(\rho(A_1 x_{2k+1}, A_1 x_{2k+1})),\\ &\quad \varphi(\rho(A_0 u, A_1 x_{2k+1})), \varphi(1/2[\rho(A_0 u, \pi_{i=1}^n S_{2i-1} x_{2k+1})\\ &+ \rho(A_1 x_{2k+1}, \pi_{i=1}^n S_{2i} u)])\}.\end{aligned}$$

Letting $k \to \infty$ and using Step 7, we see that

$$\begin{aligned}\rho(A_0u, z) &+ \rho(z, A_0u) \\ &\leq \max\{\varphi(\rho(A_0u, z)), \varphi(1/2[\rho(A_0u, z) + \rho(z, z)])\} \\ &= \varphi(\rho(A_0u, z)).\end{aligned}$$

This implies that

$$A_0u = z = \pi_{i=1}^n S_{2i}u.$$

As $(A_0, \pi_{i=2}^n S_{2i})$ is weakly compatible, we have

$$A_0z = \pi_{i=1}^n S_{2i}z = z.$$

A discussion similar to Step 3 shows that $S_{2i}z = A_0z = z$ for $i = 1, ..., n$. Thus $A_0z = A_1z = S_iz = z$ for $i = 1, ..., 2n$. That is, z is a common fixed point of $A_0, A_1, S_1, S_2, ..., S_{2n}$.

To prove the uniqueness theorem, let w be a common fixed point of $A_0, A_1, S_1, S_2, ..., S_{2n}$. Hence

$$A_0w = A_1w = S_iw = w$$

for $i = 1, ..., 2n$. Putting $u = z, v = w, x = A_0z$ and $y = A_1w$ in condition (viii), we have

$$\begin{aligned}\rho(A_0z, A_1w) &+ \rho(A_1w, A_0z) \\ &\leq \max\{\varphi(\rho(A_0z, A_0z)), \varphi(\rho(A_1w, A_1w)), \varphi(\rho(A_0z, A_1w)), \\ &\quad \varphi(1/2[\rho(A_0z, \pi_{i=1}^n S_{2i-1}w) + \rho(A_1w, \pi_{i=1}^n S_{2i}z)])\} \\ &\leq \varphi(\max\{\rho(z, w), \rho(w, z)\}).\end{aligned}$$

Therefore, $z = w$. That is, z is a unique common fixed point of the mappings. □

We conclude the paper with the following result.

Proposition 2.2 *Let $\{S_{2i}\}_{i=1}^n$ and $\{T_\alpha\}_{\alpha\in J}$ be two families of self-mappings of a $\rho-$complete quasi-metric space (X, d). If there exists $\beta \in J$ such that*

(i) $T_\beta(X) \subseteq \pi_{i=1}^n S_{2i-1}(X)$ and $T_\alpha(X) \subseteq \pi_{i=1}^n S_{2i}(X)$ *for all* $\alpha \in J$.

(ii) $\pi_{i=1}^\ell S_{2i}\pi_{i=\ell+1}^n S_{2i} = \pi_{i=\ell+1}^n S_{2i}\pi_{i=1}^\ell S_{2i}$ *for* $\ell = 1, ..., n-1$;

(iii) $T_\beta(\pi_{i=\ell}^n S_{2i}) = (\pi_{i=\ell}^n S_{2i})T_\beta$ *for* $\ell = 2, ..., n$;

(iv) $\pi_{i=1}^\ell S_{2i-1}\pi_{i=\ell+1}^n S_{2i-1} = \pi_{i=\ell+1}^n S_{2i-1}\pi_{i=1}^\ell S_{2i-1}$ *for* $\ell = 1, ..., n-1$;

(v) $T_\alpha(\pi_{i=\ell}^n S_{2i-1}) = (\pi_{i=\ell}^n S_{2i-1})T_\alpha$ *for* $\ell = 2, ..., n$;

(vi) $\pi_{i=1}^n S_{2i}$ *or* T_β *is* $\rho-$*continuous;*

(vii) *the pair* $(T_\beta, \pi_{i=1}^n S_{2i})$ *is* $\rho-$*compatible and pair* $(T_\alpha, \pi_{i=1}^n S_{2i-1})$ *is weakly compatible;*

(viii) *there exists* $\varphi \in \Phi$ *such that for every* $u, v \in X$, $x \in \overline{\pi_{i=1}^n S_{2i-1}(X)}, y \in \overline{\pi_{i=1}^n S_{2i}(X)}$ *and* $\alpha \in J$

$$\begin{aligned}\rho(T_\beta u, y) + \rho(T_\alpha v, x) \leq\ & \max\{\varphi(\rho(x, T_\beta u)), \varphi(\rho(y, T_\alpha v)), \varphi(\rho(x, y)), \\ & \varphi(1/2[\rho(x, \pi_{i=1}^n S_{2i-1}v)) + \rho(y, \pi_{i=1}^n S_{2i}u)])\},\end{aligned}$$

then $\{T_\alpha\}$ *and* $\{S_{2i}\}_{i=1}^n$ *have a unique common fixed point in* X.

Proof. Let $\alpha_0 \in J$. In Theorem 2.1, set $A_0 = T_\beta$ and $A_1 = T_{\alpha_0}$. Then $T_{\alpha_0}, T_\beta, S_1, ..., S_{2n}$ have a unique fixed point, say z. Now, let $\alpha \in J$. Then

$$\begin{aligned} \rho(T_\beta z, T_\alpha z) \quad + & \quad \rho(T_\alpha z, T_\beta z) \\ \leq & \quad \max\{\varphi(\rho(T_\beta z, T_\beta z)), \varphi(\rho(T_\alpha z, T_\alpha z)), \varphi(\rho(T_\beta z, T_\alpha z)), \\ & \quad \varphi(1/2[\rho(T_\beta z, \pi_{i=1}^n S_{2i-1} z) + \rho(T_\alpha z, \pi_{i=1}^n S_{2i} z)])\} \\ \leq & \quad \varphi(\max\{\rho(z, T_\alpha z), \rho(T_\alpha z, z)\}). \end{aligned}$$

Since $T_\beta z = z$, it follows that $T_\alpha z = z$ for all $\alpha \in J$. □

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M. Avar
Department of Mathematics,
Shiraz Branch, Islamic Azad University,
Shiraz, Iran
e-mail: moosaavar@yahoo.com
K. Jahedi
Department of Mathematics,
Shiraz Branch, Islamic Azad University,
Shiraz, Iran
e-mail: mjahedi@yahoo.com
Mohammad Javad Mehdipour
Department of Mathematics,
Shiraz University of Technology,
Shiraz 71555-313, Iran
e-mail: mehdipour@sutech.ac.ir