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Study of Convergence of HAM and its Application on the Fokker-Planck Equation

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Abstract. In this paper we prove the convergence of homotopy analysis method (HAM) and present the application of the homotopy analysis method to obtain the exact analytical solution of the Fokker-Planck equation. In the current paper this scheme will be investigated in details and efficiency of the approach will be shown by applying the procedure on several interesting and important examples.

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1. Introduction

In 1992, Liao ([6]) employed the basic ideas of the homotopy in topology to propose a general analytic method for linear and nonlinear problems, namely homotopy analysis method (HAM) ([7]). Based on homotopy of topology, the validity of the HAM is independent of whether or not there exist small parameters in the considered equation. Therefore, the HAM can overcome the foregoing restrictions and limitations of perturbation techniques. This method has been successfully applied to solve many types of nonlinear problems ([5,9,11,16]). In this paper we prove theorem

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of convergence of homotopy analysis method and apply this method to solve the Fokker-Planck equation. If a small particle of mass m is immersed in a fluid, the equation of motion for the distribution function $W(x,t)$ is given by:

$$\frac{\partial W}{\partial t} = \gamma \frac{\partial vW}{\partial v} + \gamma \frac{KT}{m} \frac{\partial^2 W}{\partial v^2}, \tag{1}$$

where v is the velocity for the Brownian motion of a small particle, t is the time, γ is the fraction constant, K is Boltzmann's constant and T is the temperature of fluid [14]. Eq. (1) is one of the simplest type of Fokker-Planck equations. By solving (1) starting with distributions function $W(x,t)$ for $t = 0$ and subject to the appropriate boundary conditions, one can obtain the distributions function $W(x,t)$ for $t > 0$. The general Fokker-Planck equation for the variable x has the form [14]:

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x} A(x) + \frac{\partial^2}{\partial x^2} B(x) \right] u, \tag{2}$$

with the initial condition given by:

$$u(x,0) = f(x), \qquad x \in R, \tag{3}$$

where $u(x,t)$ is unknown. In (2) $B(x) > 0$ is called the diffusion coefficient and $A(x)$ is the drift coefficient. The drift and diffusion coefficients may also depend on time, i.e.

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x} A(x,t) + \frac{\partial^2}{\partial x^2} B(x,t) \right] u. \tag{4}$$

Eq. (1) is seen to be a special case of the Fokker-Planck equation where the drift coefficient is linear and the diffusion coefficient is constant. Eq. (2) is an equation of motion for the distribution function $u(x,t)$. Mathematically, this equation is a linear second order partial differential equation of parabolic type. Roughly speaking, it is a diffusion equation with an additional first order derivative with respect to x. In the mathematical literatures, (2) is also called forward Kolmogorov equation. The similar partial differential equation is a backward Kolmogorov equation that is in the form [14]:

$$\frac{\partial u}{\partial t} = \left[-A(x,t)\frac{\partial}{\partial x} + B(x,t)\frac{\partial^2}{\partial x^2} \right] u. \tag{5}$$

A generalization of (2) to N variables $x_1, \cdots, x_N$ has the form:

$$\frac{\partial u}{\partial t} = \left[-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(x) + \sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}(x) \right] u, \tag{6}$$

with the initial condition:

$$u(x,0) = f(x), \qquad x \in R^N, \tag{7}$$

where $x = (x_1, \cdots, x_N)$. The drift vector A_i and diffusion tensor $B_{i,j}$ generally depend on the N variables $x_1, \cdots, x_N$. One may find analytical solutions of the Fokker-Planck equation. Generally, however, it is difficult to obtain solutions, especially if no separation of variables is possible or if the number of variables is large. Various methods of solution are: simulation methods, transformation of a Fokker-Planck equation to a Schrödinger equation, numerical integration methods and etc.[14]. There is a more general form of Fokker-Planck equation. Nonlinear Fokker-Planck equation has important applications in various areas such as plasma physics, surface physics, population dynamics, biophysics, engineering, neuroscience, nonlinear hydrodynamic, polymer physic, laser physic, pattern formation, psychology and marketing (see[2] and references therein). In one variable case the nonlinear Fokker-Planck equation is written in the following form:

$$\frac{\partial u}{\partial t} = \left[-A(x,t,u)\frac{\partial}{\partial x} + B(x,t,u)\frac{\partial^2}{\partial x^2} \right] u. \tag{8}$$

For N variables $x_1, \cdots, x_N$, it has the form:

$$\frac{\partial u}{\partial t} = \left[-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(x,t,u) + \sum_{i,j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} B_{i,j}(x,t,u) \right] u, \tag{9}$$

where $x = (x_1, \cdots, x_N)$. Notice that when $A_i(x,t,u) = A_i(x)$ and $B_{i,j}(x,t,u) = B_{i,j}(x)$ the nonlinear Fokker-Planck equation (9) reduces to the linear Fokker-Planck equation (6). Because of the large number of applications of the Fokker-Planck equation, a lot of work is done in order to find the numerical solution of this equation. For example, we refer the readers to [1, 18, 17, 12, 13, 3].

2. Basic Idea of HAM

We consider the following differential equation

$$\mathcal{N}[u(\tau)] = 0, \tag{10}$$

where $\mathcal{N}$ is a nonlinear operator, τ denotes independent variable, $u(\tau)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [8] construct the so-called zero-order deformation equation

$$(1-p)\mathcal{L}[\phi(\tau;p) - u_0(\tau)] = p\,\hbar\mathcal{H}(\tau)\mathcal{N}[\phi(\tau;p)], \tag{11}$$

where $p \in [0,1]$ is the embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter, $\mathcal{H}(\tau) \neq 0$ is an auxiliary function, $u_0(\tau)$ is an initial guess of $u(\tau)$ and $\phi(\tau;p)$ is an unknown function and $\mathcal{L}$ an auxiliary linear operator with the property

$$\mathcal{L}[f(\tau)] = 0 \qquad \textit{when} \quad f(\tau) = 0. \tag{12}$$

It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $p = 0$ and $p = 1$, it holds

$$\phi(\tau;0) = u_0(\tau), \qquad \phi(\tau;1) = u(\tau), \tag{13}$$

respectively. Thus, as p increases from 0 to 1, the solution $\phi(\tau;p)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$. Expanding $\phi(\tau;p)$ in Taylor series with respect to p, we have

$$\phi(\tau;p) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau)p^m, \tag{14}$$

where

$$u_m(\tau) = \left[\frac{1}{m!} \frac{\partial^m \phi(\tau;p)}{\partial p^m} \right]_{p=0} . \tag{15}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are so properly chosen, the series (4) converges at $p = 1$, then we have

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau), \tag{16}$$

which must be one of solutions of original nonlinear equation, as proved by [13]. As $\hbar = -1$ and $\mathcal{H}(\tau) = 1$, Eq. (2) becomes

$$(1-p)\mathcal{L}[\phi(\tau;p) - u_0(\tau)] + p\, \mathcal{N}[\phi(\tau;p)] = 0, \tag{17}$$

which is used mostly in the homotopy perturbation method [4], where as the solution obtained directly, without using Taylor series ([10]). According to the definition (5), the governing equation can be deduced from the zero-order deformation equation (2). Define the vector

$$\overrightarrow{u}_n = \{u_0(\tau), u_1(\tau), \cdots, u_n(\tau)\} .$$

Differentiating equation (2) m times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$\mathcal{L}[u_m(\tau) - \chi_m u_{m-1}(\tau)] = \hbar\, \mathcal{H}(\tau) \mathcal{R}_m(\overrightarrow{u}_{m-1}), \tag{18}$$

where

$$\mathcal{R}_m(\overrightarrow{u}_{m-1}) = \left[\frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}[\phi(\tau;p)]}{\partial p^{m-1}} \right]_{p=0} , \tag{19}$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases} \tag{20}$$

It should be emphasized that $u_m(\tau)$ for $m \geqslant 1$ is governed by the linear equation (8) under the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Matlab. For the convergence of the above method we refer the reader to Liao's work [8]. If Eq. (1) admits unique solution, then this method will produce the unique solution. If equation (1) does not possess unique solution, the HAM will give a solution among many other (possible) solutions.

3. Convergence of HAM

In this section, we will prove that, as long as the solution series (16) given by the homotopy analysis method is convergent, it must be the solution of the considered nonlinear problem.

Theorem 3.1. *As long as the series*

$$u_0(t) + \sum_{m=1}^{+\infty} u_m(t).$$

is convergent, where $u_m(t)$ is governed by the high-order deformation equation (18) under the definitions (19) and (20), it must be a solution of Equation (10).

Proof: Let

$$s(t) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t),$$

denote the convergent series. Using (18) and (20), we have

$$\begin{aligned}
&\hbar\mathcal{H}(t) \sum_{m=1}^{+\infty} \Re_m(u_{m-1}) \\
&= \sum_{m=1}^{+\infty} \mathcal{L}\left[u_m(t) - \chi_m u_{m-1}(t)\right] \\
&= \mathcal{L}\left[\sum_{m=1}^{+\infty} u_m(t) - \sum_{m=1}^{+\infty} \chi_m u_{m-1}(t)\right] \\
&= \mathcal{L}\left[(1-\chi_2) \sum_{m=1}^{+\infty} u_m(t)\right] \\
&= \mathcal{L}\left[(1-\chi_2)(s(t) - u_0(t))\right],
\end{aligned}$$

which gives, since $\hbar \neq 0$, $\mathcal{H}(t) \neq 0$ and from (12),

$$\sum_{m=1}^{+\infty} \Re_m(u_{m-1}) = 0. \tag{21}$$

On the other side, we have according to the definition (19), that

$$\sum_{m=1}^{+\infty} \Re_m(u_{m-1}) = \sum_{m=1}^{+\infty} \frac{1}{(m-1)!} \frac{\partial^{m-1} \mathcal{N}[\phi(t;q)]}{\partial q^{m-1}}|_{q=0} = 0. \tag{22}$$

In general, $\phi(t;q)$ does not satisfy the original nonlinear equation (10). Let

$$\epsilon(t;q) = \mathcal{N}[\phi(t;q)],$$

denote the residual error of Equation (10). Clearly,

$$\epsilon(t;q) = 0.$$

Corresponds to the exact solution of the original equation (10). According to the above definition, the Maclaurin series of the residual error $\epsilon(t;q)$ about the embedding parameter q is

$$\sum_{m=0}^{+\infty} \frac{1}{m!} \frac{\partial^m \epsilon(t;q)}{\partial q^m} q^m|_{q=0} = \sum_{m=0}^{+\infty} \frac{1}{m!} \frac{\partial^m \mathcal{N}[\phi(t;q)]}{\partial q^m} q^m|_{q=0}.$$

When $q = 1$, the above expression gives, using (22),

$$\epsilon(t;q) = \sum_{m=0}^{+\infty} \frac{1}{m!} \frac{\partial^m \epsilon(t;q)}{\partial q^m}|_{q=0} = 0.$$

This means, according to the definition of $\epsilon(t;q)$, that we gain the exact solution of the original equation (10) when q. Thus, as long as the series

$$u_0(t) + \sum_{m=1}^{+\infty} u_m(t),$$

is convergent, it must be one solution of the original equation (10). This ends the proof.

4. Application

In order to assess the advantages and the accuracy of homotopy analysis method for solving nonlinear systems, we will consider the following example [15].

Example 4.1. Consider (1) with

$$f(x) = x, \qquad x \in R. \tag{23}$$

Let in Eq. (2)

$$A(x) = -1, \qquad B(x) = 1. \tag{24}$$

To solve the Eq. (2) by means of homotopy analysis method, we choose the linear operator

$$\mathcal{L}[\phi(x,t;p)] = \frac{\partial \phi(x,t;p)}{\partial t}. \tag{25}$$

Now we define a nonlinear operators as

$$\mathcal{N}[\phi(x,t;p)] = \frac{\partial \phi(x,t;p)}{\partial t} - \frac{\partial \phi(x,t;p)}{\partial x} - \frac{\partial^2 \phi(x,t;p)}{\partial x^2}. \tag{26}$$

Thus, by the above definitions we obtain the mth-order deformation equations

$$\mathcal{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar \mathcal{H}(x,t) \mathcal{R}_m(\vec{u}_{m-1}) \tag{27}$$

where

$$\mathcal{R}_m(\vec{u}_{m-1}) = \frac{\partial u_{m-1}}{\partial t} - \frac{\partial u_{m-1}}{\partial x} - \frac{\partial^2 u_{m-1}}{\partial x^2}. \tag{28}$$

Now the solutions of the mth-order deformation equations (27) with $\mathcal{H}(x,t) = 1,\ \hbar = -1$

$$u_m(x,t) = \chi_m u_{m-1}(x,t) - L^{-1}[\mathcal{R}_m]. \tag{29}$$

With an initial approximations $u_0(x,t) = x$ and by means of the above iteration formula (29) we can obtain directly the other components as

$$\begin{aligned} u_1(x,t) &= t, \\ u_n(x,t) &= 0, \qquad n \geqslant 2. \end{aligned}$$

Continuing the expansion to the last term gives the solution of (2) as

$$u(x,t) = \sum_{n=0}^{+\infty} u_n(x,t) = x + t,$$

which is the exact solution of the problem.

Example 4.2. Consider (3) with

$$f(x) = \cosh(x), \qquad x \in R, \tag{30}$$

and we assume that in Eq. (4)

$$A(x) = e^t \coth(x)\cosh(x) + e^t \sinh(x) - \coth(x), \quad B(x) = e^t \cosh(x). \tag{31}$$

To solve the Eq. (4) by means of homotopy analysis method, we choose the linear operator

$$\mathcal{L}[\phi(x,t;p)] = \frac{\partial \phi(x,t;p)}{\partial t}. \tag{32}$$

Now we define a nonlinear operators as

$$\mathcal{N}[\phi(x,t;p)] = \frac{\partial \phi(x,t;p)}{\partial t} - \frac{\partial \phi(x,t;p)}{\partial x} A(x,t) - \frac{\partial^2 \phi(x,t;p)}{\partial x^2} B(x,t). \tag{33}$$

With an initial approximations $u_0(x,t) = sinh(x)$ we can obtain directly the other components as

$$\begin{aligned} u_1(x,t) &= t\sinh(x), \\ u_2(x,t) &= \frac{t^2}{2!}\sinh(x), \\ u_3(x,t) &= \frac{t^3}{3!}\sinh(x), \\ &\vdots \end{aligned}$$

Continuing the expansion to the last term gives the solution of (4) as

$$u(x,t) = \sum_{n=0}^{+\infty} u_n(x,t) = e^t \sinh(x),$$

which is the exact solution of the problem.

Example 4.3. In our final example we apply the homotopy analysis method on the nonlinear FokkerPlanck equation. Consider Eq. (9) with:

$$A(x,t,u) = \frac{4u}{x} - \frac{x}{3}, \quad B(x) = u, \tag{34}$$

and we assume that in Eq. (3)

$$f(x) = x^2, \quad x \in R. \tag{35}$$

Now we define a nonlinear operators as

$$\mathcal{N}[\phi(x,t;p)] = \frac{\partial \phi(x,t;p)}{\partial t} - \frac{\partial \phi(x,t;p)}{\partial x} A(x,t) - \frac{\partial^2 \phi(x,t;p)}{\partial x^2} B(x,t), \tag{36}$$

Thus, by the above definitions we obtain the mth-order deformation equations

$$\mathcal{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar \mathcal{H}(x,t) \mathcal{R}_m(\vec{u}_{m-1}), \tag{37}$$

where

$$\mathcal{R}_m(\vec{u}_{m-1}) = \frac{\partial u_{m-1}}{\partial t} + \frac{4}{x} \sum_{i=0}^{m-1} \frac{\partial u_i}{\partial x} u_{m-1-i} - \frac{x}{3} \frac{\partial u_{m-1}}{\partial x} - \sum_{i=0}^{m-1} \frac{\partial^2 u_i}{\partial x^2} u_{m-1-i}. \tag{38}$$

Now the solutions of the mth-order deformation equations (37) with $\mathcal{H}(x,t) = 1,\ \hbar = -1$

$$u_m(x,t) = \chi_m u_{m-1}(x,t) - L^{-1}[\mathcal{R}_m(\vec{u}_{m-1})]. \tag{39}$$

With an initial approximations $u_0(x,t) = x^2$ we can obtain directly the other components as

$$\begin{aligned} u_1(x,t) &= x^2 t, \\ u_2(x,t) &= x^2 \frac{t^2}{2!}, \\ u_3(x,t) &= x^2 \frac{t^3}{3!}, \\ &\vdots \end{aligned}$$

Continuing the expansion to the last term gives the solution of (4) as

$$u(x,t) = \sum_{n=0}^{+\infty} u_n(x,t) = x^2 e^t,$$

which is the exact solution of the problem.

5. Conclusions

Homotopy analysis method was employed successfully for solving the linear and nonlinear FokkerPlanck equation. This method finds an exact solution of the equation using the initial condition only. It is also important that the homotopy analysis method does not require discretization of the variables, i.e. time and space, it is not affected by computation round-off errors and one is not faced with the necessity for a large computer memory and time. One important objective of our research is the proof of the convergence of homotopy analysis method. The results show us the validity and great potential of the HAM for nonlinear problems in science and engineering.

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