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An Existence Result for Fractional Integro-Differential Equations in Banach Spaces

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Abstract. In this paper, we consider a class of nonlinear fractional integro-differential equations with fractional derivative of Caputo sense. We shall rely on the Krasnoselskii fixed point theorem to obtain the existence result in Banach spaces. Moreover, one of Krasnoselskii-Krein conditions is applied to establish the result. Finally, an illustrative example is also presented.

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1. Introduction

This paper is concerned with the existence result for a fractional integro-differential equations of the type

$$^{c}D_{0^{+}}^{\alpha}y(t) = h(y(t)) + f(t, y(t)) + \int_0^t K(t, s, y(s))ds, \quad t \in [0, 1], \tag{1}$$

with the initial condition

$$y(0) = y_0. \tag{2}$$

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where $0 < \alpha \leqslant 1$, ${}^cD^{\alpha}_{0+}$ denotes the Caputo fractional derivative operator, $f : [0, 1] \times X \to X$, $K : [0, 1] \times [0, 1] \times X \to X$ and $h : C([0, 1] \to X$ appropriate functions satisfying some conditions which will be stated later.

Fractional differential equations are linked with extensive applications such as continuum phenomena mechanics, electrochemistry, biophysics, biotechnology engineering and so forth. For more details see studies of Guo et al. [14], Kilbas et al. [18], Miller and Ross [20] Oldham and Spanier [23] and many other references.

Integro-differential equations emerge in many scientific and engineering specialties, oftentimes be an approximation to partial differential equations, that represent a lot of the incessant phenomena. Recently, the existence and uniqueness of solutions to fractional differential equations have studied in [1, 2, 3, 4, 10, 11, 12, 19], and the various fractional integro-differential equations have been taken into consideration by some authors, for extra information, see [5, 6, 7, 8, 9, 21, 22, 25, 27]. For example in [22] Momani et al. studied the local and global uniqueness results by applying Bihari's inequality and Gronwall's inequality for the following problem

$$ {}^cD^{\alpha}y(t) = f(t, y(t)) + \int_{t_0}^{t} K(t, s, y(s))ds, $$

$$ y(0) = y_0 $$

where $0 < \alpha \leqslant 1$, $f \in C([0, 1] \times \mathbb{R}^n, \mathbb{R}^n)$, $K \in C([0, 1] \times [0, 1] \times \mathbb{R}^n, \mathbb{R}^n)$ and ${}^cD^{\alpha}$ is the Caputo fractional operator. In [6] Ahmad and Sivasundaram considered the integro-differential equations with fractional order and nonlocal conditions

$$ {}^cD^{\alpha}y(t) = f(t, y(t)) + \int_{0}^{t} K(t, s, y(s))ds, \quad t \in [0, T], $$

$$ y(0) = y_0 - g(y), $$

where $0 < \alpha < 1$, ${}^cD^{\alpha}$ is the Caputo fractional operator, $f : [0, T] \times X \to X$, $K : [0, T] \times [0, T] \times X \to X$ are jointly continuous and $g \in C([0, T], X)$

is continuous. The authors employed the Banach contraction principle and Krasnoselskii's fixed point theorem to establish the existence and uniqueness results. Wu and Liu in [25] extended the results that have been obtained in [6,7] by employed Krasnoselskii-Krein-type conditions. In [27], Zhao discussed the collocation methods for fractional integro-differential equations with weakly singular kernels

$$^{c}D_0^{\alpha}y(t) = p(t)y(t) + g(t) + \int_0^t q(t,s)y(s)ds, \quad t \in [0,T], \tag{3}$$

$$y^{(i)}(0) = y_0^{(i)},\ i = 0, 1, ..., n-1,$$

where $0 < \alpha < 1$, $g(t)$ and $p(t)$ are bounded and continuous on $[0,T]$, and $q(t,s)$ might possess a weak singularity.

On the other hand, Heydari et al. proved the existence of a unique solution for a class of system of nonlinear singular fractional integro-differential equations and they also used many numerical methods including Chebyshev, wavelet method to solving such these equations see [15, 16, 17].

In this paper, we will prove the existence solution of the fractional integro-differential equation (1) together with the initial condition (2) via taking advantage of Krasnoselskii's fixed point theorem on the interval $[0,1]$. The existence result obtained in Banach spaces. Moreover, we also use one of the Krasnoselskii-Krein conditions.

The organization of this paper is as follows. In Section 2, we mention some known notations and definitions and also we listing the hypotheses which have advantage on this paper. The main Section 3 proves the existence of solution for the problem (1)-(2) in Banach space by Krasnoselskii fixed point theorem. Finally, an illustrative example is presented in Section 4.

2. Preliminaries

In this section, we mainly demonstrate some essential notations, definitions, and Lemmas which regard to fractional calculus and fixed point theorem. Let $J = [0,1]$, $(X, \|.\|)$ is a Banach space and $C(J,X)$ denotes the Banach space of all continuous bounded functions $g : J \to X$ equipped with the norm $\|g\|_{C(J,X)} = \sup\{|g(t)| : t \in J\}$, for any $g(t) \in X$. We consider the space $C^n(J,X)$ consisting of all real valued continuous functions which are continuously differentiable up to order $(n-1)$ on J, and $L^1(J)$ denotes the space of all real functions defined on J which are Lebesgue integrable. In the following, the Mittag-Leffler function is given by

$$E_{\alpha,\beta}(w) = \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(\alpha k + \beta)}, \quad Re(\alpha), Re(\beta) > 0.$$

Furthermore, if $0 < \alpha < 2$ and $\beta > 1$, we have [13]

$$E_{\alpha,\beta}(w) \leqslant \frac{1}{\alpha} w^{\frac{(1-\beta)}{\alpha}} e^{w^{\frac{1}{\alpha}}}.$$

Definition 2.1. *([18]). Let $\alpha > 0$ and $g : J \to X$. The left sided Riemann–Liouville fractional integral of order α of a function g is defined as*

$$I_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, \quad t \in J,$$

provided the right-hand side is pointwisely defined, where $\Gamma(.)$ is the Euler gamma function.

Definition 2.2. *([18]). Let $n-1 < \alpha < n$ and $g \in AC^n(J,X)$. The left sided Caputo fractional derivative of order α of a function g is defined as*

$${}^cD_{0^+}^{\alpha} g(t) = I_{0^+}^{n-\alpha-1} \frac{d^n}{dt^n} g(t), \quad t \in J,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.3. *([18, 24]). For $\alpha, \beta > 0$ and g, p are appropriate functions then, for $t \in J$, we have*

1. $I_{0^+}^{\alpha} I_{0^+}^{\beta} g(t) = I_{0^+}^{\alpha+\beta} g(t) = I_{0^+}^{\beta} I_{0^+}^{\alpha} g(t)$.
2. $I_{0^+}^{\alpha} (g(t) + p(t)) = I_{0^+}^{\alpha} g(t) + I_{0^+}^{\alpha} p(t)$.

3. $I^{\alpha}_{0+}\,{}^{c}D^{\alpha}_{0+}g(t) = g(t) - g(0)$, $0 < \alpha < 1$.

4. ${}^{c}D^{\alpha}_{0+}I^{\alpha}_{0+}g(t) = g(t)$.

5. ${}^{c}D^{\alpha}_{0+}g(t) = I^{1-\alpha}_{0+}\frac{d}{dt}g(t)$, $0 < \alpha < 1$.

6. ${}^{c}D^{\alpha}_{0+}C = 0$, *where* C *is a constant.*

Lemma 2.4. *([26]) (Krasnoselskii fixed point theorem). Let* E *be bounded, closed and convex subset of a Banach space* X*, and let* $T_1, T_2 : E \to E$ *satisfying the following:*

(1) $T_1x + T_2y \in E$, *for every* $x, y \in E$.

(2) T_1 *is contraction.*

(3) T_2 *is compact and continuous.*

Then, there exists $z \in E$ *such that the equation* $z = T_1z + T_2z$ *has a solution on* E.

3. Main Results

In this section, we shall demonstrate the existence result of $(1)-(2)$. Foremost, we state the subsequent lemma without proof.

Lemma 3.1. The fractional integro-differential equation (1) with the initial condition (2) is equivalent to the following nonlinear integral equation

$$\begin{aligned} y(t) &= y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(y(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\ &\quad \times f(s,y(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_s^t K(\tau,s,y(s))d\tau ds. \quad (4) \end{aligned}$$

On the other hand, each solution of the integral equation (4) is likewise a solution of the problem $(1)-(2)$ and vice versa.

For reader's comfort, we list of hypotheses is supplied as follows:

(A1) $h : C(J, X) \to X$ is continuous, bounded and there exists $0 < M < 1$ such that $\|h(u) - h(v)\| \leqslant M \|u - v\|$, for $u, v \in X$.

(A2) $f : J \times X \to X$ is continuous and there exist $\beta \in (0, 1]$, $L > 0$ such that

$$\|f(t, u) - f(t, v)\| \leqslant L \|u - v\|^{\beta}, \ t \in J, \ u, v \in X.$$

(A3) $K : D \times X \to X$, is continuous on D and there exist $\gamma \in (0, 1]$, $\rho \in L^1(J)$ such that

$$\|K(\tau, s, u(s)) - K(\tau, s, v(s))\| \leqslant \rho(\tau) \|u - v\|^{\gamma}, (\tau, s) \in D, \ u, v \in X,$$

where $D = \{(t, s) : 0 \leqslant s \leqslant t \leqslant 1\}$.

Now, we give an existence result based on the Krasnoselskii's fixed point theorem.

Theorem 3.2. Assume that the hypotheses (A1),(A2) and (A3) hold. Then the fractional integro-differential problem $(1)-(2)$ has a solution in $C(J, X)$ on J.

Proof. Transform the problem $(1) - (2)$ into a fixed point problem. Consider the operator $\digamma : C(J, X) \to C(J, X)$ defined by

$$\begin{aligned} \digamma y(t) &= y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(y(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \times f(s, y(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_s^t K(\tau, s, y(s)) d\tau ds. \end{aligned}$$

Before move ahead, we need to analyze the operator $\digamma$ into sum two operators $P + Q$ as follows

$$Py(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(y(s)) ds \tag{5}$$

and

$$\begin{aligned} Qy(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \times \int_s^t K(\tau, s, y(s)) d\tau ds. \end{aligned} \tag{6}$$

For any function $z \in C(J, X)$ and for som $j \in \mathbb{N}$, we define the norm $\|z\|_j = \sup\{e^{-jt}\|z(t)\| : t \in J\}$. Notice that the norm $\|z\|_j$ is equivalent to the norm $\|z\|_C$ for $z \in C(J, X)$. Now, we present the proof in several steps:

Step 1: We prove that $Pz + Qz^* \in S_r \subset C(J, X)$, for every $z, z^* \in S_r$. Let us set

$$\mu = \sup_{(s,z^*)\in J\times S_r} \|f(s, z^*(s))\|$$

$$\mu^* = \sup_{(\tau,s,z^*)\in D\times S_r} \int_s^t \|K(\tau, s, z^*(s))\| \, d\tau, \; \eta = \sup_{z\in S_r} \|h(z)\|$$

and there exists $r = \|z_0\| + \frac{\eta+\mu+\mu^*}{\Gamma(\alpha+1)} + 1$ such that $S_r = \{z \in C(J, X) : \|z\|_j \leqslant r\}$. From the previous assumptions, then for $z, z^* \in S_r$ and $t \in J$, we have

$$\begin{aligned}
& \|Pz(t) + Qz^*(t)\| \\
\leqslant\; & \|z_0\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|h(z(s))\| \, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, z^*(s))\| \, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
& \times \int_s^t \|K(\tau, s, z^*(s))\| \, d\tau ds \\
\leqslant\; & \|z_0\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{z\in S_r} \|h(z(s))\| \, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{(s,z^*)\in J\times S_r} \|f(s, z^*(s))\| \, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{(\tau,s,z^*)\in D\times S_r} \int_s^t \|K(\tau, s, z^*(s))\| \, d\tau ds \\
\leqslant\; & \|z_0\| + \frac{\eta t^\alpha}{\Gamma(\alpha+1)} + \frac{\mu t^\alpha}{\Gamma(\alpha+1)} + \frac{\mu^* t^\alpha}{\Gamma(\alpha+1)} \\
\leqslant\; & \|z_0\| + \frac{\eta + \mu + \mu^*}{\Gamma(\alpha+1)}.
\end{aligned}$$

Consequently,

$$\|Pz+Qz^*\|_j \leqslant e^{-j}\left(\|z_0\| + \frac{\eta+\mu+\mu^*}{\Gamma(\alpha+1)}\right) < r.$$

This means that, $Pz+Qz^* \in S_r$.

Step 2: We prove that operator P is a contraction map on S_r.
Let us make S_r as in step 1, by the preceding assumptions, then for $z, z^* \in S_r$ and for $t \in J$, we have

$$\begin{aligned}
\|Pz(t)-Pz^*(t)\| &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|h(z(s))-h(z^*(s))\|\,ds \\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} M\|z(s)-z^*(s)\|\,ds \\
&\leqslant M\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} e^{js}\sup_{s\in J} e^{-js}\|z(s)-z^*(s)\|\,ds \\
&= M\left[I_0^{\alpha}e^{jt}\right]\|z-z^*\|_j \\
&= Mt^{\alpha}E_{1,\alpha+1}(jt) \\
&\leqslant M\frac{e^{jt}}{j^{\alpha}}\|z-z^*\|_j \\
&\leqslant Me^{jt}\|z-z^*\|_j .
\end{aligned}$$

Thus,

$$\|Pz-Pz^*\|_j \leqslant M\|z-z^*\|_j .$$

Since $M<1$, we conclude that P is a contraction map on S_r.

Step 3: We show that operator Q is completely continuous on S_r.
For this end, we consider S_r defined as in step 1, and we prove that (QS_r) is uniformly bounded, $(\overline{QS_r})$ is equicontinuous and $Q: S_r \to S_r$ is continuous.

Firstly, we show that (QS_r) is uniformly bounded. By our hypotheses, then for $z \in S_r$ and $t \in J$, we have

$$\begin{aligned}
&\|Qz(t)\| \\
\leqslant\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,z(s))-f(s,0)\|\,ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,0)\|\,ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_s^t \|K(\tau,s,z(s))-K(\tau,s,0)\|\,d\tau ds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_s^t \|K(\tau,s,0)\|\,d\tau ds \\
\leqslant\ & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Le^{\beta js}\|z\|_j^{\beta}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Rds \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_s^t \rho(\tau)d\tau e^{\gamma js}\|z\|_j^{\gamma}\,ds+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}R^*ds \\
\leqslant\ & \left(L\|z\|_j^{\beta}+\|\rho\|_{L^1}\|z\|_j^{\gamma}\right)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{js}ds+\frac{R+R^*}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
\leqslant\ & \left(Lr^{\beta}+\|\rho\|_{L^1}r^{\gamma}\right)\frac{e^{jt}}{j^{\alpha}}+\frac{R+R^*}{\Gamma(\alpha+1)}t^{\alpha}.
\end{aligned}$$

Thus,

$$\|Qz\|_j \leqslant \frac{Lr^{\beta}+\|\rho\|_{L^1}r^{\gamma}}{j^{\alpha}}+\frac{R+R^*}{\Gamma(\alpha+1)e^j} := \ell,$$

where $R = \sup\limits_{s\in J}\|f(s,0)\|$ and $R^* = \sup\limits_{(\tau,s)\in D}\int_s^t \|K(\tau,s,0)\|\,d\tau$. This means that $QS_r \subset S_\ell$, for any $z \in S_r$, i.e. the set $\{Qz : z \in S_r\}$ is uniformly bounded.

Next, we will prove that $(\overline{QS_r})$ is equicontinuous. For $z \in S_r$ and for $t_1, t_2 \in J$ with $t_1 \leqslant t_2$, and also let $\delta = \left(\frac{\Gamma(\alpha+1)\epsilon}{2(\mu+\mu^*)}\right)^{\frac{1}{\alpha}}$ then, when

$|t_2 - t_1| < \delta$, we conclude that

$$
\begin{aligned}
& \|Qz(t_2) - Qz(t_1)\| \\
\leqslant\ & \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \left|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right| \|f(s, z(s))\| \, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \|f(s, z(s))\| \, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right| \int_s^{t_2} \|K(\tau, s, z(s))\| \, d\tau ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \int_s^{t_2} \|K(\tau, s, z(s))\| \, d\tau ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} ((t_1 - s)^{\alpha-1} \int_{t_1}^{t_2} \|K(\tau, s, z(s))\| \, d\tau ds \\
\leqslant\ & \frac{1}{\Gamma(\alpha)} \int_0^{t_1} ((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}) \|f(s, z(s))\| \, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \|f(s, z(s))\| \, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} ((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}) \int_s^{t_2} \|K(\tau, s, z(s))\| \, d\tau ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \int_s^{t_2} \|K(\tau, s, z(s))\| \, d\tau ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} \left[\int_s^{t_2} \|K(\tau, s, z(s))\| \, d\tau - \int_s^{t_1} \|K(\tau, s, z(s))\| \, d\tau\right] ds \\
\leqslant\ & \left[\frac{(t_1^\alpha - t_2^\alpha) + 2(t_2 - t_1)^\alpha}{\Gamma(\alpha + 1)}\right] \mu + \left[\frac{(t_1^\alpha - t_2^\alpha) + 2(t_2 - t_1)^\alpha}{\Gamma(\alpha + 1)}\right] \mu^* \\
\leqslant\ & \frac{2(\mu + \mu^*)(t_2 - t_1)^\alpha}{\Gamma(\alpha + 1)} \\
<\ & \frac{2(\mu + \mu^*)\delta^\alpha}{\Gamma(\alpha + 1)} = \epsilon.
\end{aligned}
$$

where μ and μ^* are defined as in step 1. Therefore, $(\overline{QS_r})$ is equicontinuous.

Finally, from the continuity of f and K, we can directly reach that operator $Q : S_r \to S_r$. As consequence of step 3 with Arzela-Ascoli theorem, we easily infer that (QS_r) is relatively compact set. Hence, the operator Q is completely continuous. Thus all the assumptions of Lemma 2.4 are satisfied. Consequently, the conclusion of Krasnoselskii's fixed point theorem shows that operator $F = P+Q$ has a fixed point on S_r. So the fractional integro-differential problem $(1)-(2)$ has a solution $y(t) \in C(J, X)$. This proves the required. $\square$

4. An Example

Consider the following nonlinear fractional integro-differential equation

$$^cD^{\frac{1}{2}}_{0^+}y(t) = \frac{1}{2}\sin y(t) + (\cos t + \sin t)\,[y(t)]^{\frac{1}{2}} + \int_0^t t\sin\,[y(s)]^{\frac{1}{3}}\,ds, \quad (7)$$

with the initial condition

$$y(0) = 0. \quad (8)$$

Here, $\alpha = \frac{1}{2}$, $h(y(t)) = \frac{1}{2}\sin(y(t))$, $f(t, y(t)) = (\cos t + \sin t)\,[y(t)]^{\frac{1}{2}}$, and $K(t, s, y(s)) = t\sin\,[y(s)]^{\frac{1}{3}}$. For $u, v \in X = \mathbb{R}^+$ and $t \in [0, 1]$. We can see that

$$\|h(u) - h(v)\| \leqslant \frac{1}{2}\|u - v\|,$$

$$\|f(t, u) - f(t, v)\| \leqslant 2\left\|u^{\frac{1}{2}} - v^{\frac{1}{2}}\right\| \leqslant 2\|u - v\|^{\frac{1}{2}},\ (0 < \frac{1}{2} = \beta).$$

and

$$\|K(t, s, u) - K(t, s, v)\| \leqslant t\left\|u^{\frac{1}{3}} - v^{\frac{1}{3}}\right\| \leqslant t\|u - v\|^{\frac{1}{3}},\ (\gamma = \frac{1}{3} < 1).$$

So, the conditions (A1), (A2) and (A3) are satisfied with $M = \frac{1}{2}, L = 2$, and $\rho(t) = t \in L^1[0, 1]$. By applying Theorem 3. the problem (2) (3) has a solution on $[0, 1]$.

5. Conclusions

This paper presents a class of nonlinear integro-differential equations with Caputo fractional derivative. By using famous Krasnoselskii's fixed point theorem, we have developed some adequate conditions for the existence of at least one solution to a class of nonlinear fractional integro-differential equations. The respective result has been verified by providing a suitable example.

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