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An existence result for fractional integro-differential equations on Banach space

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Abstract. In this paper, we consider integro-differential equations with fractional order including the Caputo fractional derivative. We shall rely on the Krasnoselskii fixed point theorem to obtain the existence result in Banach space. Moreover, we apply Krasnoselskii-Krein-type conditions to get the desired result.

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1 Introduction

This paper is concerned with the existence result for fractional integro-differential equations of the type

$$^{c}D^{\alpha}y(t) = h(y(t)) + f(t, y(t)) + \int_0^t K(t, s, y(s))ds, \quad t \in [0, 1], \tag{1}$$

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with initial condition

$$y(0) = y_0. \tag{2}$$

where $0 < \alpha \leq 1$, ${}^cD^\alpha$ is the Caputo fractional derivative, $f : [0,1]\times X \to X$, $K : [0,1] \times [0,1] \times X \to X)$ and $h : C([0,1], X) \to X$ appropriate functions satisfying some conditions which will be stated later.

Fractional differential equations are linked with extensive applications such as continuum phenomena mechanics, electrochemistry, biophysics, biotechnology engineering and so forth. For more details see studies of Guo et al. [11], Kilbas et al. [12], Miller and Ross [14] Oldham and Spanier [17] and many other references.

Integro-differential equations emerge in many scientific and engineering specialties, oftentimes be an approximation to partial differential equations, that represent a lot of the incessant phenomena.

Recently, the existence and uniqueness of solutions to fractional differential equations have studied in [1, 7, 8, 9, 13], and the various fractional integro-differential equations have been taken into consideration by some authors, for extra information, (see [2, 3, 4, 5, 6, 15, 16, 19]). For example in [16] Momani et al. studied the local and global uniqueness results by applying Bihari's inequality and Gronwall's inequality for the following problem

$$ {}^cD^\alpha y(t) = f(t, y(t)) + \int_{t_0}^{t} K(t, s, y(s))ds,$$
$$y(0) = y_0$$

where $0 < \alpha \leq 1$, $f \in C([0,1] \times \mathbb{R}^n, \mathbb{R}^n)$, $K \in C([0,1] \times [0,1] \times \mathbb{R}^n, \mathbb{R}^n)$ and ${}^cD^\alpha$ is the Caputo fractional operator.

In [3] Ahmad and Sivasundaram, considered the integro-differential equations with fractional order and nonlocal conditions

$$ {}^cD^\alpha y(t) = f(t, y(t)) + \int_{0}^{t} K(t, s, y(s))ds), \quad t \in [0,T],$$
$$y(0) = y_0 - g(y),$$

where $0 < \alpha < 1$, ${}^cD^\alpha$ is the Caputo fractional operator, $f : [0,T] \times X \to X$, $K : [0,T] \times [0,T] \times X \to X$ are jointly continuous and $g \in C([0,T] \times X) \to X$ is continuous. The authors employed the Banach contraction principle and Krasnoselskii's fixed point theorem to establish the existence and uniqueness results. Wu and Liu in [19], extended the results that have been obtained in [3],[4] by employed Krasnoselskii-Krein-type conditions.

In this paper, we prove the existence result of the fractional integro-differential equations (1)-(2) via taking advantage of Krasnoselskii's fixed point theorem. Moreover, we use Krasnoselskii-Krein type conditions.

The organization of this paper is as follows. In Section 2, we mention some known notations and definitions and also we listing the hypotheses which have advantage on this paper. The main Section 3 proves the existence of solutions for the problem (1)-(2) in Banach space by Krasnoselskii fixed point theorem.

2 Preliminaries

In this section, we present some essential notations and definitions concerning fractional calculus and fractional differential equations.

Let $J = [0, 1]$ and X is Banach space with norm $\|.\|$. $C(J, X)$ denotes the Banach space of all continuous bounded functions $g : J \to X$ equipped with the norm $\|g\|_{C(J,X)} = \max\{|g(t)| : t \in J\}$, for any $g(t) \in X$, we also $C^n(J, X)$ be space of all real valued continuous function which are continuously differentiable up to order $(n-1)$ on J. In the following, the Mittag-Leffler function is given by

$$E_{\alpha,\beta}(w) = \sum_{k=0}^{\infty} \frac{w^k}{\Gamma(\alpha k + \beta)}, \quad Re(\alpha), Re(\beta) > 0.$$

Furthermore, if $0 < \alpha < 1$ and $\beta > 1$, then

$$E_{\alpha,\beta}(w) \leq \frac{1}{\alpha} w^{\frac{(1-\beta)}{\alpha}} e^{w^{\frac{1}{\alpha}}}. \text{ (See [10]).}$$

Definition 2.1. ([12]). Let $\alpha > 0$ and $g \in C(J, X)$. The Riemann–Liouville fractional integral operator of order α is defined as

$$I_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) ds, \quad t \in J,$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 2.2. ([12]). Let $n-1<\alpha<n$, $n\in\mathbb{N}$ and $g\in C^n(J,X)$. The Caputo fractional derivative operator of order α is defined as

$$^cD^{\alpha}_{0^+}g(t)=I^{n-\alpha-1}_{0^+}\frac{d^n}{dt^n}g(t),\quad t\in J.$$

Lemma 2.3. *([12, 18]). For $\alpha,\beta>0$ and g,p are convenient functions then, for $t\in J$, we have,*

1. $I^{\alpha}_{0^+}I^{\beta}_{0^+}g(t)=I^{\alpha+\beta}_{0^+}g(t)=I^{\beta}_{0^+}I^{\alpha}_{0^+}g(t)$.
2. $I^{\alpha}_{0^+}(g(t)+p(t))=I^{\alpha}_{0^+}g(t)+I^{\alpha}_{0^+}p(t)$.
3. $I^{\alpha}_{0^+}\,{}^cD^{\alpha}_{0^+}g(t)=g(t)-g(0)$, $0<\alpha<1$.
4. ${}^cD^{\alpha}_{0^+}I^{\alpha}_{0^+}g(t)=g(t)$.
5. ${}^cD^{\alpha}_{0^+}g(t)=I^{1-\alpha}_{0^+}\frac{d}{dt}g(t)$, $0<\alpha<1$.
6. ${}^cD^{\alpha}_{0^+}C=0$, *where C is a constant.*

Lemma 2.4. *([20]) (Krasnoselskii fixed point theorem). Let K be bounded, closed and convex subset of a Banach space X, Let $T_1,T_2:K\to K$ satisfying the following:*

(1) *$T_1x+T_2y\in K$, for every $x,y\in K$.*

(2) *T_1 is contraction.*

(3) *T_2 is compact and continuous.*

Then, there exists $z\in K$ such that the equation $z=T_1z+T_2z$ has a solution on K.

3 Main results

In this section, we shall demonstrate the existence result of (1) − (2). Foremost, we state the subsequent lemma without proof.

Lemma 3.1. *The fractional integro-differential equations* (1) − (2) *is equivalent to the nonlinear integral equation*

$$\begin{aligned} y(t) &= y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(y(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_s^t K(\tau,s,y(s))d\tau ds, \quad t\in J. \end{aligned} \tag{3}$$

In other hand, each solution of the integral equation (3) is likewise a solution of the problem (1) − (2) *and vice versa.*

For reader's comfort, we list of hypotheses is supplied as follows:

(A1) $h : C(J,X) \to X$ is continuous, bounded and there exists $0 < M < 1$ such that $\|h(u) - h(v)\| \le M\|u - v\|$, for $u, v \in X$.

(A2) $f : J \times X \to X$ is continuous and there exist $\beta \in (0,1]$, $L > 0$ such that

$$\|f(t,u) - f(t,v)\| \le L\|u - v\|^{\beta},\ t\in J,\ u,v\in X.$$

(A3) $K : J \times J \times X \to X$, is continuous on D and there exist $\gamma \in (0,1]$, $\rho \in L^1(J)$ such that

$$\|K(\tau,s,u(s)) - K_1(\tau,s,v(s))\| \le \rho(\tau)\|u - v\|^{\gamma}, (\tau,s)\in D,\ u,v\in X,$$

where $D = \{(t,s) : 0 \le s \le t \le 1\}$.

Our first result depends on Krasnoselskii's fixed point theorem.

Theorem 3.2. *Assume that the hypotheses (A1),(A2) and (A3) hold. Then the fractional integro-differential equation* (1) − (2) *has a solution in $C(J,X)$ on J.*

Proof. In the inception, we convert the Cauchy problem (1) − (2) to be applicable to fixed point problem with the operator $\digamma : C(J,X) \to C(J,X)$ defined by

$$\begin{aligned} \digamma y(t) &= y_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h(y(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_s^t K(\tau,s,y(s))d\tau ds, \quad t\in J. \end{aligned}$$

Before move ahead, we need to analyze the operator $\digamma$ into sum two operators $P+Q$ as follows

$$Py(t)=y_0+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(y(s))ds \tag{4}$$

and

$$Qy(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s))ds+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_s^t K(\tau,s,y(s))d\tau ds. \tag{5}$$

For any function $z\in C(J,X)$ and for som $j\in\mathbb{N}$, we define the norm $\|z\|_j=\max\{e^{-jt}\|z(t)\|:t\in J\}$. Notice that the norm $\|z\|_j$ is equivalent to the norm $\|z\|_C$ for $z\in C(J,X)$. Now, we present the proof in numerous steps:

Step(1) we prove that $Pz+Qz^*\in S_r\subset C(J,X)$, for every $z,z^*\in S_r$.

Let $\mu=\sup\limits_{(s,z^*)\in J\times S_r}\|f(s,z^*(s))\|$, $\mu^*=\sup\limits_{(\tau,s,z^*)\in D\times S_r}\int_s^t\|K(\tau,s,z^*(s))\|\,d\tau$, $\eta=\sup\limits_{z\in X}\|h(z)\|$ and there exists $r=\|z_0\|+\frac{\eta+\mu+\mu^*}{\Gamma(\alpha+1)}+1$ such that $S_r=\{z\in C(J,X):\|z\|_j\le r\}$.

For $z,z^*\in S_r$ and $t\in J$, from the previous assumptions, we have

$$\begin{aligned}
&\|Pz(t)+Qz^*(t)\|\\
\le\ &\|z_0\|+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|h(z(s))\|\,ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,z^*(s))\|\,ds+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_s^t\|K(\tau,s,z^*(s))\|\,d\tau ds\\
\le\ &\|z_0\|+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sup_{z\in X}\|h(z)\|\,ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sup_{(s,z^*)\in J\times S_r}\|f(s,z^*(s))\|\,ds\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sup_{(\tau,s,z^*)\in D\times S_r}\int_s^t\|K(\tau,s,z^*(s))\|\,d\tau ds
\end{aligned}$$

$$\leq \|z_0\| + \frac{\eta t^\alpha}{\Gamma(\alpha+1)} + \frac{\mu t^\alpha}{\Gamma(\alpha+1)} + \frac{\mu^* t^\alpha}{\Gamma(\alpha+1)}$$
$$\leq \|z_0\| + \frac{\eta+\mu+\mu^*}{\Gamma(\alpha+1)}.$$

Consequently,

$$\|Pz + Qz^*\|_{j_1} \leq \|z_0\| + \frac{\eta+\mu+\mu^*}{\Gamma(\alpha+1)} < r.$$

This means that, $Pz + Qz^* \in S_r$.

Step(2) we prove that the operator P is a contraction map on S_r.

Let us make S_r as in step (1), for $z, z^* \in S_r$ and for $t \in J$, by the preceding assumptions, we have

$$\begin{aligned}
\|Pz(t) - Pz^*(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|h(z(s)) - h(z^*(s))\| \, ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M \|z(s) - z^*(s)\| \, ds \\
&\leq M \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{js} \max_{s\in J} e^{-js} \|z(s) - z^*(s)\| \, ds \\
&= M \left[I_0^\alpha e^{jt} \right] \|z - z^*\|_j \\
&= M t^\alpha E_{1,\alpha+1}(jt) \\
&\leq M \frac{e^{jt}}{j^\alpha} \|z - z^*\|_j \\
&\leq M e^{jt} \|z - z^*\|_j .
\end{aligned}$$

Since $M < 1$, we get

$$\|Pz - Pz^*\|_j \leq \|z - z^*\|_j .$$

So, P is contraction map on S_r.

Step(3) we show that the operator Q is completely continuous on S_r.

For this end, we consider S_r defined as in step (1) and we prove that (QS_r) is uniformly bounded, $(\overline{QS_r})$ is equicontinuous and $Q : S_r \to S_r$ is continuous.

Firstly, we show that (QS_r) is uniformly bounded.

For $z \in S_r$ and $t \in J$, then referring to previous assumptions, we have

$$\begin{aligned}
\|Qz(t)\| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,z(s))-f(s,0)\|\,ds+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,0)\|\,ds \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_s^t \|K(\tau,s,z(s))-K(\tau,s,0)\|\,d\tau ds \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_s^t \|K(\tau,s,0)\|\,d\tau ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Le^{\beta js}\|z\|_j^{\beta}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}Rds \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\int_s^t \rho(\tau)d\tau e^{\gamma js}\|z\|_j^{\gamma}\,ds+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}R^{*}ds \\
&\leq \left(L\|z\|_j^{\beta}+\|\rho\|_{L^1}\|z\|_j^{\gamma}\right)\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{js}ds+\frac{R+R^{*}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&\leq \left(Lr^{\beta}+\|\rho\|_{L^1}r^{\gamma}\right)\frac{e^{jt}}{(tj)^{\alpha}}+\frac{R+R^{*}}{\Gamma(\alpha+1)}t^{\alpha}.
\end{aligned}$$

Thus,

$$\|Qz\|_j \leq \frac{Lr^{\beta}+\|\rho\|_{L^1}r^{\gamma}}{j^{\alpha}}+\frac{R+R^{*}}{\Gamma(\alpha+1)} := \ell,$$

where $R=\sup_{s\in J}\|f(s,0)\|$ and $R^{*}=\sup_{(\tau,s)\in D}\int_s^t\|K(\tau,s,0)\|\,d\tau$. This means that $QS_r\subset S_\ell$, for any $z\in S_r$, i.e. the set $\{Qz : z\in S_r\}$ is uniformly bounded.

Next, we will prove that $(\overline{QS_r})$ is equicontinuous.

For $z\in S_r$ and for $t_1,t_2\in J$ with $t_1\leq t_2$, and also let $\delta=\left(\frac{\Gamma(\alpha+1)\epsilon}{2(\mu+\mu^{*})}\right)^{\frac{1}{\alpha}}$

then, when $|t_2 - t_1| < \delta$, we conclude that

$$\begin{aligned}
& \|Qz(t_2) - Qz(t_1)\| \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_0^{t_2} \left|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right| \|f(s, z(s))\| \, ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \|f(s, z(s))\| \, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left|(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right| \int_s^{t_2} \|K(\tau, s, z(s))\| \, d\tau ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \int_s^{t_2} \|K(\tau, s, z(s))\| \, d\tau ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} ((t_1 - s)^{\alpha-1} \int_{t_1}^{t_2} \|K(\tau, s, z(s))\| \, d\tau ds \\
\leq\ & \frac{1}{\Gamma(\alpha)} \int_0^{t_1} ((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}) \|f(s, z(s))\| \, ds + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \|f(s, z(s))\| \, ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} ((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}) \int_s^{t_2} \|K(\tau, s, z(s))\| \, d\tau ds \\
& + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \int_s^{t_2} \|K(\tau, s, z(s))\| \, d\tau ds \\
& + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} \left[\int_s^{t_2} \|K(\tau, s, z(s))\| \, d\tau - \int_s^{t_1} \|K(\tau, s, z(s))\| \, d\tau \right] ds \\
\leq\ & \left[\frac{(t_1^\alpha - t_2^\alpha) + 2(t_2 - t_1)^\alpha}{\Gamma(\alpha + 1)} \right] \mu + \left[\frac{(t_1^\alpha - t_2^\alpha) + 2(t_2 - t_1)^\alpha}{\Gamma(\alpha + 1)} \right] \mu^* \\
\leq\ & \frac{2(\mu + \mu^*)(t_2 - t_1)^\alpha}{\Gamma(\alpha + 1)} \\
<\ & \frac{2(\mu + \mu^*)\delta^\alpha}{\Gamma(\alpha + 1)} = \epsilon.
\end{aligned}$$

where μ and μ^* are defined as in step (1). Therefore, $(\overline{QS_r})$ is equicontinuous.

Finally, from the continuity of f and K, we can directly reach that the operator $Q : S_r \to S_r$.

As consequence of step 3 with Ascoli-Arzela theorem, we easily infer that (QS_r) is relatively compact set. Hence, the operator Q is complete continuous.

Thus, Krasnoselskii's fixed point theorem shows that the operator

$F = P + Q$ has a fixed point on S_r and hence the fractional integro-differential equation (1) − (2). has a solution $y(t) \in C(J, X)$. This proves the required. □

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