Journal of Mathematical Extension
Vol. 5, No. 2 (2), (2011), 13-21

Some Model Theoretic Remarks on Bass Modules

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Abstract. We study Bass modules, Bass rings, and related concepts from a model theoretic point of view. We observe that the class of Bass modules (over a fixed ring) is not stable under elementary equivalence. We observe that under which conditions the class of Bass rings are stable under elementary equivalence.

AMS Subject Classification: 16D90; 16D50.
Keywords and Phrases: Bass module, Max module, regular module, elementary equivalence, axiomatizability.

1. Introduction

In this paper by R we mean an associative ring with identity, by module we mean a unitary module and by R-Mod we mean the category of all left R-modules. By language we mean the first order language of modules over a fixed ring in the sense of [9] or [6]. Let R be a fixed ring and M and N belong to R-Mod. Then M and N is said to be *elementary equivalent* and denoted by $M \equiv N$ if M and N satisfy in the same (first order) sentences or equivalently when M satisfies in ϕ, N satisfies in ϕ as well, where by ϕ we mean a first order sentence in the language of modules over a fixed ring. We say that a (an algebraic) property P is an elementary property whenever $M \equiv N$ and M has the property P then N has the property P too (for a systematic study of these concepts see [9] and [6]). Throughout this article, by $\sigma[M]$, we mean the category

Received January 2011; Final Revised October 2011

of M-*subgenerated* modules or *Wisbauer category*[1]. The construction of $\sigma[M]$ is quite simple: for any module M, take direct sums $M^{(\Lambda)}$ for any index set Λ, factor modules of these (M-*generated modules*), and then submodules (M-*subgenerated modules*), hence Wisbauer category is the smallest Grothendieck category subgenerated by M. The reader is referred to [11], for a systematic study of module theory in the $\sigma[M]$ frame.

A submodule N of a nonzero module M is called maximal if it is a proper submodule and is not properly contained in any other proper submodule of M. A module M is called a *Max module* if every nonzero submodule of M has a maximal submodule and it is called a *Bass module* if every nonzero module in $\sigma[M]$ has a maximal submodule. The ring R is called left Max if ${}_RR$ is a Max module, i.e., if every nonzero left ideal contains a maximal left subideal. The ring R is called left Bass if ${}_RR$ is a Bass module. Bass (Max) rings and Bass (Max) modules have been extensively studied in the literature (see for example [10], [4] and [2] and their references). In [2], it has been observed that $\oplus M_i$ is a Max module if and only if $\prod M_i$ is a Max module, where $\{M_i\}$ is the family of left Max R-modules (see [2, 2.20]). On the other hand it is well known that direct sum of a family of modules (over a fixed ring R) is *elementary equivalent* with the direct product of the class. Now a natural question raise: Is the property of being a Bass (Max) module, an elementary property ? In the sequel we will answer the question in a negative way. While working with Bass modules and Bass rings it is very natural to think about the dual of these objects, that is, semi-artinian modules and semi-artinian rings. These important algebraic objects will be also studied in this note. We need the fourth part of the next result from [2, 2.21].

Theorem 1.1. *For M the following are equivalent:*

1. *M is a Bass module.*

2. *for every $0 \neq N \in \sigma[M]$, $RadN$ is small in N.*

[1] Here we follow a recent suggestion made by Patrik. F. Smith.

3. *every self-injective module in $\sigma[M]$ has a Maximal submodule.*
4. *the M-injective hull of any simple module in $\sigma[M]$ is a Max module.*
5. *There is a cogenerator in $\sigma[M]$ which is Max.*

We need also the following important result which is an example due to V. Camilo and K. Fuller (see [3] or [10, Example 2. 25]):

Example 1.2. There exists a right and left semi-Artinian ring R that is not a left Bass ring and is a regular ring with the minimum condition on ideals.
Now we are ready to prove our first result in this article.

Proposition 1.3. *The property of being a Bass module is not an elementary property.*

Proof. Let M be a module and $PE(M)$ the pure injective hull of M. It is well-known that every module is elementarily equivalent to its pure injective hull (see [9, 2.27]), hence M and $PE(M)$ are elementary equivalent. Now suppose that M is a regular module, then every module in $\sigma[M]$ is absolutely pure and so is pure (see [11, 35.2]). This implies that $PE(M) = E(M)$, where by $E(M)$ we mean the injective hull of M in $\sigma[M]$ and therefore $M \equiv E(M)$. Now for a moment suppose that R is the ring of Example 2 and $M =_R R$. Since $_RR$ is not a left Bass module, by Theorem 1, there exists a simple module S in $\sigma[M]$ such that $E(S)$ is not a Max module. It is clear that S is a Bass module. If the property of being a Bass module were an elementary property, then $E(S)$ had to be a Bass module, which is not the case (it is not even a Max module). $\square$

The above result gives rise to this natural question that when the class of Bass-rings are stable under elementary equivalence. We need the next two extra-results and some model theoretic notions to answer this question partially.
Let $\{M_\alpha\}_{\alpha\in I}$ be a family of R-modules. Furthermore, let $\mathcal{F}$ be an ultra-filter on I. In catesian product $\prod_{\alpha\in I} M_\alpha$ we introduce an equivalence

relation by setting $(m_\alpha) \sim (m'_\alpha)$ if the set $\{\alpha| \ m_\alpha = m'_\alpha\}$ belongs to $\mathcal{F}$. The equivalence class represented by an element (m_α) is denoted by $[m_\alpha]$. By componentwise operations these equivalence classes form a module, called the *ultraproduct* of $\{M_\alpha\}_{\alpha\in I}$ with respect to $\mathcal{F}$ is denoted by $\prod_{\alpha\in I} M_\alpha/\mathcal{F}$. A class $\mathcal{C}$ of rings (modules) is called *axiomatizable* if there exists a family of first order sentences in the corresponding language such that $\mathcal{C}$ consists exactly of the rings (modules) satisfying these first order sentences. According to the following well-known result there is a connection between axiomatizability on one hand and the ultraproduct and elementary equivalence. It is well-known that a class $\mathcal{C}$ of rings (modules) is axiomatizable if and only if $\mathcal{C}$ is closed under elementary equivalence and under formation of ultraproducts (see [6, Theorem 2.12]).

Theorem 1.4. *(Los's principal) Let $(R_\alpha)_{\alpha\in I}$ be a family of rings, (resp. fields, modules,$\cdots$) and $\mathcal{F}$ an ultrafilter on I. A first order sentence σ in the language of rings (resp. fields, modules,$\cdots$) holds for the ultraproduct $\prod_{\alpha\in I} R_\alpha/\mathcal{F}$ if and only if σ holds in R_α for almost all $\alpha \in I$.*

Theorem 1.5. *(R. Hamsher) Let R be a commutative ring, the following are equivalent:*

1. *R is a Bass ring;*
2. *$R/\mathrm{rad}(R)$ is von Neumann regular and $\mathrm{rad}(R)$ is T-nilpotent.*

Proposition 1.6. *Let R be a left Bass ring with identity. Consider the following assertions on R:*

1. *Every ring S, elementarily equivalent to R, is a left Bass ring.*
2. *$R^* = R^{\mathbb{N}}/\mathcal{F}$ is a left Bass ring for any non-trivial ultrafilter $\mathcal{F}$ on $\mathbb{N}$.*
3. *$\mathrm{rad}(R)$ is nilpotent.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Moreover (3) implies (1) if R is commutative.

Proof. (1)$\Rightarrow$ (2) By Los's principal is obvious.
(2) $\Rightarrow$ (3) From $R^* \equiv R$ we deduce that $R/\mathrm{rad}(R) \equiv R^*/\mathrm{rad}(R^*)$. Since R is a left Bass ring then so is $R/\mathrm{rad}(R)$. Now we claim that $\mathrm{rad}(R)$ is nilpotent. Suppose not, i.e., $\mathrm{rad}(R)$ is not nilpotent. This means that for any natural number n there is a nonzero product

$$a_{n0}a_{n1}a_{n2}\cdots a_{nn} \neq 0$$

of elements $a_{ij} \in \mathrm{rad}(R)$, $i \geqslant j$. Clearly this allows to define elements

$$a_k^* = [a_{nk}]_{n\geqslant k} \in \mathrm{rad}(R^*)$$

for any $k \in \mathbb{N}$ with the property $a_0^*a_1^*\cdots a_k^* \neq 0$ for all $k \in \mathbb{N}$, thus contradicting left nilpotency of rad(R*).
Suppose that R is commutative, we show that (3)$\Rightarrow$ (1) Notice first that $x \in \mathrm{rad}(R)$ means that $1 - rx$ is invertible in R for every $r \in R$. Hence $x \in (R)$ is definable by a first order formula. Since $(\mathrm{rad}(R))^t = 0$ means thar $x_1x_2\cdots x_t = 0$ for all $x_1, \cdots, x_t \in \mathrm{rad}(R)$, this also defines a first order property. On the other hand, by commutativity of R and Hamsher's Theorem (see [5]), we see that $R/\mathrm{rad}(R)$ is a von Neumann regular ring, which is also definable by a first order formula (i.e. for each $x \in R/\mathrm{rad}(R)$ there exists $y \in R/\mathrm{rad}(R)$ such that $xyx = x$). Now if S is a ring which is elementary equivalent to R (in oder words $S \equiv R$), then $S/\mathrm{rad}(S)$ is von Neumann regular and $\mathrm{rad}(S)$ is nilpotent.

Example 1.7. (Osofsky's example) This example can be served in the next result as an example of a left Bass ring whose Jacobson radical is not nilpotent. Let $(I, <)$ be any partially ordered set. Let R be the set of all $I \times I$ matrices $(a_{i,j})$ with coefficients in a field F such that $a_{i,j} = a_{j,j}$ for all $i, j \in I$, and if $i \neq j$ then $a_{i,j} = 0$ except for a finite set of pairs (i, j) where $i < j$. Then R is a ring under matrix addition and multiplication, and $N = \{(a_{i,j}) \in R \mid a_{i,i} = 0 \text{ for all } i \in I\}$ is the Jacobson radical of R. Then if I has d.c.c. (respectively a.c.c), then R is right (left)perfect (see [8]).

Corollary 1.8. *The class of left Bass rings are not stable under elementary equivalence.*

Proof. Let $\mathcal{B}$ be the class of all left Bass rings. Suppose that R is a member of $\mathcal{B}$ with non-nilpotent Jacobson radical (let us say Osofsky's example). By Proposition 6 there exists a ring S which is elementary equivalent to R but does not belong to $\mathcal{B}$ (otherwise $\mathrm{rad}(R)$ must be nilpotent).

We have already seen that the classes of Bass modules and Bass rings are not closed under elementary equivalence. Hence we come immediately to this conclusion that:

Corollary 1.9. *The following classes are not axiomatizable:*

1. *The class of left Bass modules over a fixed ring R.*
2. *The class of Bass rings.*

Remark 1.10. *The cartesian product of Bass rings is not necessarily a Bass ring. For example $\prod_{n=1}^{\infty} \mathbb{Z}_{2^n}$ is not a Bass ring because its Jacobson radical is not T-nilpotent (but it is a Max ring, see Lemma 12). Now let M be a Bass module, i.e., every $N \in \sigma[M]$ has a maximal submodule. If K is a submodule of M, then does it imply that every module in $\sigma[M/K]$ has a maximal submodule ? As suc as $M/K \in \sigma[M]$, the smallest Grothendieck category subgenerated by M/K is also a full subcategory of $\sigma[M]$, hence every memeber of $\sigma[M/K]$ has a maximal submodule since it is so in $\sigma[M]$.*

2. Max Modules and Max Rings

We conclude the previous section with this remark that Bass rings are not closed under cartesian products. But we have already observed that the factor modules (and submodules)of Bass modules are Bass modules. On the other hand, as we mentioned in the introduction, arbitrary direct products (and direct sums) of Max R-modules are Max R-modules. But contrary to Bass modules, factor modules of Max modules are not in general Max modules. The next examples show that there are Max rings which are not Bass rings.

Example 2.1. Since any Noetherian module is a Max module, any left noetherian ring is a left Max ring. On the otherhand, a ring is a perfect ring if and only if it is a semilocal Bass ring (see [11, 43.9]). Hence any noetherian semilocal ring that is not perfect is an example of a Max ring that is not a Bass ring (e.g. the localization of integers by two primes, say $\mathbb{Z}_{(3,5)}$). The second way to construct a left (right) Max ring which is not a left (right) Bass ring is as follows: find an infinite class of Bass rings, and construct their direct product, if the radical of their direct product is not T-nilpotent (which happens frequently as we see in Remark 10), then the direct product is an example of a left (right) Max ring (see Lemma 12) which is not a left (right) Bass ring.

Lemma 2.2. *Let $\{R_i\}_{i\in I}$ be an arbitrary family of left Max rings. Then $\prod_{i\in I} R_i$ is a left Max ring.*

Proof. Let $R = \prod_{i\in I} R_i$, then for every non-zero left ideal of U of R at least one of the projections $\pi_i : R \longrightarrow R_i$ should be non-zero. Looking at R_i as a left R-module, where all other components act as zero except the ith component that act as the multiplication in R_i, the projection is left R-linear and the image $\pi_i(U)$ is a left ideal of R_i. If R_i is a left Max ring, then there exists a maximal left ideal M in $\pi_i(U)$. If $\pi : \pi_i(U) \longrightarrow \pi_i(U)/M$ denotes the canonical projection, the composition $\pi\circ\pi_i : U \longrightarrow \pi_i(U)/M$ is an epimorphism of left R-modules to a simple R-module, i.e., U has a maximal left subideal. Hence R is a left Max ring. $\square$

Now it is natural to ask:

Question 2.3. *When (and under which condition(s)) is a left Max ring, a left Bass ring?*

Based on the previous facts it is very natural to ask if Max modules are closed under formation of ultraproducts.

Proposition 2.4. *The class of Max modules over a fixed ring R, and the class of Max rings are not closed under ultraproducts.*

3. Elementary Submodules of Max Modules and Bass Modules

Definition 3.1. *Let R be a ring and ${}_RM$ a left R-module. We say that a property P is preserved under* **elementary descent** *if (S,N) is an elementary submodule of (R,M) (that is $S \subseteq R$ ($N \subseteq M$) and $S \equiv R$ ($N \equiv M$)), then (S,N) has the property P whenever (R,M) does so.*

The following lemma is a useful tool in this section.

Lemma 3.2. *Suppose (S,N) is an elementary submodule of (R,M) and U is an S-submodule of N. Then the following holds:*

1. $U = N \cap RU$
2. *There is an R-submodule V of M with the property (S,U) is an elementary submodule of (R,V).*

Proof. See [6]. □

Since every submodule of a Max module is again a Max module and if M is a Bass module, then any module $N \in [M]$ is a Bass module. Hence we have:

Corollary 3.3. *Bass modules and Max modules are stable under elementary descent.*

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