Semi E-pseudoinvex and Semi E-quasiinvex Functions and Applications

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Abstract. In this paper, we introduce three new types of generalized convex functions, called semi E-pseudoinvex, strictly semi E-pseudoinvex and semi E-quasiinvex functions. Then some of their basic properties are studied, We used directional derivative and obtain the new results in this class of functions. As applications of our results, we obtain optimal solutions for multiobjective programming.

1. Introduction

Convexity and generalized convexity play a central role in mathematical economics, engineering, management sciences and optimization. In recent years several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is invex functions introduced by Hanson 1981. His initial result inspired a great deal of subsequent work which has greatly expanded the role and applications of invexity in nonlinear optimization and other branches of

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pure and applied sciences. In fact he has shown that the Kuhn-Tuker conditions are sufficient for optimality of nonlinear programming problems under invexity conditions. Some different kinds of invexity and their characterizations have been studied in Garzon et al.2003, Jabarootian and Zafarani 2006, Luo and, Xu (2004), Mohan and Neogy 1995, Yang 2001, Yang, and Li 2001, Yang, et al. 2001. Another recent generalization of convex functions is E-convex functions introduced by Youness 1999 and many results for convex sets and convex functions actually hold for a wider class of sets and functions, called E-convex set and E-convex functions, see also Yang 2001. In this paper, by introducing the concept of E-invexity, we unify these two notions of generalized convexity. Furthermore, we obtain optimal solutions for multiobjective programming problems to illustrate the applications of our results.

2.   E-invex Sets and Semi E-preinvex Functions

Let X be a normed linear space endowed with a norm $\|.\|$. Let M be a non-empty subset of X, $\eta : X \times X \to X$, $E : X \longrightarrow X$, and $f : X \to \mathbb{R}^m$ be vector-valued functions.

Throughout this paper, the following convention for vectors
$x = (x_1, ..., x_m),\ y = (y_1, ..., y_m) \in \mathbb{R}^m$ will be followed:

$x < y$ if and only if $x_i < y_i,\ \ i = 1, 2, ..., m;$

$x \leqq y$ if and only if $x_i \leqslant y_i,\ \ i = 1, 2, ..., m;$

$x \leqslant y$ if and only if $x_i \leqslant y_i,\ \ i = 1, 2, ..., m,$ but $\ x \neq y.$

Definition 2.1. *(a) Let us recall that a subset M of X is said to be invex with respect to $\eta : X \times X \to X$ if, for any $x, y \in M$ and $\lambda \in [0, 1]$, $y + \lambda \eta(x, y) \in M$.*

(b) A subset M of X is said to be E-invex with respect to η if there exist, a vector valued function $E : X \longrightarrow X$ such that for any $x, y \in M$ and $\lambda \in [0, 1]$,

$$E(y) + \lambda \eta(E(x), E(y)) \in M. \tag{1}$$

Remark 2.2. *(a) If $E : X \longrightarrow X$ is the identity map and, M is an E-*

invex subset of X with respect to η, then M is an invex set with respect to η.

(b) If $\eta(x, y) = x - y$ and, M is an E-invex subset of X with respect to η, then M is a E-convex subset of X.

(c) If a set $M \subset X$ is E-invex with respect to η, then $E(M) \subset M$.

Example 2.3. Let $M = [0, 2] \cup [5, 7]$,

$$\eta(x, y) = x - y \qquad \text{if} \quad x, y \in [0, 2] \text{ or } x, y \in [5, 7];$$

$$= 5 - y \qquad \text{if} \quad x \in [0, 2] \text{ and } y \in [5, 7];$$

$$= 2 - y \qquad \text{if} \quad x \in [5, 7] \text{ and } y \in [0, 1];$$

$$= x - y \qquad \text{if} \quad x \in [5, 7] \text{ and } y \in [1, 2];$$

and define $E : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$E(x) = x/2 \qquad \text{if} \quad x < 4;$$

$$= x \qquad \text{if} \quad x \geqslant 4.$$

Then the set M is not convex and not invex with respect to η. $E(M) = [0, 1] \cup [5, 7]$, therefore M is not E-convex and $E(M)$ is not invex with respect to η, however M is an E-invex set with respect to η.

Definition 2.4. *Let M be an E-invex subset of a normed linear space X, then a function $f : M \longrightarrow \mathbb{R}^m$ is said to be semi E-preinvex with respect to η on M, if for any $x, y \in M$ and $\lambda \in [0, 1]$, one has*

$$f(E(y) + \lambda \eta(E(x), E(y))) \leqq \lambda f(x) + (1 - \lambda) f(y); \qquad (2)$$

Example 2.5. Let $U = [0, 1] \cup [2, 3] \cup [4, 5]$,

$$\eta(x, y) = 1 - y \qquad \text{if} \quad 0 \leqslant y \leqslant 1 \text{ and} \quad 2 \leqslant x \leqslant 3;$$

$$= 3 - y \qquad \text{if} \quad 2 \leqslant y \leqslant 3 \text{ and} \quad 0 \leqslant x \leqslant 1;$$

$$= x - y \qquad \qquad \text{other ways,}$$

and $E : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as

$$E(x) = 0 \qquad\qquad \text{if} \quad 0 \leqslant x \leqslant 1;$$

$$= 3 - x \qquad\qquad \text{if} \quad 2 \leqslant x \leqslant 3;$$

$$= x - 2 \qquad\qquad \text{if} \quad 4 \leqslant x \leqslant 5;$$

$$= x \qquad\qquad\qquad \text{other\ \ ways}$$

Then set U is E-invex with respect to η and

$$E(U) = [0, 1] \cup [2, 3] \subset U.$$

Now let $f : U \longrightarrow \mathbb{R}$ be defined as

$$f(x) = x \qquad\qquad \text{if} \quad 0 \leqslant x \leqslant 1;$$

$$= 3 - x \qquad\qquad \text{if} \quad 2 \leqslant x \leqslant 3;$$

$$= x - 3 \qquad\qquad \text{if} \quad 4 \leqslant x \leqslant 5,$$

Thus, function f is a semi E-preinvex function with respect to η on U.

Proposition 2.6. *Let M be an E-invex subset of a normed linear space X. If function $f : M \longrightarrow \mathbb{R}^m$ is semi E-preinvex, then for any vector $r \in \mathbb{R}^m$, the level set $S(f, r) = \{x \in M | f(x) \leqq r\}$ is an E-invex set.*

Proof. Suppose that $r \in \mathbb{R}^m$ and $x, y \in S(f, r)$, then for each $\lambda \in [0, 1]$, we have

$$f(E(y) + \lambda \eta(E(x), E(y))) \leqq \lambda f(x) + (1 - \lambda) f(y) \leqq r.$$

This means that $E(y) + \lambda \eta(E(x), E(y)) \in S(f, r)$ and therefore the set $S(f, r)$ is E-invex. $\square$

Lemma 2.7. *Assume that $E : X \longrightarrow X$ is a vector valued function, and $(E \times I) : X \times \mathbb{R}^m \longrightarrow X \times \mathbb{R}^m$ is defined by $(E \times I)(x, r) = (E(x), r)$. Then, $M \subset X$ is E-invex with respect to η if and only if $M \times \mathbb{R}^m$ is $(E \times I)$-invex with respect to η', where $\eta'((x, r), (y, s)) = (\eta(x, y), r - s)$, for all $(x, r), (y, s) \in M \times \mathbb{R}^m$.*

Proof. $M \subset X$ is E-invex with respect to η if and only if for each $x, y \in M$, and $\lambda \in [0, 1]$,

$$E(y) + \lambda \eta(E(x), E(y)) \in M,$$

if and only if for each $x, y \in M$, $\lambda \in [0, 1]$, and $r, s \in \mathbb{R}^m$,

$$(E(y) + \lambda \eta(E(x), E(y)), s + \lambda(r - s)) \in M \times \mathbb{R}^m.$$

This equal with, for each $(x, r), (y, s) \in M \times \mathbb{R}^m$, and $\lambda \in [0, 1]$,

$$(E(y), s) + \lambda \eta'((E(x), r), (E(y), s)) \in M \times \mathbb{R}^m.$$

Hence, $M \times \mathbb{R}^m$ is $(E \times I)$-invex with respect to η'. $\square$

In the following result, we obtain a characterization of semi E-preinvex function f in term of invexity of its epi(f), where

$$\mathrm{epi}(f) = \{(x, r) \in M \times \mathbb{R}^m : f(x) \leqq r\}.$$

Theorem 2.8. *Suppose that M is an E-invex subset of a normed linear space X. Then f is a semi E-preinvex function with respect to η on M, if and only if its epi(f) is $(E \times I)$-invex with respect to η'.*

Proof. Assume that f is a semi E-preinvex function with respect to η on M. If $(x, r), (y, s) \in \mathrm{epi}(f)$ and $\lambda \in [0, 1]$, then by semi E-preinvexity of f we have

$$f(E(y) + \lambda \eta(E(x), E(y))) \leqq \lambda f(x) + (1 - \lambda) f(y) \leqq \lambda r + (1 - \lambda) s.$$

Thus $(E(y) + \lambda \eta(E(x), E(y)), \lambda r + (1 - \lambda) s) \in \mathrm{epi}(f)$, hence

$$(E \times I)(y, s) + \lambda \eta'((E \times I)(x, r), (E \times I)(y, s)) \in \mathrm{epi}(f).$$

Conversely, suppose that epi(f) is $(E \times I)$-invex with respect η'. If $x, y \in M$ and $\lambda \in [0, 1]$, then $(x, f(x)), (y, f(y)) \in \mathrm{epi}(f)$ and by $(E \times I)$-invexity of epi(f) with respect to η', we deduce

$$(E(y) + \lambda \eta(E(x), E(y)), \lambda f(x) + (1 - \lambda) f(y))$$

$$= (E(y), f(y)) + \lambda\eta'((E(x), f(x)), (E(y), f(y))) \in \mathrm{epi}(f).$$

Therefore,

$$f(E(y) + \lambda\eta(E(x), E(y))) \leqq \lambda f(x) + (1 - \lambda)f(y).$$

This means that f is a semi E-preinvex function with respect to η. $\square$

3.   Semi E-pseudoinvex and Semi E-quasiinvex Functions

Let $f : M \longrightarrow \mathbb{R}^m$ throughout the paper, if the following limit exist, we consider the directional derivative of function f in direction $\eta(x, y)$ at point y by;

$$f'(y, \eta(x, y)) = \lim_{\lambda \longrightarrow 0^+} \frac{f(y + \lambda\eta(x, y)) - f(y)}{\lambda},$$

when the limit exists.

Definition 3.1. *Let M be a subset of a normed linear space X, then a function $f : M \longrightarrow \mathbb{R}^m$ is said to be*
(a) semi E-pseudoinvex with respect to η at a point $y \in M$ if for any $x \in M$, one has

$$f(x) - f(y) \leqslant 0 \implies f'(E(y), \eta(E(x), E(y))) \leqslant 0. \tag{3}$$

(b) strictly semi E-pseudoinvex with respect to η at a point $y \in M$ if for any $x \in M$, one has

$$f(x) - f(y) < 0 \implies f'(E(y), \eta(E(x), E(y))) < 0. \tag{4}$$

(c) semi E-quasiinvex with respect to η at a point $y \in M$ if for any $x \in M$, one has

$$f(x) - f(y) \leqq 0 \implies f'(E(y), \eta(E(x), E(y))) \leqq 0. \tag{5}$$

Proposition 3.2. *If f is a semi E-preinvex function with respect to*

η on M, directional derivative of function f at y exists, and $f(E(y)) = f(y)$. Then f is semi E-pseudoinvex, strictly semi E-pseudoinvex and semi E-quasiinvex with respect to η at y.

Proof. Let f be semi E-preinvex with respect to η on M, then

$$f'(E(y), \eta(E(x), E(y))) = \lim_{\lambda \longrightarrow 0^+} \frac{f(E(y) + \lambda \eta(E(x), E(y))) - f(E(y))}{\lambda}.$$

$$\leqq \lim_{\lambda \longrightarrow 0^+} \frac{\lambda f(x) + (1 - \lambda)f(y) - f(E(y))}{\lambda} = f(x) - f(y).$$

Then the results are proved. $\square$

Example 3.3. In Example 2.5., for each $y \in (2, 3)$ we have $f(E(y)) = f(y)$ and function f is semi E-preinvex, therefore f is semi E-pseudoinvex, strictly semi E-pseudoinvex and semi E-quasiinvex with respect to η at y.

4.   Applications

As a consequence of our results in preceding sections, now we obtain optimal solutions for a multiobjective programming problem.
Consider the following multiobjective programming problem:

$$\min f(x)$$

$$(P) \qquad\qquad g(x) \leqq 0$$

$$x \in X$$

where X be a normed linear space, $f : X \longrightarrow \mathbb{R}^m$, $g : X \longrightarrow \mathbb{R}^n$, are vector valued functions on X, and $M = \{x \in X | g(x) \leqq 0\}$ is the set of feasible solution of (P).

Definition 4.1. *A point $\bar{x} \in M$ is called a global efficient solution of (P) if there does not exist any point $y \in M$ such that $f(y) \leqslant f(\bar{x})$. A point $\bar{x} \in M$ is called a local efficient solution of (P) if there is a neighborhood $N(\bar{x})$ of $\bar{x}$ such that there does not exist any point $y \in M \cap N(\bar{x})$ such that $f(y) \leqslant f(\bar{x})$.*

Definition 4.2. *A point $\bar{x} \in M$ is called a global weakly efficient solution of (P) if there does not exist any point $y \in M$ such that $f(y) < f(\bar{x})$. A point $\bar{x} \in M$ is called a local weakly efficient solution of (P) if there is a neighborhood $N(\bar{x})$ of $\bar{x}$ such that there does not exist any point $y \in M \cap N(\bar{x})$ such that $f(y) < f(\bar{x})$.*

Theorem 4.3. *Let $\bar{x}$ be a feasible solution for (P), directional derivative of f and g at $E(\bar{x}) \in M$ exist and there exists $\bar{\mu} \in \mathbb{R}_+^n$ such that $(\bar{x}, \bar{\mu})$ satisfies the following conditions:*
(i) $f'(E(\bar{x}), \eta(E(x), E(\bar{x}))) + \bar{\mu}g'(E(\bar{x}), \eta(E(x), E(\bar{x}))) \geqq 0, \ \forall x \in X$,
(ii) $\bar{\mu}^T g(E(\bar{x})) = 0$,
(iii) $g(E(x)) \leqq g(x)$ for all $x \in M$.
Then,
(a) If f is semi E-pseudoinvex and $\bar{\mu}^T g$ is semi E-quasiinvex at $\bar{x} \in M$ then $\bar{x}$ is a global efficient solution of (P),
(b) If f is strictly semi E-pseudoinvex and $\bar{\mu}^T g$ is semi E-quasiinvex, at $\bar{x} \in M$ then $\bar{x}$ is a global weakly efficient solution of (P).

Proof. (a) We proceed by contradiction. Assume that $\bar{x}$ is not a global efficient solution of (P). Then there is a feasible solution x of (P) such that $f(x) \leqslant f(\bar{x})$, then by semi E-pseudoinvexity of f we deduce

$$f'(E(\bar{x}), \eta(E(x), E(\bar{x}))) \leqslant 0. \tag{6}$$

On the other hand by feasibility of x and (ii), we get

$$\bar{\mu}^T(g(E(x)) - g(E(\bar{x}))) \leqq 0.$$

Then by semi E-quasiinvexity of $\bar{\mu}^T g$ we deduce

$$\bar{\mu}^T g'(E(\bar{x}), \eta(E(x), E(\bar{x}))) \leqq 0. \tag{7}$$

By (6) and (7) we deduce

$$f'(E(\bar{x}), \eta(E(x), E(\bar{x}))) + \bar{\mu}^T g'(E(\bar{x}), \eta(E(x), E(\bar{x}))) \leqslant 0. \tag{8}$$

Which contradicts (i).
(b) We proceed by contradiction. Assume that $\bar{x}$ is not a global weakly

efficient solution of (P). Then there is a feasible solution x of (P) such that $f(x) < f(\bar{x})$, then by strictly semi E-pseudoinvexity of f we deduce

$$f'(E(\bar{x}), \eta(E(x), E(\bar{x}))) < 0. \tag{9}$$

By (9) and (7) we deduce

$$f'(E(\bar{x}), \eta(E(x), E(\bar{x}))) + \bar{\mu}^T g'(E(\bar{x}), \eta(E(x), E(\bar{x}))) < 0. \tag{10}$$

Which contradicts (i). $\square$

In relation to (P), we consider the following dual problem which is in the form of Mond-Weir(1981).

$$(MWD) \qquad \max f(y)$$

$$s.t. \qquad (\zeta^T f' + \mu^T g')(E(y), \eta(E(x), E(y))) \geqq 0, \quad \text{for al1l} \ x \in M$$

$$\mu^T g(E(y)) \geqq 0,$$

$$\zeta^T e = 1,$$

$$\zeta \geqslant 0, \mu \geqslant 0,$$

where $e = (1, 1, ..., 1) \in \mathbb{R}^m$. Let

$$W = \{(y, \zeta, \mu) \in X \times \mathbb{R}^m \times \mathbb{R}^n : (\zeta^T f' + \mu^T g')(E(y), \eta(E(x), E(y))) \geqq 0,$$

$$\mu^T g(E(y)) \geqq 0, \zeta^T e = 1, \zeta \geqslant 0, \mu \geqslant 0\},$$

denotes the set of all the feasible solutions of (MWD).

We denote by $pr_X W$ the projection of, the set W on X.

Theorem 4.4. *Suppose that directional derivatives of f and g on M exist and $g(E(x)) \leqq g(x)$ for all $x \in M$. Let $\bar{x}$ and (y, ζ, μ) be feasible solutions for (P) and (MWD), respectively.*

(a) If f is semi E-pseudoinvex, and $\bar{\mu}^T g$ are semi E-quasiinvex, at y on $M \cap pr_X W$ then, $f(x) \nleqq f(y)$.

(b) If f is strictly semi E-pseudoinvex and $\bar{\mu}^T g$ is semi E-quasiinvex, at y on $M \cap pr_X W$ then, $f(x) \nless f(y)$.

Proof.   (a) We proceed by contradiction. Assume that $f(x) \leqslant f(y)$, then by semi E-pseudoinvexity of f we deduce

$$f'(E(y), \eta(E(x), E(y))) \leqslant 0. \tag{11}$$

Since x is feasible for (P) and (y, ζ, μ) is a feasible solution for (MWD), we get

$$\mu^T(g(E(x)) - g(E(y))) \leqq 0.$$

Then by semi E-quasiinvexity of $\bar{\mu}^T g$ we deduce

$$\mu^T g'(E(y), \eta(E(x), E(y))) \leqq 0. \tag{12}$$

By (11) and (12) we deduce

$$\zeta^T f'(E(y), \eta(E(x), E(y))) + \mu^T g'(E(y), \eta(E(x), E(y))) \leqslant 0. \tag{13}$$

Which contradicts with the feasibility of (y, ζ, μ).
(b) We proceed by contradiction. Assume that $f(x) < f(y)$, then by strictly semi E-pseudoinvexity of f we deduce

$$f'(E(y), \eta(E(x), E(y))) < 0. \tag{14}$$

By (12) and (14) we deduce

$$\zeta^T f'(E(y), \eta(E(x), E(y))) + \mu^T g'(E(y), \eta(E(x), E(y))) \leqslant 0. \tag{15}$$

Which contradicts with the feasibility of (y, ζ, μ). $\square$

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