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A Sequence of Fourier Partial Sums not Containing $2\pi\mathbb{Q}$

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Abstract. In this paper, we focus on the bounded sequences of Fourier partial sums. Our interest is on the sequences which do not contain any rational multiple of 2π. We construct a function on $\mathbb{C}[\mathbb{T}]$ where its set of points is one of such sequences. We will show that this set is of the first category in $\mathbb{R}$. Moreover the complement of this set in any arbitrary real closed interval form an uncountable set.

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1. Introduction

In what follows $\mathbb{R}$ and $\mathbb{Z}$ are respectively the sets of real numbers and integers. Define the *circle group* to be $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and we shall regard a function on $\mathbb{T}$ as being a 2π-periodic function on $\mathbb{R}$ and vice-versa. Let $p \geqslant 1$ be a real number and for any complex measurable function f on $\mathbb{T}$, let $\|f\|_p$, $\|f\|_u$, and $\|f\|_\infty$ be the usual, sup, and infinity norms respectively. We also define $L_p(\mathbb{T})$, the set of all measurable functions f on $\mathbb{T}$ so that $\|f\|_p < \infty$ and $C(\mathbb{T})$, the set of all continuous functions on $\mathbb{T}$ (or continuous 2π-periodic on $\mathbb{R}$).
A *trigonometric series* is any series of the form

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$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad c_n \in \mathbb{C}, \qquad x \in \mathbb{R}. \tag{1}$$

The n-th partial sum of this series is

$$S_n(x) = \sum_{j=-n}^{n} c_n e^{ijx}. \tag{2}$$

In view of the Euler's formula, we also have

$$S_n(x) = \frac{a_0}{2} + \sum_{j=1}^{n} (a_j \cos(jx) + b_j \sin(jx)), \tag{3}$$

where $a_n = c_n + c_{-n}$ and $b_n = i(c_n - c_{-n})$. So we also write (1) as $\frac{a_0}{2} + \sum_{j=1}^{\infty} (a_j \cos(jx) + b_j \sin(jx))$. Any function of the form (2) or (3) is called a *trigonometric polynomial.* In case $|c_n| + |c_{-n}| \neq 0$ (or $|a_n| + |b_n| \neq 0$), we say that s_n has *order* n. The proof of the following theorem can be found in any of the references [2], [5] or [8].

Theorem 1.1. *Consider any trigonometric series having partial sums of type (2) or (3). Suppose that there is a function f and a subsequence S_{n_k} such that either*
(i): $f \in C(\mathbb{T})$ *and* $lim_k \| f - S_{n_k} \|_u = 0$ *or*
(ii): $f \in L_p(\mathbb{T})$ *for some* $1 \leqslant p \leqslant \infty$ *and* $lim_k \| f - S_{n_k} \|_p = 0$.
Then for each $r \in \mathbb{Z}$, c_r in (2) equals to $\frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-irx} dx$. *Moreover for appropriate r, we have* $a_r = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos rx dx$ *and* $b_r = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin rx dx$.
The numbers a_r, b_r, and c_r are called the Fourier coefficients of f. Denote $a_r(f) = a_r$, $b_r(f) = b_r$, and $c_r(f) = c_r$. We also write $f^{\wedge}(j) = c_j$. Thus $f^{\wedge}$ is a complex function on $\mathbb{Z}$. The trigonometric series having the Fourier coefficients of f as coefficients is called a Fourier series of f and is denoted by $S(f)$. The partial sums of the Fourier series of f are written as

$$S_n(f, x) = \sum_{k=-n}^{n} f^{\wedge}(k) e^{ikx}.$$

2. The Main Result

Theorem 2.1. *If there exists a real-valued even function $f \in C(\mathbb{T})$ for which $B = \{x \in \mathbb{R} : \sup_{n \geqslant 0} |S_n(f, x)| < \infty\}$, then B is of the first category in $\mathbb{R}$ and does not contain any rational multiple of 2π. Moreover if $\alpha < \beta$ in $\mathbb{R}$, then the set $[\alpha, \beta] - B$ is uncountable.*

Note that B in the statement of theorem is the set of points that the Fourier partial sums of f form a bounded sequence. Before proof of theorem we need the following lemma.

Lemma 2.2. *Let $\{b_n\}_{n \geqslant 1}$ be a decreasing sequence of positive real numbers and for each $k \geqslant 0$, put $\delta_k = \sup\limits_{j>k} b_j$. If $0 \leqslant k < n$, then*

$$|\sum_{j=k+1}^{n} b_j \sin jx| \leqslant (\pi + 2)\delta_k, \quad \text{for all } \ x \in \mathbb{R}.$$

Proof. For any $j \geqslant 1$, let $\Delta b_j = b_j - b_{j+1}$ (so $\Delta b_j \geqslant 0$). For any $n \geqslant 1$, define $f_n(x) = \sum\limits_{j=1}^{n} \sin jx$ and put $f_0(x) = 0$. Then

$$f_n(x) = \frac{\cos \frac{1}{2}x - \cos(n + \frac{1}{2})x}{2 \sin \frac{1}{2}x}, \quad \text{for all } \ x \in \mathbb{R} \backslash 2\pi\mathbb{Z}.$$

Note that for every j we have $b_j \sin jx = \Delta b_j f_j(x) + (b_{j+1} f_j(x) - b_j f_{j-1}(x))$ and so

$$\begin{aligned} |\sum_{j=k+1}^{n} b_j \sin jx| &= |\sum_{j=k+1}^{n} \Delta b_j f_j(x) + b_{n+1} f_n(x) - b_{k+1} f_k(x)| \\ &\leqslant \sum_{j=k+1}^{n} \Delta b_j |f_j(x)| + b_{n+1} |f_n(x)| + b_{k+1} |f_k(x)|. \end{aligned}$$

Therefore for every $x \in \mathbb{R} \backslash 2\pi\mathbb{Z}$ we have

$$\begin{aligned} |\sum_{j=k+1}^{n} b_j \sin jx| &\leqslant \frac{2}{2\sin\frac{1}{2}x}[\sum_{j=k+1}^{n} \Delta b_j + b_{n+1} + b_{k+1}] \\ &= \frac{1}{\sin\frac{1}{2}x}[b_{k+1} - b_{n+1} + b_{n+1} + b_{k+1}]. \end{aligned}$$

Thus for every $x \in \mathbb{R}\backslash 2\pi\mathbb{Z}$,

$$|\sum_{j=k+1}^{n} b_j \sin jx| \leqslant \frac{2b_{k+1}}{\sin\frac{1}{2}x}. \tag{4}$$

Suppose that $0 \leqslant k < n$ and fix $x \in \mathbb{R}$ so that $0 < x < \pi$. Choose the positive integer m so that $\frac{\pi}{m+1} \leqslant x < \frac{\pi}{m}$. If $0 \leqslant k < n \leqslant m$, then

$$\begin{aligned} |\sum_{j=k+1}^{n} b_j \sin jx| &\leqslant \sum_{j=k+1}^{n} b_j |\sin jx| \leqslant \sum_{j=k+1}^{n} b_j x \\ &\leqslant \delta_k \sum_{j=k+1}^{n} x \leqslant \delta_k \sum_{j=k+1}^{n} \frac{\pi}{m} \leqslant \pi\delta_k. \end{aligned} \tag{5}$$

If $0 \leqslant m \leqslant k < n$, then

$$\begin{aligned} |\sum_{j=k+1}^{n} b_j \sin jx| &\leqslant \frac{2b_{k+1}}{\sin\frac{1}{2}x} \leqslant \frac{2\pi b_{k+1}}{x} \\ &\leqslant 2(m+1)b_{k+1} \leqslant 2(k+1)b_{k+1} \leqslant 2\delta_k. \end{aligned} \tag{6}$$

Finally, if $0 \leqslant k < m < n$, we combine (5) and (6) to get

$$\begin{aligned} |\sum_{j=k+1}^{m} b_j \sin jx| &\leqslant |\sum_{j=m+1}^{n} b_j \sin jx| + |\sum_{j=k+1}^{n} b_j \sin jx| \\ &\leqslant \pi\delta_k + 2\delta_k = (\pi+2)\delta_k. \quad \square \end{aligned}$$

Now we are able to prove Theorem 2.1.

Proof of Theorem 2.1. For each positive integer n and $x \in \mathbb{R}$, put

$$q_n(x) = 2\sin(2nx)\sum_{j=1}^{n}\frac{\sin(jx)}{j}.$$

By the Lemma 2.2., for every $x \in \mathbb{R}$ and $n \geqslant 1$, we have

$$|q_n(x)| \leqslant 2(\pi + 2).$$

For each n and $x \in \mathbb{R}$, put $p_n(x) = S_{2n}(q_n)$. Then by the definition, we have

$$p_n(x) = \sum_{j=n}^{2n-1}\frac{\cos jx}{2n - j}.$$

Moreover, since $\log\frac{k+1}{k} < \frac{1}{k}$ for every k, we have $p_n(0) > \log n$.
Next, let $\{n_k\}$ be a sequence of positive integers such that $(k+1)n_{k+1} > 3n_k$ and $\limsup \frac{1}{k^2}\log n_k = \infty$ (such sequence exists for instance $n_k = 2^{k^3}$). Next for each k let $f_k(x) = q_{n_k}(k!x)$ and define

$$f(x) = \sum_{k=1}^{\infty}\frac{1}{k^2}f_k(x).$$

Since $|f_k(x)| \leqslant 2(\pi + 2)$ for every k and every $x \in \mathbb{R}$, the above series converges uniformly on $\mathbb{R}$ and hence $f \in C(\mathbb{T})$. Also for all $n \geqslant 0$, we have

$$S_n(f) = \sum_{k=1}^{\infty}\frac{1}{k^2}S_n(f_k).$$

Next, writing

$$q_n(x) = \sum_{j=n}^{2n-1}\frac{\cos(jx)}{2n - j} - \sum_{j=2n+1}^{3n}\frac{\cos(jx)}{j - 2n},$$

we see that for all $r \in \mathbb{Z}$, $f_k^{\wedge}(r) = 0$ unless $|r| = k!j$, where $n_k \leqslant j \leqslant 3n_k$. Therefore, by The Theorem 1.1.,

$$S_n(f_k) = \begin{cases} 0 & \text{if } n < k!n_k \\ f_k & \text{if } n \geqslant k!n_k. \end{cases} \tag{7}$$

By the definition of p_n, we have $S_{k!2n_k}(f_k, x) = \frac{1}{k^2} p_{n_k}(k!x)$. So if for each positive integer r we define $p_r = r!n_r - 1$ and $q_r = r!2n_r$, then

$$S_{q_r}(f_k, x) - S_{p_r}(f_k, x) = \begin{cases} p_{n_k}(k!x) & \text{if } r = k \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

Thus $S_{q_r}(f_k, x) - S_{p_r}(f_k, x) = \frac{1}{r^2} p_{n_r}(r!x)$. Now suppose that a is an integer and b is a positive integer. Then since $P_n(0) > \log n$, we have

$$S_{q_r}(f_k, \frac{2a\pi}{b}) - S_{p_r}(f_k, \frac{2a\pi}{b}) = \frac{1}{r^2} P_{n_r}(0) > \frac{1}{r^2} \log n_r. \tag{9}$$

The right side of (9) approach to ∞ as $r \to \infty$ (because $\overline{\lim} \frac{1}{k^2} \log n_k = \infty$) and so $\frac{a}{b} \times 2\pi$ is not in B. This means that B does not contain any rational multiple of 2π.

To show that B is of the first category in $\mathbb{R}$, for each positive integer k, let

$$B_k = \cap_{n=1}^{\infty} \{x \in \mathbb{R} : |S_n(f, x)| \leqslant k\}.$$

Clearly each B_k is closed in $\mathbb{R}$ and $B = \cup_{k=1}^{\infty} B_k$.

Finally, if there are real numbers $\alpha < \beta$ so that $\{x_k\}_{k=1}^{\infty}$ is the set of points in $[\alpha, \beta]$ which are not in B, then each of the sets $A_k = (B_k \cap [\alpha, \beta]) \cup \{x_k\}$ is nowhere dense in $[\alpha, \beta]$ and $\cup_1^{\infty} A_k = [\alpha, \beta]$. Since $[\alpha, \beta]$ is a complete metric space, this contradicts the Bair's category theorem. $\square$

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