Journal of Mathematical Extension
Vol. 5, No. 2 (1), (2011), 101-119

Measuring the Relative Efficiency in Multi-Component Decision Making Units and its Application to Bank Branches

A. A. Noora
Sistan and Baluchestan University

F. Hosseinzadeh Lotfi
Science and Research Branch, Islamic Azad University

A. Payan
Zahedan Branch, Islamic Azad University

Abstract. In many cases of data envelopment analysis (DEA), decision making units (DMUs) can be separated into different components. These DMUs are called multi-component DMUs, and studying them is known as multi-component DEA. In multi-component DEA some inputs are shared among the components of a DMU, and some components involve into producing some outputs of the DMU. In this paper, we survey measuring the relative efficiency in multi-component DEA. It is shown that using common idea for measuring the efficiency of multi-component DMUs, the relative efficiency of an evaluating DMU may be not obtained. Therefore, present paper proposes a new DEA model which can obtain the relative efficiencies of multi-component DMUs. Some facts about the proposed approach are also provided by theorems. Moreover, the proposed DEA model is compared to another approach in literature utilizing a set of data about 19 bank branches.

AMS Subject Classification: 90C05.

Keywords and Phrases: Data envelopment analysis (DEA), multi-component decision making units, relative efficiency, absolute efficiency.

Received September 2010; Final Revised February 2011

1. Introduction

Performance evaluation is one of the most serious concerns for managers, since it can be utilized as a reference in decision making with regard to performance improvement. Performance is conventionally defined either as organizational outputs or inputs, or as a relationship between them. Because the evaluation characteristics are generally multi-dimensional, there is no appropriate aggregation schema for them and the basic problem of performance measurement is how to evaluate the relative performance of units. To overcome this difficulty, data envelopment analysis (DEA) is a widely employed technique for efficiency evaluation within a group of decision making units (DMUs).

DEA is a mathematical programming technique, which is used to evaluate the relative efficiency of homogeneous DMUs on the basic of multiple inputs and outputs and has been suggested by Charnes et al. ([3]) (CCR model). The concept of the relative efficiency in the CCR model ([3]) was indicated by Thompson et al. ([14]) (Maximin DEA model). Then, CCR model ([3]) has been expanded by Banker et al. ([2]) (BCC model). DEA is an important analysis tool and research way in management science, operational research, system engineers, decision analysis and so on. A thorough review upon DEA to 2009 can be found in Cook and Seiford ([4]).

In many DEA models, DMUs have different tasks and so are divided into different components. In this situation, some inputs are shared among some components. Some components are also shared to produce some outputs. DMUs with this structure are called multi-component DMUs. For example, universities, as decision making units, have two education and research tasks. Teacher as input factor shares into education and research tasks. So, universities can be considered as multi-component DMUs. Bank branches can be also considered as multi-component DMUs ([1, 5]). Cook et al. ([5]) suggested a method to measure the efficiency of multi-component DMUs with shared inputs. Cook and Green ([6]) also proposed an approach to obtain the efficiency in multiplant firms that were considered as multi-component DMUs. Jahanshahloo et al. ([8]) provided a method for measuring the efficiency of multi-component DMUs with shared inputs and outputs. According to this approach, they ([9]) suggested a method for surveying progress and regress of multi-component DMUs with shared inputs and outputs in DEA and applied their approach for calculating productivity in commercial banks.

To evaluate the efficiencies of multi-component DMUs multiple constraints are added to models. These constraints are the efficiencies of the components that must be less than or equal to one. The constraints are incorporated to models to preserve a relation between the efficiency of the total process and the efficiencies of the components. Adding the constraints into multi-component DEA models is similar to considering weight restrictions in DEA models. DEA models with weight restrictions maximize the absolute efficiency of a unit which may not equal to the relative efficiency of the unit. In addition, in this case, distinguishing DEA frontier and determining target points for inefficient DMUs may not easy. Podinovski ([10]) stated that if non-homogeneous weight restrictions in the form $Au \leqslant b, Cv \leqslant d$ are incorporated in the CCR model or its linear analogous, the relative efficiency of assessed DMU may not be attained. This is notable that if homogeneous weight restrictions in the form $Au \leqslant 0, Cv \leqslant 0$ are incorporated in the models, the obtained absolute efficiency of assessed DMU is equal to its relative efficiency. Although, weight restrictions incorporated into multi-component DEA models are homogeneous, but have not the mentioned homogeneous structure in work of Podinovski ([10]). Therefore, incorporating them into multi-component DEA models may not lead to calculate relative efficiency. In the next section, a numerical example shows that the calculated efficiency in multi-component DEA model may not be the relative efficiency of Multi-component DMU. Full details about DEA models with weight restrictions can be found in the works of Podinovski and Athanassopoulos ([11]) and Podinovski ([10, 12, 13]).

To avoid the aforementioned problems, in this paper, the Maximin DEA model ([14]) is used for measuring the relative efficiencies of the multi-component DMUs. An example is also used to compare our proposed method with the previous method in the literature to evaluate the performance of 19 bank branches.

The rest of the paper is organized as follows. In Section 2, multi-component DEA is introduced, briefly. A new DEA model to assess the multi-component DMUs is suggested in Section 3, which explicitly measures the relative efficiency of a multi-component DMUs. Some facts related to the proposed models are also provided in this section. In Section 4, the proposed multi-component DEA method is compared with another approach in the literature. The last section comprises our conclusions.

2. Background of Multi-Component DEA

Consider n DMUs as multi-component DMUs and each DMU has b components. Let, $Y_k^{(i)}$, $(i = 1, ..., b)$ be the output vector of the ith component of DMU_k in which $Y_k^{(i)} = (Y_{k1}^{(i)}, ..., Y_{kJ_i}^{(i)})$, $(i = 1, ..., b)$. Also, $X_k^{(i)}$ $(i = 1, ..., b)$ is I_i-dimensional input vector of the ith component of DMU_k. Consider, $X_k^{(c)}$ and $Y_k^{(c)}$ are a I_c-dimensional and J_c-dimensional vectors of shared inputs and outputs, respectively. A schematic of multi-component DMU is given in Fig. 1.
Let, α_i $X_k^{(c)}$ is the portion the ith component of shared input $X_k^{(c)}$ and β_i $Y_k^{(c)}$ is the portion produced of the shared output $Y_k^{(c)}$. Note that

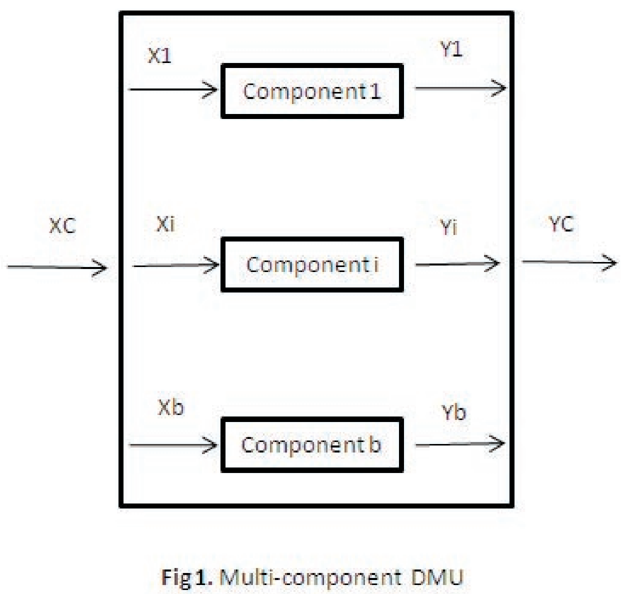

Fig1. Multi-component DMU

$\alpha_i \geqslant 0$, $\beta_i \geqslant 0$ and $\sum_{i=1}^{b} \alpha_i = \sum_{i=1}^{b} \beta_i = 1$. In proposed model α_i and β_i are decision variables which must be determined. According to Cook et al. ([5]) and Jahanshahloo et al. ([8]), a measure of aggregate performance $e_k^{(a)}$ can be represented by

$$e_k^{(a)} = \frac{\sum_{i=1}^{b} U^{(i)} Y_k^{(i)} + \sum_{i=1}^{b} U^{(s_i)} (\beta_i \; Y_k^{(c)})}{\sum_{i=1}^{b} V^{(i)} X_k^{(i)} + \sum_{i=1}^{b} V^{(s_i)} (\alpha_i \; X_k^{(c)})} \qquad (1)$$

where the vectors U and V would be determined in a DEA manner to be discussed below. Performance measures for each components of DMU_k can be represented by

$$e_k^{(i)} = \frac{U^{(i)} Y_k^{(i)} + U^{(s_i)} (\beta_i \; Y_k^{(c)})}{V^{(i)} X_k^{(i)} + V^{(s_i)} (\alpha_i \; X_k^{(c)})} \qquad i = 1, ..., b \qquad (2)$$

This is shows in ([8]) that the aggregate performance measure $e_k^{(a)}$ is a convex combination of $e_k^{(i)}$ s. To derive $e_k^{(a)}, e_k^{(1)}, ..., e_k^{(b)}$, a mathematical program was suggested in ([5, 8]) as follows:

$$\begin{aligned}
Max \quad & e_k^{(a)}, \\
s.t \quad & e_j^{(a)} \leqslant 1, \quad j \in Y_o, \\
& e_j^{(i)} \leqslant 1, \quad i = 1, ..., b, \quad j \in Y_o, \\
& \sum_{i=1}^{b} \alpha_i = 1, \\
& \sum_{i=1}^{b} \beta_i = 1, \\
& U^{(i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& V^{(i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& U^{(s_i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& V^{(s_i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& \alpha_i \geqslant 0, \quad i = 1, ..., b, \\
& \beta_i \geqslant 0, \quad i = 1, ..., b, \qquad (3)
\end{aligned}$$

where $Y_o = \{1, ..., n\}$ and $\epsilon > 0$ is a nonarchimedean number. Model ([2]) is ratio and utilizing a simple transformation variable, this model can be expressed in following form:

$$
\begin{aligned}
Max \quad & \sum_{i=1}^{b} U^{(i)} Y_k^{(i)} + \sum_{i=1}^{b} U^{(s_i)} (\beta_i \ Y_k^{(c)}), \\
s.t \quad & \sum_{i=1}^{b} V^{(i)} X_k^{(i)} + \sum_{i=1}^{b} V^{(s_i)} (\alpha_i \ X_k^{(c)}) = 1, \\
& \sum_{i=1}^{b} U^{(i)} Y_j^{(i)} + \sum_{i=1}^{b} U^{(s_i)} (\beta_i \ Y_j^{(c)}) \\
& - \sum_{i=1}^{b} V^{(i)} X_j^{(i)} - \sum_{i=1}^{b} V^{(s_i)} (\alpha_i \ X_j^{(c)}) \leqslant 0, \quad j \in Y_o, \\
& U^{(i)} Y_j^{(i)} + U^{(s_i)} (\beta_i \ Y_j^{(c)}) - V^{(i)} X_j^{(i)} - V^{(s_i)} (\alpha_i \ X_j^{(c)}) \leqslant 0, \\
& i = 1, ..., b, \quad j \in Y_o, \\
& \sum_{i=1}^{b} \alpha_i = 1, \\
& \sum_{i=1}^{b} \beta_i = 1, \\
& U^{(i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& V^{(i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& U^{(s_i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& V^{(s_i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& \alpha_i \geqslant 0, \quad i = 1, ..., b, \\
& \beta_i \geqslant 0, \quad i = 1, ..., b, \qquad (4)
\end{aligned}
$$

Since α_i and β_i $(i = 1, ..., b)$ are decision variables, this problem is clearly nonlinear. If we make the change of variables $\bar{V}^{(s_i)} = V^{(s_i)} \alpha_i$ $(i = 1, ..., b)$ and $\bar{U}^{(s_i)} = U^{(s_i)} \beta_i$ $(i = 1, ..., b)$ problem ([4]) reduces to the following form:

$$
\begin{aligned}
Max \quad & \sum_{i=1}^{b} U^{(i)} Y_k^{(i)} + \sum_{i=1}^{b} \bar{U}^{(s_i)} Y_k^{(c)}, \\
s.t \quad & \sum_{i=1}^{b} V^{(i)} X_k^{(i)} + \sum_{i=1}^{b} \bar{V}^{(s_i)} X_k^{(c)} = 1, \\
& \sum_{i=1}^{b} U^{(i)} Y_j^{(i)} + \sum_{i=1}^{b} \bar{U}^{(s_i)} Y_j^{(c)} - \sum_{i=1}^{b} V^{(i)} X_j^{(i)} - \sum_{i=1}^{b} \bar{V}^{(s_i)} X_j^{(c)} \leqslant 0, \\
& j \in Y_o, \\
& U^{(i)} Y_j^{(i)} + \bar{U}^{(s_i)} Y_j^{(c)} - V^{(i)} X_j^{(i)} - \bar{V}^{(s_i)} X_j^{(c)} \leqslant 0, \quad i = 1, ..., b, \\
& j \in Y_o, \\
& U^{(i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& V^{(i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& \bar{U}^{(s_i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& \bar{V}^{(s_i)} \geqslant \epsilon, \quad i = 1, ..., b, \qquad (5)
\end{aligned}
$$

The optimal value of the objective function of the model ([1]) is equal to that of the model ([4]). This value is considered as the efficiency score of DMU_k. In the model ([2]), to preserve a relation between the efficiency of multi-component DMU and the efficiencies of the components, constraints $\frac{U^{(i)} Y_j^{(i)} + U^{(s_i)} (\beta_i \ Y_j^{(c)})}{V^{(i)} X_j^{(i)} + V^{(s_i)} (\alpha_i \ X_j^{(c)})} \leqslant 1 \quad (i = 1, ..., b)(j \in Y_o)$ were incorporated to the model. Adding these constraints to the model is similar to consider weight restrictions in the CCR model ([7]). However, incorporating weight restrictions in the CCR model ([7]) leads to the absolute efficiency of the evaluating DMU which is not always equal to its relative efficiency. In other words, the primary goal in DEA is obtaining the relative efficiency for evaluating DMU, while the value of the optimal objective function of the model ([2]) is the absolute efficiency. In fact, all DMUs with the model ([2]) may have the scores strictly less than one. As an example, consider two DMUs A and B with two components. Components use one shared input and each component produces one output. The data are provided in Table 1. Using the model ([1]), the efficiency scores of DMUs A and B are 0.871 and 0.944, respectively. Note that none of the scores is equal to one. Thus, the model cannot measure the relative efficiencies of the units. In what follows, to overcome this difficulty, based on the definition of the relative efficiency ([7]) and using the Maximin DEA model ([14]), we provide an approach to measure the relative efficiency of multi-component DMUs.

Table 1: The data set

DMU	*Shared input*	*Output of component* 1	*Output of component* 2	*Efficiency*
A	400	8000	500	0.871
B	300	400	10000	0.944

3. Methodology

In classic DEA, to evaluate DMU_k $(k = 1, ..., n)$ with m inputs and s outputs, is formed a virtual input by weights v_i $(i = 1, ..., m)$ as $\sum_{i=1}^{m} v_i x_{ik}$ and a virtual output by weights u_r $(r = 1, ..., s)$ as $\sum_{r=1}^{s} u_r y_{rk}$. The absolute efficiency of DMU_k is defined by the ratio of virtual output to virtual input, which is shown by E_k, as:

$$E_k = \frac{\sum_{r=1}^{s} u_r y_{rk}}{\sum_{i=1}^{m} v_i x_{ik}} \qquad (6)$$

To measure the relative efficiency of DMU_k in comparison to the other DMUs is used the ratio of the absolute efficiency of DMU_k to the maximum absolute efficiencies all DMUs, which is shown by RE_k, as:

$$RE_k = \frac{\frac{\sum_{r=1}^{s} u_r y_{rk}}{\sum_{i=1}^{m} v_i x_{ik}}}{\max\limits_{j=1,...,n} \{\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}}\}} \qquad (7)$$

According to definitions (1) and (7), the relative efficiency of DMU_k ($k \in Y_o$) in multi-component DEA is as:

$$\frac{e_k^{(a)}}{\max\limits_{j \in Y_o}\{e_j^{(a)}\}} \qquad (8)$$

We must determine the multipliers to measure the relative efficiency of DMU_k. To determine the multipliers, according to Thompson et al. ([14]), the relative efficiency of DMU_k must be maximized. To preserve a relation between efficiencies of components and the efficiency of the total process, it must be maximized under the assumption that the efficiency of each component must be less than or equal to one. So, to measure the relative efficiency of DMU_k, we have a fractional program as:

$$\begin{array}{ll} Max & \dfrac{e_k^{(a)}}{\max\limits_{j \in Y_o}\{e_j^{(a)}\}}, \\ s.t & e_j^{(i)} \leqslant 1, \quad i = 1, ..., b, \quad j \in Y_o, \\ & \sum\limits_{i=1}^{b} \alpha_i = 1, \\ & \sum\limits_{i=1}^{b} \beta_i = 1, \\ & U^{(i)} \geqslant \epsilon, \quad i = 1, ..., b, \\ & V^{(i)} \geqslant \epsilon, \quad i = 1, ..., b, \\ & U^{(s_i)} \geqslant \epsilon, \quad i = 1, ..., b, \\ & V^{(s_i)} \geqslant \epsilon, \quad i = 1, ..., b, \\ & \alpha_i \geqslant 0, \quad i = 1, ..., b, \\ & \beta_i \geqslant 0, \quad i = 1, ..., b, \qquad (9) \end{array}$$

Using substitution

$$\frac{1}{t} = \max_{j \in Y_o}\{e_j^{(a)}\} = \max_{j \in Y_o}\{\frac{\sum_{i=1}^{b} U^{(i)} Y_j^{(i)} + \sum_{i=1}^{b} U^{(s_i)}(\beta_i \ Y_j^{(c)})}{\sum_{i=1}^{b} V^{(i)} X_j^{(i)} + \sum_{i=1}^{b} V^{(s_i)}(\alpha_i \ X_j^{(c)})}\}$$

and variable transformations $\bar{U}^{(i)} = t\ U^{(i)},\ \bar{U}^{(s_i)} = t\ U^{(s_i)}\ (i = 1, ..., b)$ and $\bar{V}^{(i)} = t\ V^{(i)},\ \bar{V}^{(s_i)} = t\ V^{(s_i)}\ (i = 1, ..., b)$ the above problem is converted to another fractional program as:

$$
\begin{aligned}
Max \quad & \frac{\sum_{i=1}^{b} \bar{U}^{(i)} Y_k^{(i)} + \sum_{i=1}^{b} \bar{U}^{(s_i)} (\beta_i\ Y_k^{(c)})}{\sum_{i=1}^{b} V^{(i)} X_k^{(i)} + \sum_{i=1}^{b} V^{(s_i)} (\alpha_i\ X_k^{(c)})}, \\
s.t \quad & \frac{\sum_{i=1}^{b} \bar{U}^{(i)} Y_j^{(i)} + \sum_{i=1}^{b} \bar{U}^{(s_i)} (\beta_i\ Y_j^{(c)})}{\sum_{i=1}^{b} V^{(i)} X_j^{(i)} + \sum_{i=1}^{b} V^{(s_i)} (\alpha_i\ X_j^{(c)})} \leqslant 1, j \in Y_o \qquad (10-1) \\
& \frac{\bar{U}^{(i)} Y_j^{(i)} + \bar{U}^{(s_i)} (\beta_i\ Y_j^{(c)})}{\bar{V}^{(i)} X_j^{(i)} + \bar{V}^{(s_i)} (\alpha_i\ X_j^{(c)})} \leqslant 1, \quad i = 1, ..., b, \quad j \in Y_o, \ (10-2) \\
& \sum_{i=1}^{b} \alpha_i = 1, \\
& \sum_{i=1}^{b} \beta_i = 1, \\
& \bar{U}^{(i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& \bar{V}^{(i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& \bar{U}^{(s_i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& \bar{V}^{(s_i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& V^{(i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& V^{(s_i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& \alpha_i \geqslant 0, \quad i = 1, ..., b, \\
& \beta_i \geqslant 0, \quad i = 1, ..., b, \qquad (10)
\end{aligned}
$$

Then, this fractional program by definition a new variable $\frac{1}{t'} = \sum_{i=1}^{b} V^{(i)} X_k^{(i)} + \sum_{i=1}^{b} V^{(s_i)} (\alpha_i\ X_k^{(c)})$ and variable transformations $\tilde{U}^{(i)} = t'\ \bar{U}^{(i)},\ \tilde{U}^{(s_i)} = t'\ \bar{U}^{(s_i)}\ (i = 1, ..., b)$ and $\tilde{V}^{(i)} = t'\ \bar{V}^{(i)},\ \tilde{V}^{(s_i)} = t'\ \bar{V}^{(s_i)}\ (i = 1, ..., b)$ and $\hat{V}^{(i)} = t'\ V^{(i)},\ \hat{V}^{(s_i)} = t'\ V^{(s_i)}\ (i = 1, ..., b)$ is converted to an equivalent program as:

$$
\begin{aligned}
Max \quad & \sum_{i=1}^{b} \tilde{U}^{(i)} Y_k^{(i)} + \sum_{i=1}^{b} \tilde{U}^{(s_i)} (\beta_i \ Y_k^{(c)}), \\
s.t \quad & \sum_{i=1}^{b} \hat{V}^{(i)} X_k^{(i)} + \sum_{i=1}^{b} \hat{V}^{(s_i)} (\alpha_i \ X_k^{(c)}) = 1, \qquad (11-1) \\
& \sum_{i=1}^{b} \tilde{U}^{(i)} Y_j^{(i)} + \sum_{i=1}^{b} \tilde{U}^{(s_i)} (\beta_i \ Y_j^{(c)}) - \\
& \sum_{i=1}^{b} \hat{V}^{(i)} X_j^{(i)} - \sum_{i=1}^{b} \hat{V}^{(s_i)} (\alpha_i \ X_j^{(c)}) \leqslant 0, j \in Y_o \qquad (11-2) \\
& \tilde{U}^{(i)} Y_j^{(i)} + \tilde{U}^{(s_i)} (\beta_i \ Y_j^{(c)}) - \tilde{V}^{(i)} X_j^{(i)} - \tilde{V}^{(s_i)} (\alpha_i \ X_j^{(c)}) \leqslant 0, \\
& i = 1, ..., b, \quad j \in Y_o, \qquad (11-3) \\
& \sum_{i=1}^{b} \alpha_i = 1, \\
& \sum_{i=1}^{b} \beta_i = 1, \\
& \tilde{U}^{(i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& \tilde{V}^{(i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& \tilde{U}^{(s_i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& \tilde{V}^{(s_i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& \hat{V}^{(i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& \hat{V}^{(s_i)} \geqslant \epsilon, \quad i = 1, ..., b, \\
& \alpha_i \geqslant 0, \quad i = 1, ..., b, \\
& \beta_i \geqslant 0, \quad i = 1, ..., b, \qquad (11)
\end{aligned}
$$

Theorem 3.1. *At least one of the constraints (10-1) in the model (10) is binding in optimality.*

Proof. By contrapositive assumption, consider

$$\frac{\sum_{i=1}^{b} \bar{U}^{(i)^*} Y_j^{(i)} + \sum_{i=1}^{b} \bar{U}^{(s_i)^*}(\beta_i \ Y_j^{(c)})}{\sum_{i=1}^{b} V^{(i)^*} X_j^{(i)} + \sum_{i=1}^{b} V^{(s_i)^*}(\alpha_i \ X_j^{(c)})} < 1, \quad j \in Y_o$$

where $\bar{U}^{(i)^*}$, $\bar{U}^{(s_i)^*}$, $V^{(i)^*}$, $V^{(s_i)^*}$, $\bar{V}^{(i)^*}$, $\bar{V}^{(s_i)^*}$ $(i = 1, ..., b)$ are the optimal multipliers of the problem (10). So, there exist slack variables $\Delta_j > 0$ $(j \in Y_o)$ that

$$\frac{\sum_{i=1}^{b} \bar{U}^{(i)^*} Y_j^{(i)} + \sum_{i=1}^{b} \bar{U}^{(s_i)^*}(\beta_i \ Y_j^{(c)})}{\sum_{i=1}^{b} V^{(i)^*} X_j^{(i)} + \sum_{i=1}^{b} V^{(s_i)^*}(\alpha_i \ X_j^{(c)})} + \frac{\Delta_j \sum_{i=1}^{b} 1_Y^{(i)} Y_j^{(i)}}{\sum_{i=1}^{b} V^{(i)^*} X_j^{(i)} + \sum_{i=1}^{b} V^{(s_i)^*}(\alpha_i \ X_j^{(c)})} = 1, \quad j \in Y_o.$$

Let, $\Delta = \min_{j \in Y_o}\{\Delta_j\}$, then $\Delta > 0$ and $\Delta \leqslant \Delta_j$ $(j \in Y_o)$. Using definition Δ, we have

$$\frac{\sum_{i=1}^{b} \bar{U}^{(i)^*} Y_j^{(i)} + \sum_{i=1}^{b} \bar{U}^{(s_i)^*}(\beta_i \ Y_j^{(c)}) + \Delta \sum_{i=1}^{b} 1_Y^{(i)} Y_j^{(i)}}{\sum_{i=1}^{b} V^{(i)^*} X_j^{(i)} + \sum_{i=1}^{b} V^{(s_i)^*}(\alpha_i \ X_j^{(c)})} \leqslant 1, \quad j \in Y_o$$

Therefore

$$\frac{\sum_{i=1}^{b} (\bar{U}^{(i)^*} + \Delta 1_Y^{(i)}) Y_j^{(i)} + \sum_{i=1}^{b} \bar{U}^{(s_i)^*}(\beta_i \ Y_j^{(c)})}{\sum_{i=1}^{b} V^{(i)^*} X_j^{(i)} + \sum_{i=1}^{b} V^{(s_i)^*}(\alpha_i \ X_j^{(c)})} \leqslant 1, \quad j \in Y_o$$

Beside, according to the model (10),

$$\frac{\bar{U}^{(i)^*} Y_j^{(i)} + \bar{U}^{(s_i)^*}(\beta_i \ Y_j^{(c)})}{\bar{V}^{(i)^*} X_j^{(i)} + \bar{V}^{(s_i)^*}(\alpha_i \ X_j^{(c)})} \leqslant 1, \quad i = 1, ..., b, \quad j \in Y_o$$

So,

$$\bar{U}^{(i)^*} Y_j^{(i)} + \bar{U}^{(s_i)^*}(\beta_i \ Y_j^{(c)}) \leqslant \bar{V}^{(i)^*} X_j^{(i)} + \bar{V}^{(s_i)^*}(\alpha_i \ X_j^{(c)}),$$

$$i = 1, ..., b, \quad j \in Y_o$$

Therefore,

$$\bar{U}^{(i)^*} Y_j^{(i)} + \bar{U}^{(s_i)^*}(\beta_i \, Y_j^{(c)}) + \Delta 1_Y^{(i)} Y_j^{(i)} \leqslant \bar{V}^{(i)^*} X_j^{(i)} + \bar{V}^{(s_i)^*}(\alpha_i \, X_j^{(c)}) + \xi_j^{(i)},$$

$$i = 1, ..., b, \quad j \in Y_o$$

where

$$\xi_j^{(i)} = \Delta 1_Y^{(i)} Y_j^{(i)} = \eta_j^{(i)} 1_X^{(i)} X_j^{(i)}.$$

Consider $\eta = \max\limits_{j \in Y_o} \max\limits_{i=1,...,b} \{\eta_j^{(i)}\}$ Thus,

$$\bar{U}^{(i)^*} Y_j^{(i)} + \bar{U}^{(s_i)^*}(\beta_i \, Y_j^{(c)}) + \Delta 1_Y^{(i)} Y_j^{(i)} \leqslant \bar{V}^{(i)^*} X_j^{(i)} + \bar{V}^{(s_i)^*}(\alpha_i \, X_j^{(c)}) + \eta 1_X^{(i)} X_j^{(i)},$$

$$i = 1, ..., b, \quad j \in Y_o$$

and so,

$$(\bar{U}^{(i)^*} + \Delta 1_Y^{(i)}) Y_j^{(i)} + \bar{U}^{(s_i)^*}(\beta_i \, Y_j^{(c)}) \leqslant (\bar{V}^{(i)^*} + \eta 1_X^{(i)}) X_j^{(i)} + \bar{V}^{(s_i)^*}(\alpha_i \, X_j^{(c)}),$$

$$i = 1, ..., b, \quad j \in Y_o$$

Hence,

$$\frac{(\bar{U}^{(i)^*} + \Delta 1_Y^{(i)}) Y_j^{(i)} + \bar{U}^{(s_i)^*}(\beta_i \, Y_j^{(c)})}{(\bar{V}^{(i)^*} + \eta 1_X^{(i)}) X_j^{(i)} + \bar{V}^{(s_i)^*}(\alpha_i \, X_j^{(c)})} \leqslant 1, i = 1, ..., b, \quad j \in Y_o$$

Thereupon, $\bar{U}^{(i)^*} + \Delta 1_Y^{(i)}, \ \bar{U}^{(s_i)^*}, \ V^{(i)^*}, \ V^{(s_i)^*}, \ \bar{V}^{(i)^*} + \eta 1_X^{(i)} \ , \bar{V}^{(s_i)^*}$, is a feasible solution of the problem (10). But, we have

$$\frac{\sum_{i=1}^b \bar{U}^{(i)^*} Y_k^{(i)} + \sum_{i=1}^b \bar{U}^{(s_i)^*}(\beta_i \, Y_k^{(c)})}{\sum_{i=1}^b V^{(i)^*} X_k^{(i)} + \sum_{i=1}^b V^{(s_i)^*}(\alpha_i \, X_k^{(c)})} <$$

$$\frac{\sum_{i=1}^b (\bar{U}^{(i)^*} + \Delta 1_Y^{(i)}) Y_k^{(i)} + \sum_{i=1}^b \bar{U}^{(s_i)*}(\beta_i \, Y_k^{(c)})}{\sum_{i=1}^b V^{(i)*} X_k^{(i)} + \sum_{i=1}^b V^{(s_i)^*}(\alpha_i \, X_k^{(c)})}$$

So, the value of objective function for feasible solution $\bar{U}^{(i)^*} + \Delta 1_Y^{(i)}, \ \bar{U}^{(s_i)^*}$, $V^{(i)^*}, \ V^{(s_i)^*}, \ \bar{V}^{(i)^*} + \eta 1_X^{(i)}, \ \bar{V}^{(s_i)^*}$ is more than that for optimal solution of problem (10) and this is not true. Thus, in optimality, at least one of the constraints ([10-3]) is binding. □

Theorem 3.2. *Optimal value of the model ([10]) is the relative efficiency of multi-component DMU_k.*

Proof. Let, $\bar{U}^{(i)^*}$, $\bar{U}^{(s_i)^*}$, $V^{(i)^*}$, $V^{(s_i)^*}$, $\bar{V}^{(i)^*}$, $\bar{V}^{(s_i)^*}$ $(i = 1, ..., b)$ are the optimal multipliers of the problem ([10]). According to Theorem (1), at least one of the constraints ([10-3]) is binding. So,

$$\max_{j \in Y_o}\{\frac{\sum_{i=1}^{b} \bar{U}^{(i)^*} Y_j^{(i)} + \sum_{i=1}^{b} \bar{U}^{(s_i)^*}(\beta_i \ Y_j^{(c)})}{\sum_{i=1}^{b} V^{(i)^*} X_j^{(i)} + \sum_{i=1}^{b} V^{(s_i)^*}(\alpha_i \ X_j^{(c)})}\} = 1$$

Therefore,

$$\frac{\sum_{i=1}^{b} \bar{U}^{(i)^*} Y_k^{(i)} + \sum_{i=1}^{b} \bar{U}^{(s_i)^*}(\beta_i \ Y_k^{(c)})}{\sum_{i=1}^{b} V^{(i)^*} X_k^{(i)} + \sum_{i=1}^{b} V^{(s_i)^*}(\alpha_i \ X_k^{(c)})} = \frac{\frac{\sum_{i=1}^{b} \bar{U}^{(i)^*} Y_k^{(i)} + \sum_{i=1}^{b} \bar{U}^{(s_i)^*}(\beta_i \ Y_k^{(c)})}{\sum_{i=1}^{b} V^{(i)^*} X_k^{(i)} + \sum_{i=1}^{b} V^{(s_i)^*}(\alpha_i \ X_k^{(c)})}}{1}$$

$$= \frac{\frac{\sum_{i=1}^{b} \bar{U}^{(i)^*} Y_k^{(i)} + \sum_{i=1}^{b} \bar{U}^{(s_i)^*}(\beta_i \ Y_k^{(c)})}{\sum_{i=1}^{b} V^{(i)^*} X_k^{(i)} + \sum_{i=1}^{b} V^{(s_i)^*}(\alpha_i \ X_k^{(c)})}}{\max_{j \in Y_o}\{\frac{\sum_{i=1}^{b} \bar{U}^{(i)^*} Y_j^{(i)} + \sum_{i=1}^{b} \bar{U}^{(s_i)^*}(\beta_i \ Y_j^{(c)})}{\sum_{i=1}^{b} V^{(i)^*} X_j^{(i)} + \sum_{i=1}^{b} V^{(s_i)^*}(\alpha_i \ X_j^{(c)})}\}}$$

Thus, optimal value of the objective function ([10]) is the relative efficiency of multi-component DMU_k. □

Theorem 3.3. *At least one of the constraints ([11-14]) in the model ([11]) is binding in optimality.*

Proof. Proof is similar to Theorem 3.1. □

Theorem 3.4. *Optimal value of the model ([11]) is the relative efficiency of multi-component DMU_k.*

Proof. Proof is similar to Theorem 3.2. □

4. Example

In order to have a better understanding of robustness the proposed method in compare to the classic method to evaluate multi-component DMUs, a hypothesis data set on 19 commercial bank branches is considered in this section. The data from a study by Amirteimoori and Kordrostami ([1]) with some changes have been taken. Input factors are the number of staff, the number of computer terminals, and square meters of premises. The amount of deposits, the amount of loans are considered as output factors. The branches are considered as two-component DMUs. In this structure, inputs are shared among two components. The deposits are produced by a component, and the loans are produced by another component. The data are provide in Table 2.

Table 2: Data of the bank branches

Branch	*Staff*	*Computer terminals*	*Space* (m^2)	*Deposits*	*Loans*
B_1	10.0	12.61	31.10	700	60
B_2	9.0	11.29	34.46	600	40
B_3	8.0	7.66	21.26	300	50
B_4	8.5	8.81	31.72	500	30
B_5	11.0	7.43	41.00	400	250
B_6	9.5	9.88	44.26	650	470
B_7	10.0	13.82	40.54	1000	900
B_8	9.0	15.10	65.46	850	850
B_9	7.5	9.74	61.08	700	410
B_{10}	8.5	11.31	24.58	450	820
B_{11}	9.0	12.28	23.04	900	630
B_{12}	11.0	20.55	80.38	1000	950
B_{13}	12.0	15.78	38.54	800	470
B_{14}	16.5	22.69	90.82	700	530
B_{15}	4.5	21.34	85.04	550	620
B_{16}	10.0	14.78	42.90	700	430
B_{17}	15.0	15.52	60.34	1500	850
B_{18}	9.0	23.45	98.02	600	670
B_{19}	6.5	20.11	70.00	900	430

Table 3 represents the numerical results of applying the proposed method and the classic method for evaluating the branches as two-component DMUs. The second and third columns of Table 3 consist of the efficiencies obtained from these two methods. As shown in the second column of Table 3, the efficiencies of the branches by applying the classic method are less than one. Therefore, the obtained efficiencies are not the relative efficiencies of branches. In fact, these efficiencies are absolute efficiencies of the branches. So, the branches can not be compared according to these scores. Based on the third column of Table 3, branches 7, 10, 11, 15, 17, and 19 have the scores equal to one and so are efficient. Also, this shows the fact that the scores in the third column of Table 3 are the relative efficiencies of branches.

Table 3: Efficiencies of the bank branches using different methods

DMU	*classic method*	*Our method*
B_1	0.6926	0.6944
B_2	0.6492	0.6510
B_3	0.4656	0.4663
B_4	0.6045	0.6056
B_5	0.5562	0.5895
B_6	0.6831	0.7877
B_7	0.9628	1.0000
B_8	0.9368	0.9912
B_9	0.8867	0.8945
B_{10}	0.9959	1.0000
B_{11}	0.9985	1.0000
B_{12}	0.8377	0.8894
B_{13}	0.6568	0.6592
B_{14}	0.3976	0.4141
B_{15}	0.9988	1.0000
B_{16}	0.6618	0.6645
B_{17}	0.9969	1.0000
B_{18}	0.6632	0.6827
B_{19}	0.9967	1.0000

5. Conclusion

In this paper, the evaluation of DMUs in multi-component DEA was studied. Measuring the relative efficiencies of decision making units is the primary aim in DEA literature. The suggested models to acquire the efficiencies of DMUs in multi-component DEA are based on the CCR model. However, the CCR model obtains the absolute efficiency of assessed DMU and this efficiency in classic DEA is coincided to the relative efficiency of the DMU. In the present paper, a numerical example was used to show that previous multi-component DEA method may not measure the relative efficiency of evaluating unit. To overcome this problem, in this paper a fractional model for obtaining the relative efficiencies of DMUs is proposed in multi-component DEA using the definition of the relative efficiency. The model was transformed to a fractional program, and the program was converted to an equivalent program by simple variable transformations. It is proven that the fractional program and its equivalent problem measure the relative efficiency of an assessed DMU in multi-component DEA. Furthermore, to indicate the ability of the proposed method, the method was applied to evaluate the performance of bank branches. Finally, considering the evaluated method in this paper to study other versions of DEA such as BCC model and determining return to scale in multi- component DMUs can be suggested for further research.

Acknowledgment:
The authors would like to thank referees for their useful comments on an earlier version of this paper.

References

[1] A. R. Amirteimoori and S. Kordrostami, Multi-component efficiency measurement with imprecise data, *Applied Mathematics and Computation,* 3 (2005), 1265-1277.

[2] R. D. Banker, A. Charnes, and W. W. Cooper, Some models for estimating technical and scale inefficiencies in data envelopment analysis, *Management Science,* 9 (1984), 1078-1092.

[3] A. Charnes, W. W. Cooper, and E. Rhodes, Measuring the efficiency of decision making units, *European Journal of Operational Research,* 2 (1978), 429-444.

[4] W. D. Cook and L. M. Seiford, Data envelopment analysis(DEA)-Thirty years on, *European Journal of Operational Research,* 1 (2009), 1-17.

[5] W. D. Cook, M. Hababou, and H. J. H. Tuenter, Multicomponent efficiency measurement and shared inputs in data envelopment analysis: an application to sales and service performance in bank branches, *Journal of Productivity Analysis,* 3 (2000), 209-224.

[6] W. D. Cook and R. H. Green, Multicomponent efficiency measurement and core business identification in multiplant firms: A DEA model, *European Journal of Operational Research,* 3 (2004), 540-551.

[7] W. W. Cooper, L. M. Seiford, and T. Tone, *Data Envelopment Analysis,* Springer: New York, 2007.

[8] G. R. Jahanshahloo, A. R. Amirteimoori, and S. Kordrostami, Measuring the multi-component efficiency with shared inputs and outputs in data envelopment analysis, *Applied Mathematics and Computation,* 1 (2004), 283-293.

[9] G. R. Jahanshahloo, A. R. Amirteimoori, and S. Kordrostami, Multicomponent performance, progress and regress measurement and shared inputs and outputs in DEA for panel data: an application in commercial bank branches, *Applied Mathematics and Computation,* 1 (2004), 1-16.

[10] V. V. Podinovski, Suitability and redundancy of non-homogeneous weight restrictions for measuring the relative efficiency in DEA, *European Journal of Operational Research,* 2 (2004), 380-395.

[11] V. V. Podinovski and A. Athanassopoulos, Assessing the relative efficiency of decision making units using DEA models with weight restrictions, *Journal of the Operational Research Society,* 49 (1998), 500-508.

[12] V. V. Podinovski, Side effects of absolute weight bounds in DEA models, *European Journal of Operational Research,* 3 (1999), 583-595.

[13] V. V. Podinovski, DEA models for explicit maximisation of relative efficiency, *European Journal of Operational Research,* 3 (2001), 572-586.

[14] R. G. Thompson, P. S. Dharmapala, and M. R. Thrall, Importance for DEA of zeros in data, multipliers, and solutions, *Journal of Productivity Analysis,* 4 (1993), 379-390.

Abbas Ali Noora
Department of Mathematics
Assistant Professor of Mathematics
Sistan and Baluchestan University
Zahedan, Iran
E-mail: a_noora@yahoo.com

Farhad Hosseinzadeh Lotfi
Department of Mathematics
Professor of Applied Mathematics
Science and Research Branch
Islamic Azad University
Tehran, Iran
E-mail: hosseinzadeh_lotfi@yahoo.com

Ali Payan
Department of Mathematics
Assistant Professor of Applied Mathematics
Islamic Azad University-Zahedan Branch
Zahedan, Iran
E-mail:apayan_srb@yahoo.com