Some Fixed Point Theorems for Pointwise Contractions

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Abstract. In this paper we give some fixed point theorems in cone metric spaces and the purpose of this paper is the investigation of some manners for finding the fixed point of pointwise contraction and asymptotic pointwise contraction mappings.

1. Introduction

Let E be a normed linear space. the subset P of E is called a cone if
(i) P is closed, non-empty and $P \neq \{0\}$,
(ii) $ax + by \in P$ for all $x, y \in P$ and all non-negative real numbers a, b,
(iii) $P \cap -P = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\leqslant$ with respect to P by $x \leqslant y$ if and only if $y - x \in P$. $x < y$ will stand for $x \leqslant y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P. The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in E$, $0 \leqslant x \leqslant y$ implies $\|x\| \leqslant M\|y\|$. The least positive number satisfying the above conditions is called the normal constant of P.

Definition 1.1. *Let X be a non-empty set, $(E, \|.\|)$ a normed space that is ordered by a normal cone P with constant normal M=1, $intP \neq \emptyset$ and $\leqslant$ is a partial ordering with respect to P. Also suppose that and the mapping $d : X \times X \to E$ satisfies*
(i) $0 \leqslant d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if x=y,
(ii) $d(x, y) = d(y, x)$ for all for all $x, y \in X$
(iii) $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then d is called a cone metric on X, and (X, d) is called a cone metric space.

For example suppose $E = C_R[0, 1]$, $P = \{\varphi \in E : \varphi \geqslant 0\}$, $X = R$ and $d : X \times X \to E$ defined by $d(x, y) = |x - y|\varphi$ where $\varphi : [0, 1] \to R$ such that $\varphi(t) = e^t$. It easy to see that (X, d) is a cone metric space.

Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X , $x \in X$ and also for every $c \in E$ with $0 \ll c$, there is an integer $N > 0$ such that for all $n > N$, $d(x_n, x) \ll c$. Then $\{x_n\}$ is said to be convergent to x. We denote this by $\lim_{n \to \infty} x_n = x$.

If for every $c \in E$ with $0 \ll c$ there is an integer $N > 0$ such that for all $n, m > N$, $d(x_n, x_m) \ll c$, Then $\{x_n\}$ is said to be Cauchy sequence. If every Cauchy sequence is convergence, then X is called a complete cone metric space.
Also sequence $\{x_n\}$ is said to be bounded if there is $M \gg 0$ such that for all $n \in N$ we have

$$d(0, x_n) \ll M.$$

Fixed point theory is an important tool for solving equations $T(x) = x$. However, if T does not have fixed points, then one often tries to find an element x which is in some sense closest to $T(x)$. A classical result in this direction is a best approximation theorem due to Ky Fan ([3]).
In this paper we consider sufficient conditions that ensure the existence of fixed point in cone metric space. It is notable that we use the results of [1-4].

2.   Main Results

In this section we present some fixed point theorems on cone metric spaces.

Let M be a cone metric space. A mapping $T : M \to M$ is called a pointwise contraction if there exists a mapping $\alpha : M \to [0, 1)$ such that

$$d(Tx, Ty) \leqslant \alpha(x)d(x, y) \qquad for \ any \ x, y \in M.$$

Let M be a cone metric space. A mapping $T : M \to M$ is called an asymptotic pointwise mapping if there exists a sequence of mappings $\alpha_n : M \to [0, 1)$ such that

$$d(T^n(x), T^n(y)) \leqslant \alpha_n(x)d(x, y) \quad for \ any \ x, y \in M$$

Now if $\{\alpha_n\}$ converges pointwise to $\alpha : M \to [0, 1)$, then T is called an asymptotic pointwise contraction.

Proposition 2.1. *Let M be a complete cone metric space , P a normal cone with normal constant K and $T : M \to M$ a pointwise contraction. If there is $b \in (0, \frac{1}{2})$ and $n_0 \in N$ such that*

$$1) \quad \alpha(x_n) < b, \ for \ all \ n > n_0, \ x_0 \in M, \ x_{n+1} = T^{n+1}x_0$$

$$2) \quad \lim_{n \to \infty} \alpha(x_n) = 0,$$

then T has a unique fixed point in X.

Proof. Choose $x_0 \in X$ and $n \geqslant 1$. Set

$$x_1 = Tx_0, x_2 = Tx_1 = T^2 x_0, ..., x_{n+1} = Tx_n = T^{n+1}x_0, \cdots .$$

We have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leqslant \alpha(x_{n-1})d(x_{n-1}, x_n).$$

For $n > m > n_0$,We have

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + ... + d(x_{m+1}, x_m) \\ &\leqslant (b^{n-1} + ... + b^m)d(x_0, x_1), \end{aligned}$$

thus $d(x_n, x_m) \leqslant \frac{b^m}{1-b} d(x_0, x_1)$. Since P is normal we obtain,

$$\|d(x_n, x_m)\| \leqslant \frac{b^m}{1-b} K \|d(x_0, x_1)\|.$$

This implies $d(x_n, x_m) \to 0$. Hence $\{x_n\}$ is a Cauchy sequence. By completeness of M, there is $x^* \in M$ such that $x_n \to x^*$. Thus, $d(Tx^*, x^*) \leqslant d(Tx_n, Tx^*) + d(Tx_n, x^*) \leqslant \alpha(x_n) d(x_n, x^*) + d(Tx_n, x^*)$. Since $x_n \to x^*$, thus

$$\|d(Tx^*, x^*)\| \leqslant K\alpha(x_n)\|d(Tx_n, x_n)\| + K\|d(Tx_n, x^*)\| \to 0.$$

Therefore, $d(Tx^*, x^*) = 0$, and so $Tx^* = x^*$. Now, if y^* is another fixed point of T, then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leqslant \alpha(x^*) d(x^*, y^*).$$

Hence $d(x^*, y^*) = 0$, and so $x^* = y^*$. Therefore, the fixed point of T is unique. $\square$

Let M be a cone metric space and $\mathcal{F}$ a family of subsets of M. Then we say that $\mathcal{F}$ defines a convexity structure on M if it contains the closed balls and is closed under intersection. We will say that $\mathcal{F}$ is compact if any family $(A_\alpha)_{\alpha \in \Gamma}$ of elements of $\mathcal{F}$, has a nonempty intersection provided $\bigcap_{\alpha \in F} A_\alpha \neq \emptyset$ for any finite subset $F \subset \Gamma$.
We will say that a function $\phi : M \to [0, \infty)$ is $\mathcal{F}$-convex if $\{x; \phi(x) \leqslant r\} \in \mathcal{F}$ for any $r \geqslant 0$. We will say that a convexity structure $\mathcal{F}$ on M is T-stable if mappings

$$\phi(u) := \limsup_{n \to \infty} d(x_n, u) \qquad for~all~u \in M,$$

are $\mathcal{F}$-convex .

Lemma 2.2. *Let M be a cone metric space and $\mathcal{F}$ a compact convexity structure on M which is T-stable. Then for any mappings ϕ, there exists $x_0 \in M$ such that*

$$\phi(x_0) = \inf\{\phi(x), x \in M\}.$$

The proof is easy and will be omitted.

Proposition 2.3. *Let M be a complete cone metric space , P a normal cone with normal constant K and $T : M \to M$ a asymptotic pointwise contraction. Assume that there exists a convexity structure $\mathcal{F}$ which is compact and T -stable. Then T has a unique fixed point in X.*

Proof. Since $\mathcal{F}$ is compact and T -stable, then there exists $x_0 \in M$ such that

$$\limsup_{n\to\infty} d(T^n(x), x_0) = \inf\{\limsup_{n\to\infty} d(T^n(x), u); u \in M\}.$$

Let us show that $\limsup_{n\to\infty} d(T^n(x), x_0) = 0$. Indeed we have

$$d(T^{n+m}(x), T^m(x_0)) \leqslant \alpha_m(x_0)d(T^n(x), x_0).$$

for any $n, m \geqslant 1$. If we let n go to infinity, we get

$$\limsup_{n\to\infty} d(T^n(x), T^m(x_0)) \leqslant \alpha_m(x_0)\phi(x_0).$$

which implies that

$$\limsup_{n\to\infty} d(T^n(x), x_0) \leqslant \limsup_{n\to\infty} d(T^n(x), T^m(x_0))$$

$$\leqslant \alpha_m(x_0) \limsup_{n\to\infty} d(T^n(x), x_0).$$

If we let m go to infinity, we get

$$\limsup_{n\to\infty} d(T^n(x), x_0) \leqslant \alpha(x_0) \limsup_{n\to\infty} d(T^n(x), x_0).$$

Since $\alpha(x_0) < 1$, we get $\limsup_{n\to\infty} d(T^n(x), x_0) = 0$, which implies that $\{T^n(x)\}$ converges to x_0. Indeed, since T is asymptotic pointwise contraction, we have, for all n,

$$d(T(x_0), T^{n+1}(x_0)) \leqslant \alpha_1(x_0)d(x_0, T^n(x_0)).$$

Since M be a complete cone metric space, we have

$$\|d(T(x_0), T^{n+1}(x_0))\| \leqslant K\|\alpha_1(x_0)d(x_0, T^n(x_0))\|.$$

Therefore,$d(T(x_0), x_0) = 0$, and so $Tx_0 = x_0$. Now, if y_0 is another fixed point of T, then

$$d(x_0, y_0) = d(T^n(x_0), T^n(y_0)) \leqslant \alpha_n(x_0)d(x_0, y_0).$$

Next we let n go to infinity to get

$$d(x_0, y_0) \leqslant \alpha(x_0)d(x_0, y_0),$$

and so $x_0 = y_0$. Therefor, the fixed point of T is unique. $\square$

Proposition 2.4. *Let X be a complete cone metric space, P a normal cone with normal constant K and $T : X \to X$. If there is a continuous map $\varphi : P \to [0, \frac{1}{2})$, $b \in (0, \frac{1}{2})$ and $n_0 \in N$ such that*

1) $d(Tx, Ty) \leqslant \varphi(d(x, y))(d(Tx, y) + d(x, Ty))$

2) $\varphi(d(x_n, x_{n+1})) < b,\ for\ all\ n > n_0,\ x_0 \in X,\ x_{n+1} = T^{n+1}x_0$

3) $\varphi(0) = 0,$

then T has a unique fixed point in X.

Proof.   Choose $x_0 \in X$ and $n \geqslant 1$.   Set $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, ..., x_{n+1} = Tx_n = T^{n+1}x_0, \cdots$. We have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leqslant \varphi(d(x_{n-1}, x_n))(d(Tx_n, x_{n-1}) + d(Tx_{n-1}, x_n)).$$

Therefore $d(x_n, x_{n+1}) \leqslant b(d(x_{n+1}, x_n) + d(x_n, x_{n-1}))$. Thus $d(x_n, x_{n+1}) \leqslant \frac{b}{1-b}d(x_n, x_{n-1})$, where $h = \frac{b}{1-b}$.   For $n > m > n_0$,

$$\begin{aligned} d(x_n, x_m) &\leqslant & d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + ... + d(x_{m+1}, x_m) \\ &\leqslant & (h^{n-1} + ... + h^m)d(x_0, x_1), \end{aligned}$$

then $d(x_n, x_m) \leqslant \frac{h^m}{1-h}d(x_0, x_1)$. Since P is normal we obtain,

$$\|d(x_n, x_m)\| \leqslant \frac{h^m}{1-h}K\|d(x_0, x_1)\|.$$

This implies $d(x_n, x_m) \to 0$, hence $\{x_n\}$ is a Cauchy sequence.   By completeness of X, there is $x^* \in X$ such that $x_n \to x^*$. Thus,

$$\begin{aligned} d(Tx^*, x^*) &\leqslant & d(Tx_n, Tx^*) + d(Tx_n, x^*) \\ &\leqslant & \varphi(d(x_n, x^*))(d(Tx_n, x^*) + d(Tx^*, x_n)) + d(Tx_n, x^*). \end{aligned}$$

Since $x_n \to x^*$ and $\varphi(0) = 0$, thus

$\|d(Tx^*, x^*)\| \leqslant$

$\quad K\varphi(d(x_n, x^*))(\|d(Tx_n, x_n)\| + \|d(Tx^*, x^*)\|) + K\|d(Tx_n, x^*)\| \to 0.$

Therefore, $d(Tx^*, x^*) = 0$, and so $Tx^* = x^*$. Now, if y^* is another fixed point of T, then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leqslant \varphi(d(x^*, y^*))(d(Tx^*, y^*) + d(Ty^*, x^*)).$$

Hence $d(x^*, y^*) = 0$, and so $x^* = y^*$. Therefor, the fixed point of T is unique. $\square$

Example 2.5. Suppose $E = R^2$, $P = \{(x, y) \in E : x, \ y \geqslant 0\}$,

$$X = \{(x, 0) : x \in [0, 1]\} \cup \{(0, x) : x \in [0, 1]\}.$$

The mapping $d : X \times X \to E$ is defined by

$$d((x, 0), (y, 0)) = d((x, 0), (0, y)) = (\frac{4}{3}|x - y|, |x - y|)$$

$$d((0, x), (0, y)) = (|x - y|, \frac{2}{3}|x - y|).$$

Then (X, d) is a complete cone metric space. Let $T : X \to X$ define by $T(x, 0) = (0, x)$ and $T(0, x) = (x, 0)$, also and let $\varphi : P \to [0, \frac{1}{2})$ define by

$$\varphi(t, s) = \begin{cases} \frac{3}{8}t & 0 \leqslant t < 1 \\ 0 & t \geqslant 1 \end{cases}$$

then we have

$d(T((x_1, x_2)), T((y_1, y_2))) \leqslant \varphi(d((x_1, x_2), (y_1, y_2))[d(T((x_1, x_2))) + d(T((y_1, y_2)))]$

for all $(x_1, x_2), (y_1, y_2) \in X$ and so by Proposition 2.4. T has a unique fixed point in X.

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