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DEA Sensitivity Analysis for Parallel Production Systems

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Abstract. In this paper, we introduce systems consisting of several production units, each of which include several subunits working in parallel. Meanwhile, each subunit is working independently. The input and output of each production unit are the sums of the inputs and outputs of its subunits, respectively. We consider each of these subunits as an independent decision making unit(DMU) and create the production possibility set(PPS) produced by these DMUs, in which the frontier points are considered as efficient DMUs. Then we introduce models for obtaining the efficiency of the production subunits. Using super-efficiency models, we categorize all efficient subunits into different efficiency classes. Then we follow by presenting the sensitivity analysis and stability problem for efficient subunits, including extreme efficient and non-extreme efficient subunits, assuming simultaneous perturbations in all inputs and outputs of subunits such that the efficiency of the subunit under evaluation declines while the efficiencies of other subunits improve.

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1. Introduction

Evaluating the efficiency of the production units of a system is an important issue for managers. Charnes et al. ([1]) introduced data envelopment analysis(DEA) to measure the relative efficiency of a set decision making units(DMUs) consuming similar inputs to produce similar outputs. Consider n DMUs in a production system, such that the *kth* production unit uses m inputs x_{ik} to produce s output y_{rk}. The efficiency of this prodution unit ,E_K, is calculated as follows (Charnes et al. [1]).

$$
\begin{aligned}
E_K = \max \quad & \sum_{r=1}^{s} u_r y_{rk} \\
s.t \quad & \sum_{i=1}^{s} v_i x_{ik} = 1 \\
& \sum_{r=1}^{n} u_r y_{rj} - v_i x_{ij} \leqslant 0, \qquad j = 1, \ldots, n \\
& u_r, v_i \geqslant \epsilon \qquad r = 1, \ldots, s \;\; i = 1, \ldots, m,
\end{aligned} \tag{1}
$$

where u_r and v_i are the most favorable multipliers to be applied to the *rth* output and *ith* input for DMU k in calculating its efficiency, and ϵ is a small non-Archimedean quantity(Charnes et al. [2]).

In the real word, there are systems composed of production units, each of which, in turn, is composed of production subunits. Consider, for instance, a large company including several production workshops to manufacture the parts that the factory needs. The input and output of each factory are sums of the inputs and outputs of its workshops, respectively.

One case of such production units are those in which the subunits of each unit work in parallel to each other, such that the input of the unit is distributed among its subunits, and the outputs of its subunits make up the output of the unit. Assume we have a production system. The *kth* production unit of the sestem is composed of q production subunits, as shown in Fig.1. The ith input and output of the pth subunit of *kth* unit $x_{ik}^{(p)}$ and $y_{rk}^{(p)}$, respectively. Then, the input and output of the *kth* unit are $\sum_{p=1}^{q} x_{ik}^{(p)}$ and $\sum_{p=1}^{q} y_{rk}^{(p)}$ respectively.

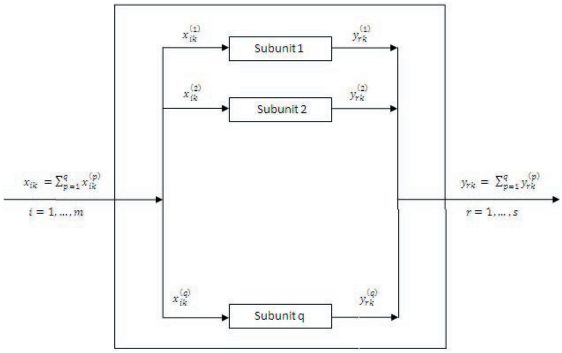

Figure 1: The parallel system, each unit includes q subunit production that every subunit utilizes the same m inputs to produce the same s outputs.

Färe and Grosskopf ([3,4]) and Färe et al. ([5]) developed several network models for evaluating the efficiency of production units composed of subunits. These models can be used to discuss variations of the standard DEA models. Kao and Hwang ([6]) introduced a DEA model to evaluate the efficiency of network systems in which each production unit is composed of two suitable and the output of the first subunit is the input of the second subunit the efficiency of each unit is computed as the product of the efficiencies of its subunits. Kao ([7]) developed a model for evaluating the efficiency of production systems composed of units whose subunits work in parallel towards the aims of production. This model was developed by using the concept of the inefficiency slack.
Kao ([8]) introduced a relational network DEA model to evaluate the efficiency of production systems composed of units whose subunits worked both in parallel and in series.
In this paper, we consider production systems composed of units whose sununits operate in parallel to each other. Meanwhile, each subunit working independently. Then, we introduce models for evaluating the efficiency of these subunits. Next, we address the sensitivity analysis of these efficient subunits. The paper investigates sensitivity analysis using super-efficiency models.

Zhu and Seiford ([9,10]) and Zhu ([11]) introduced sensitivity analysis and stability of efficiency classification of efficient DMUs using super-efficiency models, and provided the necessary and sufficient conditions for the stability of efficiency classification of efficient DMUs after simultaneous perturbations in all data.
After introducing models for obtaining the efficiency of production subunits in this paper, we deal with the sensitivity analysis and stability of efficiency classification of efficient subunits. Also, we present the necessary and sufficient conditions for the stability of efficiency state of the production subunit by causing perturbations in the inputs and outputs of all subunits. We consider the worst-case the subunit under evaluation declines, while the the efficiencies of all other subunits improve. This paper is organized as follows. In Section 2, we present models for the efficiency evaluation of production subunits, and introduce super-efficiency models. Section 3, deals with the sensitivity analysis and stability of efficiency state of efficient subunits. Finally, Section 4 contains the conclusion.

2. Preliminaries

Suppose we have n production units, considered as DMUs, each using the input vector x_j to produce the output vector , y_j. Each unit is composed of production subunits with input and output vectors $x_j^{(p)} \neq 0$ and $y_j^{(p)} \neq 0$, respectively, as shown in Fig.1. The subunits of each unit operate in parallel, that is, the input and output of each DMU are the sums of the inputs and outputs of its subunits, respectively. So, we will have $x_j = \sum_{p=1}^{q} x_j^{(p)}$ and $y_j = \sum_{p=1}^{q} y_j^{(p)}$.
To evaluate the efficiency of each subunit, the production possibility set T is produced as follows.

$$T = \{(X,Y) \mid \sum_{j=1}^{n}\sum_{p=1}^{q} \lambda_j^{(p)} X_j^{(p)} \leqslant X,\ \sum_{j=1}^{n}\sum_{p=1}^{q} \lambda_j^{(p)} Y_j^{(p)} \geqslant Y, \lambda_j^{(p)} \geqslant 0,$$
$$j = 1, \ldots, n,$$
$$p = 1, \ldots, q\}.$$

The frontier and non-frontier points of the above set are considered efficient and inefficient points, respectively.
The model for evaluating the efficiency of the *wth* subunit of the *kth* unit is as follows.

$$\begin{aligned}
\theta^{\star(w)} = \min \quad & \theta^{(w)} \\
s.t \quad & \sum_{j=1}^{n}\sum_{p=1}^{q}\lambda_j^{(p)}x_{ij}^{(p)} \leqslant \theta^{(w)}x_{ik}^{(w)}, \quad i=1,\ldots,m \\
& \sum_{j=1}^{n}\sum_{p=1}^{q}\lambda_j^{(p)}y_{rj}^{(p)} \geqslant y_{kj}^{(w)}, \quad r=1,\ldots,s, \\
& \lambda_j^{(p)} \geqslant 0, \quad j=1,\ldots,n,\ p=1,\ldots,q,
\end{aligned} \tag{2}$$

Definition 2.1. *Subunit* DMU_w^k *is efficient if* $\theta^{\star(w)} = 1$

Definition 2.2. *Subunit* DMU_w^k *is strongly efficient if* $\theta^{\star(w)} = 1$ *and all slack variables in every optimal solution are equal to zero.*

Definition 2.3. *Subunit* DMU_w^k *is called extreme strongly efficient if it is strongly efficient and cannot be expressed as a convex combination of two efficient subunits.*

Definition 2.4. *Subunit* DMU_w^k *is called non-extreme strongly efficient if it is a strongly efficient DMU, but not an extreme DMU.*

Definition 2.5. *Subunit* DMU_w^k *is called weakly efficient if* $\theta^{\star(w)} = 1$ *and we have non-zero slacks in the optimal solution.*

Theorem 2.6. *In any optimal solution of model (2), we will have* $0 < \theta^{\star(w)} \leqslant 1$.

Proof. The Proof is clear.

We can compute the efficiency of DMU_k as follows.

$$\begin{array}{ll}\min & \theta^{(w)} \\ s.t & \sum_{j=1}^{n}\sum_{p=1}^{q}\lambda_j^{(p)}x_{ij}^{(p)} \leqslant \theta^{(w)}(\sum_{p=1}^{q}x_{ik}^{(p)}), \quad i=1,\dots,m \\ & \sum_{j=1}^{n}\sum_{p=1}^{q}\lambda_j^{(p)}y_{rj}^{(p)} \geqslant \sum_{p=1}^{q}y_{kj}^{(p)}, \quad r=1,\dots,s \\ & \lambda_j^{(p)} \geqslant 0, \quad j=1,\dots,n,\ p=1,\dots,q,\end{array} \tag{3}$$

The above model (3) is obtained by setting DMU_k as the DMU under evaluation in model (2).
The efficiency values of DMU_k^w and DMU_k can be obtained by stating the duals of models (2) and (3), respectively as follows.

$$\begin{array}{ll}\max & \sum_{r=1}^{s}u_r y_{rk}^{(w)} \\ s.t & \sum_{i=1}^{s}v_i x_{ik}^{(w)} = 1 \\ & \sum_{r=1}^{n}u_r y_{rj}^{(p)} - v_i x_{ij}^{(p)} \leqslant 0, \qquad j=1,\dots,n,\ \ p=1,\dots,q \\ & u_r, v_i \geqslant \epsilon \qquad r=1,\dots,s\ \ i=1,\dots,m,\end{array} \tag{4}$$

$$\begin{array}{ll}\max & \sum_{r=1}^{s}\sum_{p=1}^{q}u_r y_{rk}^{(p)} \\ s.t & \sum_{i=1}^{s}\sum_{p=1}^{q}v_i x_{ik}^{(p)} = 1 \\ & \sum_{r=1}^{n}u_r y_{rj}^{(p)} - \sum_{i=1}^{m}v_i x_{ij}^{(p)} \leqslant 0, \qquad j=1,\dots,n,\ \ p=1,\dots,q \\ & u_r, v_i \geqslant \epsilon \qquad r=1,\dots,s\ \ i=1,\dots,m,\end{array} \tag{5}$$

Regarding the optimal value of model (2), if we have $\sum_{r=1}^{s}u_r^{\star}y_{rk}^{(w)} = 1$ in the optimal solution of model (4), then DMU_k^w is efficient. Similarly, if

$\sum_{r=1}^{s}\sum_{p=1}^{q} u_r^\star y_{rk}^{(w)} = 1$ then DMU_k is efficient. $\square$

Definition 2.7. *DMU_k is efficient if all of its subunits are efficient, that is, $\theta^{\star(w)} = 1$ for all $w \in \{1, 2, \dots, m\}$.*
It can be readily shown if all subunits of a unit are efficient, then the optimal value of model (3) is equal to one.
Consider 6 production units, each composed of 3 subunits. The data for these units are presented in Table 1.

Table 1.
The data of the six unit by three subunit.

DMU	x_1	x_2	y	DMU	x_1	x_2	y	DMU	x_1	x_2	y
A	2	5	1	B	3	3	1	E	3	2	1
a_1	2/3	2	1/3	b_1	1/2	4/3	1/3	e_1	1	5/12	1/3
a_2	2/3	2	1/3	b_2	1	4/3	1/3	e_2	3/4	11/24	1/3
a_3	2/3	1	1/3	b_3	3/2	1/3	1/3	e_3	5/4	27/24	1/3
C	6	2.5	1	D	2	7	1	F	2	6	1
c_1	1	3/2	1/3	d_1	1/3	2	1/3	f_1	1/3	3	1/3
c_2	3	2/3	1/3	d_2	7/6	9/2	1/3	f_2	5/12	3/2	1/3
c_3	2	1/3	1/3	d_3	1/2	1/2	1/3	f_3	15/12	3/2	1/3

We employ model (1) to evaluate the efficiency of the subunits. Table 2 contains the efficiency values for all these subunits. To evaluate the efficiency of the units we can use model (3).

Table 2.
The optimal values of model (2) for evaluating data of Table 1.

DMU	$\theta^{\star(w)}$	DMU	$\theta^{\star(w)}$	DMU	$\theta^{\star(w)}$	DMU	$\theta^{\star(w)}$	DMU	$\theta^{\star(w)}$	DMU	$\theta^{\star(w)}$
A	0.6222	B	0.5000	C	0.5826	D	0.6000	E	0.7000	F	0.6250
a_1	0.6248	b_1	0.8588	c_1	0.5000	d_1	1	e_1	1	f_1	1
a_2	0.6250	b_2	0.4848	c_2	0.4991	d_2	0.3333	e_2	1	f_2	0.9523
a_3	0.7143	b_3	1	c_3	1	d_3	1	e_3	0.4374	f_3	0.3922

As can be observed, subunits $b_3, c_3, d_1, d_3, e_1, e_2, f_1$ are efficient. The sets of extreme strongly efficient, non-extreme strongly efficient, and weakly efficient points are denoted by E, $\bar{E}$, and F, respectively. For the data in Table 1. the above sets will be as follows.
$E = \{b_3, d_1, d_3\}$, $\bar{E} = \{e_1, e_2\}$, $F = \{c_3, f_1\}$
As can be seen in Fig. 2. the efficiency frontier includes the following segments.

d_1d_3, d_3b_3, the line that starts from b_3 and passes c_3 and is parallel to the first input axis, the line that starts from d_1 and passes f_1 and is parallel to the second input axis.
If we omit extreme efficient subunit d_3, the new efficient frontier in inputs space will include the following segments.
d_1e_2, e_2b_3, the line that starts from b_3 and passes c_3 and is parallel to the first input axis, the line that starts from d_1 and passes f_1 and is parallel to the second input axis.
The efficiency classification of the production units can be obtained by super-efficiency models. The super-efficiency model corresponding to model (2) is introduced as follows.

$$\begin{array}{lll}\theta^{\star(w)}_{sup} = \min & \theta^{(w)}_{sup} & \\ & s.t & \sum_{j=1}^{n}\sum_{p=1,\ (j,p)\neq(k,w)}^{q} \lambda_j^{(p)} x_{ij}^{(p)} \leqslant \theta^{(w)}_{sup} x_{ik}^{(w)}, \quad i,\ldots,m \\ & & \sum_{j=1}^{n}\sum_{p=1,\ (j,p)\neq(k,w)}^{q} \lambda_j^{(p)} y_{rj}^{(p)} \geqslant y_{kj}^{(w)}, \quad r=1,\ldots,s, \\ & & \lambda_j^{(p)} \geqslant 0, \quad j=1,\ldots,n,\ p=1,\ldots,q,\ (j,p)\neq(k,w),\end{array} \qquad (6)$$

Based on Thrall ([12]), we have
1)$\theta^{\star(w)}_{sup} > 1$ or (6) is infeasible if and only if $DMU_k^w \in E$.
2) $\theta^{\star(w)}_{sup} = 1$ if and only if $DMU_k^w \in \bar{E} \cup F$.
In 2, if non-zero input/output slack values are detected in (2), then $DMU_k^w \in F$.
We solve the super-efficiency model (6) for the data in Table 1. The results are provided in Table 3.

Table 3.
The optimal values of model (6) for evaluating data of Table(1).

DMU	$\theta^{\star}_{sup}$	DMU	$\theta^{\star}_{sup}$	DMU	$\theta^{\star}_{sup}$	DMU	$\theta^{\star}_{sup}$	DMU	$\theta^{\star}_{sup}$	DMU	$\theta^{\star}_{sup}$
A	0.6522	B	0.5000	C	0.5826	D	0.6000	E	0.7000	F	0.6250
a_1	0.6248	b_1	0.8588	c_1	0.5	d_1	1.1251	e_1	1	f_1	1
a_2	0.625	b_2	0.4848	c_2	0.4991	d_2	0.3333	e_2	1	f_2	0.9523
a_3	0.7143	b_3	1.0953	c_3	1	d_3	1.3585	e_3	0.4374	f_3	0.3922

The supper-efficiency value of DMU_k, i.e., $\theta^{\star(w)}_{sup}$, is computed by setting DMU_k in model (6) as the DMU under evaluation. In the supper-efficiency model, The DMU under evaluation is omitted from the production possibility set T, and obtain the efficiency of the DMU under evaluation using the new PPS produced by the remaining DMU.

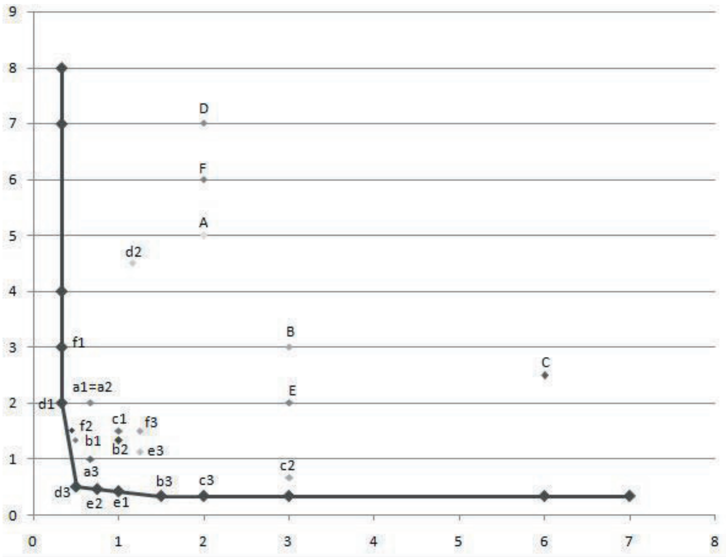

Figure 2: The project of pps in the space inputs.

3. Sensitivity Analysis of Efficient Subunits

Now, we consider sensitivity analysis for the frontier points of the production possibility set T. For this purpose, we deal with the sensitivity analysis of two groups of efficient subunits:
a)extreme strongly efficient subunits
b) non-extreme strongly efficient subunits and weakly efficient subunits.
Since an increase of any output or a decrease of any input cannot worsen the efficiency of DMU_k^w, we only focus on increase in the inputs and decrease in the output of DMU_k^w. To do so, based on zhu ([11]), we consider simultaneous perturbations in all subunits. So, we deal with the worst-case scenario, that is, the inputs and outputs of DMU_k^w are increased and decreased, respectively, while the inputs and outputs of the other subunits are decreased and increased, respectively, so that the efficiency of DMU_k^w declines while the efficiencies of other subunits improve. Suppose I and O are subsets of inputs and outputs in which we are interested. We consider input and output perturbation for inputs and outputs corresponding to set I and O. Simultaneous perturbations

in the inputs and output of the subunits (percentage data perturbations) are assumed as follows.
for DMU_k^w,
$\hat{x}_{ik}^{(w)} = \delta_i x_{ik}^{(w)} \;\; \delta_i \geqslant 1, \;\; i \in I$ and $\hat{x}_{ik}^{(w)} = x_{ik}^{(w)} \;\; i \in \{1, 2, \ldots, m\} - I$
and
$\hat{y}_{rk}^{(w)} = \tau_r y_{rk}^{(w)} \;\; \tau_r \leqslant 1, \;\; r \in O$ and $\hat{y}_{rk}^{(w)} = y_{rk}^{(w)} \;\; r \in \{1, 2, \ldots, s\} - O$
for DMU_j^p, $((j, p) \neq (k, w), \; j = 1, \ldots, n, \; p = 1, \ldots, q)$,
$\hat{x}_{ij}^{(p)} = x_{ij}^{(p)} / \tilde{\delta}_i, \;\; \tilde{\delta}_i \geqslant 1, \;\; i \in I$ and $\hat{x}_{ij}^{(p)} = x_{ij}^{(p)}, \;\; i \in \{1, 2, \ldots, m\} - I$
and
$\hat{y}_{rj}^{(p)} = y_{rj}^{(p)} / \tau_r, \;\; \tau_r \leqslant 1, \;\; r \in O$ and $\hat{y}_{rj}^{(p)} = y_{rj}^{(p)}, \;\; r \in \{1, 2, \ldots, s\} - O$

Where $\hat{()}$ represents adjusted data. The super-efficiency models corresponding to sets I will be as follows.

$$
\begin{aligned}
\theta_I^{\star(w)} = \min \quad & \theta_I^{(w)} \\
s.t \quad & \textstyle\sum_{j=1}^{n} \sum_{p=1, \; (j,p) \neq (k,w)}^{q} \lambda_j^{(p)} x_{ij}^{(p)} \leqslant \theta_I^{(w)} x_{ik}^{(w)}, \quad i \in I \\
& \textstyle\sum_{j=1}^{n} \sum_{p=1, \; (j,p) \neq (k,w)}^{q} \lambda_j^{(p)} x_{ij}^{(p)} \leqslant x_{ik}^{(w)}, \quad i \in \{1, 2, \ldots, m\} - I \\
& \textstyle\sum_{j=1}^{n} \sum_{p=1, \; (j,p) \neq (k,w)}^{q} \lambda_j^{(p)} y_{rj}^{(p)} \geqslant y_{kj}^{(w)}, \quad r = 1, \ldots, s \\
& \lambda_j^{(p)} \geqslant 0, \quad j = 1, \ldots, n, \; p = 1, \ldots, q, \; (j, p) \neq (k, w),
\end{aligned} \tag{7}
$$

Model (7) above projects only the inputs to corresponding to set I on to the new pps produced by the remaining subunits after omitting DMU_k^w.
Using above model for the data in Table 1, the result for sets $I = \{1\}$, $I = \{2\}$, and $I = \{1, 2\}$ are presented in Tables 3,4, and 5. As can be observed, model (7) computes the maximum input increase corresponding to set I necessary for to reach the new pps. For instance, consider subunit d_3. After omitting d_3, which is an extreme strongly efficient subunit, the new pps will include the line segment passing through d_1 and e_2 , that is shown as a light line in Fig. 3.
If $I = \{1\}$, we have $\theta_1^{\star(w)} = 1.4730$, which means d_3 reaches d_{31} on new pps and its first input has increase to reach the input of d_{31}, i.e., 0.7365.
If $I = \{2\}$, we have $\theta_2^{\star(w)} = 2.4790$, which means d_3 reaches d_{33} on new pps and its second input has increase to reach the input of d_{33}, i.e.,

1.2395. If $I = \{1, 2\}$, we have $\theta^{\star(w)}_{1,2} = \theta^{\star(w)}_{sup} = 1.3580$, which shows a simultaneous increase in two inputs of d_3, so that d_3 reaches d_{32} on the new pps, as shown in Fig. 3.

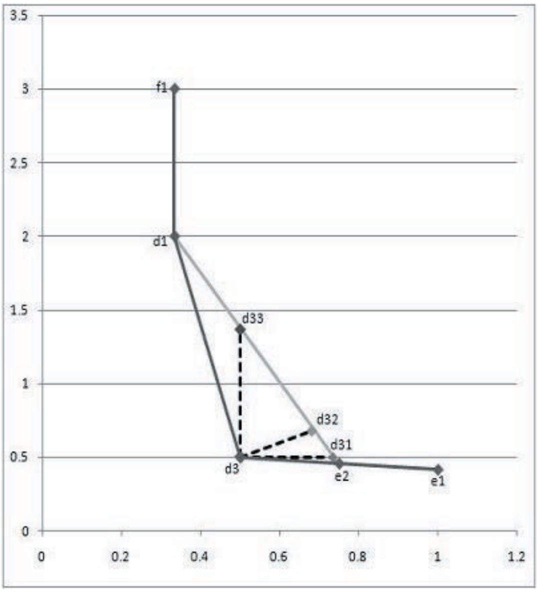

Figure 3: A part of efficient frontier in inputs space.

Table 4.
The optimal values of model (7) for $I = \{1\}$.

DMU	$\theta^{\star}_{\{1\}}$	DMU	$\theta^{\star}_{\{1\}}$	DMU	$\theta^{\star}_{\{1\}}$	DMU	$\theta^{\star}_{\{1\}}$	DMU	$\theta^{\star}_{\{1\}}$	DMU	$\theta^{\star}_{\{1\}}$
A	0.6522	B	0.5000	C	0.5826	D	0.6	E	0.7000	F	0.6250
a_1	0.5	b_1	0.8166	c_1	0.4444	d_1	1.1666	e_1	0.9762	f_1	1
a_2	0.5	b_2	0.4094	c_2	0.16	d_2	0.2859	e_2	1	f_2	0.9334
a_3	0.6666	b_3	1.3359	c_3	0.7469	d_3	1.4734	e_3	0.3444	f_3	0.3112

Table 5.
The optimal values of model (7) for $I = \{2\}$.

DMU	$\theta^{\star}_{\{2\}}$	DMU	$\theta^{\star}_{\{2\}}$	DMU	$\theta^{\star}_{\{2\}}$	DMU	$\theta^{\star}_{\{2\}}$	DMU	$\theta^{\star}_{\{2\}}$	DMU	$\theta^{\star}_{\{2\}}$
A	0.6522	B	0.5000	C	0.5826	D	0.6000	E	0.7000	F	0.6250
a_1	0.2359	b_1	0.3829	c_1	0.4444	d_1	1.1666	e_1	0.9762	f_1	1
a_2	0.2359	b_2	0.3132	c_2	0.4998	d_2	0.0859	e_2	1	f_2	0.8233
a_3	0.4719	b_3	1.1333	c_3	1	d_3	2.4789	e_3	0.3333	f_3	0.2489

Now , we address the sensitivity analysis of the group of weakly efficient subunits.

Theorem 3.1. *suppose* $\theta^{\star(w)}_{sup} = 1$, $\theta^{\star(w)}_{I} < 1$ *then for all* $\delta_i \geqslant 1$, $\tilde{\delta}_i \geqslant 1$, $i \in I$, DMU^w_k *will remain in set* F.

Proof. Considering $\theta^{\star(w)}_{sup} < 1$ and since we have $\theta^{\star(w)}_{I} \leqslant \theta^{\star(w)}_{sup}$, then $\theta^{\star(w)}_{I} \leqslant 1$ as $\theta^{\star(w)}_{I} < 1$, in any optimal solution of model (7) we will have non-zero slack values for any $i \in I$, in the input constraints. So $DMU^w_k \in F$.
Now we show that DMU^w_k will still remain in set F after the above perturbations. Since $DMU^w_k \in F$ then the optimal value of model (4) will be equal to one. That is , there exists $(u^\star, v^\star)$ such that $\sum_{r=1}^{s} u^\star_r y^{(w)}_{rk} = 1$. $(u^\star, v^\star)$ is the feasible solution of model (4), and in the optimal solution of model (2) $s^\star_i > 0$ for all $i \in I$. Based on the complementary slackness theorem for models (2) and (4) , we have $v^\star_i = 0$ for all $i \in I$. Therefore if $\hat{x}^{(p)}_{ij}$, $p \in \{1, 2, \ldots, q\}$, $j \in \{1, 2, \ldots, n\}$, $i \in \{1, 2, \ldots, m\}$, are the adjusted input values, then $(u^\star, v^\star)$ is the feasible solution for model (4) with adjusted data , and the optimal value will be equal to one. Therefore, $DMU^w_k \in F$ and the proof is complete. □

Consider $DMU^3_3 = c_3$. Regarding Table 2, this DMU is a weakly efficient subunit.
By setting $I = \{1\}$ and solving model (7) to evaluate this DMU , we have $\theta^\star_{\{1\}} = 0.7469$. So, c_3 has a non-zero slack variable in this first input. With regard to Fig. 2. this DMU will remain efficient for any amount of increase in its first input while other DMUs decrease their first input. The reverse of Theorem 3.1 does not hold, because DMU^w_k will remain in set F for any amount of increase in its inputs corresponding to set I while $\theta^{\star(w)}_{sup} = 1$ and $\theta^{\star(w)}_{I} \leqslant 1$. That is , for some subunits in set F we have a slack variable equal to zero in some inputs , and these subunits will remain in set F with any amount of increase in their inputs.
As we know, if $\theta^{\star(w)}_{sup} = 1$ then $DMU^w_k \in F \cup \bar{E}$. The sensitivity analysis for the subunits in set $\bar{E}$ was carried out. Now we consider the subunits in set $\bar{E}$. Since the omission these subunits does not change the pps frontier, any increase/decrease in their input/output will make them inefficient. Now, we address the sensitivity analysis of the second group of

efficient production subunits,i.e., the extreme efficient production subunits. Obviously, if $\theta^{\star(w)}_{sup} > 1$ or model (6) is infeasible then the subunit under evaluation is an extreme efficient subunit, and the pps from their will change by omitting this subunit. Therefore, there is the possibility of perturbations in the inputs and outputs of such subunits. regarding Fig. 2. d_1, d_3 and b_3 are extreme efficient subunits.
The efficiency frontier will remain unchanged by omitting weakly efficient and non-extreme strongly efficient subunits and model (6) for the evaluation of these subunits will remain feasible. Thus, the infeasibility of models (6) and (7) will remain feasible. Thus, the infeasibility of models (6), (7) will occur only in the evaluation of extreme efficient subunits.

Theorem 3.2. *Suppose DMU^w_k is an extreme efficient subunit. Model (7) for the evaluation of DMU^w_k is infeasibel if and only if for any $\delta_i \geqslant 1, \tilde{\delta}_i \geqslant 1 (i \in I)$, DMU^w_k remains extreme efficient.*

Proof. To prove the if part, suppose model (7) is feasible. then, since the optimal value of model (7) will the maximum increase proportion in the input of DMU^w_k corresponding to set I, thus the inputs corresponding to set I can not increase infinitely. Therefore, model (7) will be infeasible.
In order to prove the only if part, suppose the *ith* input $(i \in I)$ increase by M^i and DMU^w_k is not extreme efficient. Therefore, DMU^w_k is either inefficient or non-extreme efficient. By setting DMU^w_k in model (2) and solving the model, we will obtain the optimal solution $\theta^{\star(w)} \leqslant 1$, $\lambda^{\star(p)}_j (j = 1, \ldots, n,\ p = 1, \ldots, q,\ (j,p) \neq (k,w))$, and $\lambda^{\star(w)}_k$. Therefore, we will have

$$\begin{array}{l}
\sum_{j=1}^{n} \sum_{p=1,\ (j,p)\neq(k,w)}^{q} \lambda^{\star(p)}_j x^{(p)}_{ij} \leqslant \theta^{\star(w)} M^i x^{(w)}_{ik}, \quad i \in I \\
\sum_{j=1}^{n} \sum_{p=1,\ (j,p)\neq(k,w)}^{q} \lambda^{\star(p)}_j x^{(p)}_{ij} \leqslant \theta^{\star(w)} x^{(w)}_{ik} \leqslant x^{(w)}_{ik},\ i \in \{1,2,\ldots,m\} - I \\
\sum_{j=1}^{n} \sum_{p=1,\ (j,p)\neq(k,w)}^{q} \lambda^{\star(p)}_j y^{(p)}_{rj} \geqslant y^{(w)}_{kj}, \quad r = 1, \ldots, s, \\
\lambda^{\star(p)}_j \geqslant 0, \quad j = 1, \ldots, n,\ p = 1, \ldots, q,\ (j,p) \neq (k,w),
\end{array} \tag{8}$$

Therefore, $\theta^{\star(w)}_i = \theta^\star M^i, i \in I,\ \lambda^{\star(p)}_j (j = 1, \ldots, n,\ p = 1, \ldots, q,\ (j,p) \neq (k,w))$ will be a feasible solution for model (7), which contradicts the infeasibility assumption of model (7). Since M^i, $i \in I$, are arbitrary, then the *ith* input $i \in I$ can increase infinitely while remains extreme efficient. The proof is complete. □

For the data in Table 1., model (7) for the evaluation of all production subunits is feasible. If c_3, is not included in the production subunits and there is not any production subunit with a first input equal to 1/3, then model (7) for the evaluation of b_3, setting $I = \{1\}$, will be infeasible, while model (7) will be feasible here. So, the first input of b_3 can be increased by any amount while the first input of the other subunits are decreased, keeping b_3 still extreme efficient.

Regarding the fact that in the infeasibility of model (7) the efficiency classification of the DMU under evaluation remains unchanged, we suppose that model (7) is feasible.

Theorem 3.3. *Suppose DMU_k^w is an extreme strongly efficient subunit and Model (7) for the evaluation of DMU_k^w is feasible. If $1 \leqslant \delta_i\tilde{\delta}_i \leqslant \theta_i^{\star(w)}, (i \in I)$, then DMU_k^w remains extreme efficient. Furthermore, if equality holds, that is, $\delta_i\tilde{\delta}_i = \theta_i^{\star(w)}, (i \in I)$, then DMU_k^w remains on the frontier.*

Proof. suppose $1 \leqslant \delta_i\tilde{\delta}_i \leqslant \theta_i^{\star(w)}, (i \in I)$, and DMU_k^w does not remain extreme efficient when $\hat{x}_{ik}^{(w)} = \delta_i x_{ik}^{(w)}$ and $\hat{x}_{ij}^{(p)} = x_{ij}^{(p)}/\tilde{\delta}_i$, $i \in I$. Therefore, regarding model (6), we have $\theta_{sup}^{\star(w)} \leqslant 1$ and there exist $\lambda_j^{(p)}(j = 1, \dots, n,\ p = 1, \dots, q,\ (j,p) \neq (k,w))$ such that

$$\begin{array}{l}
\sum_{j=1}^{n}\sum_{p=1,\ (j,p)\neq(k,w)}^{q} \lambda_j^{\star(p)}(x_{ij}^{(p)}/\tilde{\delta}_i) \leqslant \theta_{sup}^{\star(w)}\delta_i x_{ik}^{(w)}, \quad i \in I \\
\sum_{j=1}^{n}\sum_{p=1,\ (j,p)\neq(k,w)}^{q} \lambda_j^{\star(p)} x_{ij}^{(p)} \leqslant \theta_{sup}^{\star(w)} x_{ik}^{(w)} \leqslant x_{ik}^{(w)}, \quad i \in \{1, 2, \dots, m\} - I \\
\sum_{j=1}^{n}\sum_{p=1,\ (j,p)\neq(k,w)}^{q} \lambda_j^{\star(p)} y_{rj}^{(p)} \geqslant y_{kj}^{(w)}, \quad r = 1, \dots, s, \\
\lambda_j^{\star(p)} \geqslant 0, \quad j = 1, \dots, n,\ p = 1, \dots, q,\ (j,p) \neq (k,w),
\end{array} \tag{9}$$

This means that $\lambda_j^{(p)}(j = 1, \dots, n,\ p = 1, \dots, q,\ (j,p) \neq (k,w)), \theta_{sup}^{\star(w)}\delta_i\tilde{\delta}_i$ is a feasible solution for model (7). However $\theta_{sup}^{\star(w)}\delta_i\tilde{\delta}_i \leqslant \theta_{sup}^{\star(w)}\theta_I^{\star(w)} \leqslant \theta_I^{\star(w)}$, which contradicts the optimality of $\theta_I^{\star(w)}$. Therefore, DMU_k^w remains extreme strong efficient.

Suppose $\delta_i\tilde{\delta}_i = \theta_i^{\star(w)}, (i \in I)$, and DMU_k^w does not remain efficient.

Then, regarding model (6) we have $\theta_{sup}^{\star(w)} < 1.$ with regard to constraints of model (7), we have $\theta_{sup}^{\star(w)}\delta_i\tilde{\delta}_i \leqslant \theta_{sup}^{\star(w)}\theta_I^{\star(w)} < \theta_I^{\star(w)}$ which is a contradiction to the optimality of $\theta_I^{\star(w)}$. Therefore, DMU_k^w will remain remain efficient efficient, and the proof is complete. □

Now, consider the extreme efficient subunits $b_3, d_1, d_3,$. Model (7) for the evaluation of these subunits, setting $I = \{1\}$ or $I = \{2\}$ or $I = \{1, 2\}$, is feasible.

d_1 remains extreme efficient when it increases its first input while other subunits decrease their first inputs and inequality $1 \leqslant \delta_1\tilde{\delta}_1 \leqslant 1.669$ holds. Regarding its second input, it can preserve its efficiency classification when it increases its second input while other subunits decrease their second inputs such that inequality $1 \leqslant \delta_2\tilde{\delta}_2 \leqslant 1.5$ holds. If $1 \leqslant \delta_i\tilde{\delta}_i = 1.5$, then d_1 coincides f_1 and both become extreme efficient, and the efficiency classification of d_1 is preserved. If d_1 aims only to remain on the frontier, its second input can be increased by any amount. Considering d_3, which is an extreme efficient subunit, the first input can be increased while other subunit decrease their first inputs while inequality $1 \leqslant \delta_1\tilde{\delta}_1 \leqslant 1.473$ holds, in order for d_3 to preserve its efficiency classification. The second input of d_3 can be increased while those of other subunits are decrease as long as inequality $1 \leqslant \delta_2\tilde{\delta}_2 \leqslant 2.479$ holds, so that d_3 preserves its efficiency classification.

As regards b_3, it can preserve its efficiency classification if its first input is increased while the first inputs of other subunits are decreased and equality $1 \leqslant \delta_1\tilde{\delta}_1 \leqslant 1.336$ holds. If $\delta_1\tilde{\delta}_1 = 1.336$, then b_3 coincides c_3 and both become extreme efficient. Moreover, b_3 can remain in its efficiency classification, if its second input is increased while the second input of other subunits are decreased and inequality $1 \leqslant \delta_2\tilde{\delta}_2 \leqslant 1.113$ holds. If $\delta_2\tilde{\delta}_2 = 1.113$ then b_3 is projected onto a non-extreme strong efficient point on the frontier.

By considering output-oriented models similar to the input-oriented ones, we can deal with the sensitivity analysis of these models and obtain results similar to those of the input-oriented models.

4. Conclusion

This paper addressed networks composed of several production units , each one composed of subunits operating in parallel to each other. Since each subunits is working independently of other subunits , it can be considered as a separate DMU. By considering the pps produced by these subunits , we introduced model for obtaining the efficiency classifications , using super-efficiency models , and later we addressed the issue of seusitivity analysis and stability of the efficiency classification. In future works, sensitivity analysis of these models , assuming variable returns to scale with absolute perturbations in the data. The problem can also be extended and developed by considering the subunits operating in series or a mix of series and parallel operation models.

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