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Eigenfunctions of the Weighted Composition Operators

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Abstract. In the present paper, we characterize the eigenfunctions of a weighted composition operator on space of holomorphic function on the unit disk.

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1. Introduction

A weighted composition operator $C_{\varphi,\psi}$ is an operator that maps $f \in H(\mathbb{U})$, the space of holomorphic functions on the unit disk $\mathbb{U}$, into $C_{\varphi,\psi}(f)(z) = \varphi(z)f(\psi(z))$, where φ and ψ are analytic functions defined in $\mathbb{U}$ such that $\psi(\mathbb{U}) \subseteq \mathbb{U}$. When $\varphi \equiv 1$, we just have the composition operator C_ψ defined by $C_\psi(f) = f \circ \psi$.
The eigenfunctions of a composition operator on the classical Hardy space H^2, induced by a hyperbolic disk automorphism, are considered in [2, 4, 5] where it has been shown that many eigenfunctions of a composition operator can be found in the doubly cyclic subspace generated by special functions in H^2.
Studying the eigenfunctions of weighted composition operators entails a study of the iterate behavior of holomorphic self maps. The holomorphic self maps of $\mathbb{U}$ are divided into classes of elliptic and non-elliptic type. The elliptic type is an automorphism and has a fixed point in $\mathbb{U}$. It is

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well known that this map is conjugate to a rotation $z \to \lambda z$ for some complex number λ with $|\lambda| = 1$. The iterate of a non-elliptic map can be characterized by the Denjoy-Wolff Iteration Theorem [1, 6, 7]. In the following notation " $\stackrel{k}{\to}$ " means uniformly converges on compact subsets of $\mathbb{U}$ and ψ_n denotes the composition of ψ with itself n-times.

Denjoy-Wolff Iteration Theorem. *Suppose ψ is a holomorphic self-map of $\mathbb{U}$ that is not an elliptic automorphism. Then*

(*i*) *If ψ has a fixed point $w \in \mathbb{U}$, then $\psi_n \stackrel{k}{\to} w$ and $|\psi'(w)| < 1$.*

(*ii*) *If ψ has no fixed point in $\mathbb{U}$, then there is a point $w \in \partial\mathbb{U}$ such that $\psi_n \stackrel{k}{\to} w$ and the angular derivative of ψ exists at w, with $0 < \psi'(w) \leqslant 1$.*

We call the unique attracting point w, the Denjoy-Wolff point of ψ. By the Denjoy-Wolff Iteration Theorem, a general classification of a non-elliptic holomorphic self maps of $\mathbb{U}$ can be given: let w be the Denjoy-Wolff point of a holomorphic self-map of $\mathbb{U}$. We say ψ is of dilation type if $w \in \mathbb{U}$, of hyperbolic type if $w \in \partial\mathbb{U}$ and $\psi'(w) < 1$, and of parabolic type if $w \in \partial\mathbb{U}$ and $\psi'(w) = 1$.

In the present paper we characterize the eigenfunctions of a weighted composition operators on $H(\mathbb{U})$.

2. Main Result

From now on, we assume that w is the Denjoy-Wolff point of non-elliptic holomorphic self-map ψ and φ is a holomorphic function on $\mathbb{U}$ which is continuous at w and $\varphi(w) \neq 0$.

We characterize the eigenfunctions of $C_{\varphi,\psi}$ in $H(\mathbb{U})$. In fact, if the infinite product $\prod_{n=0}^{\infty} \frac{1}{\varphi(w)} \varphi(\psi_n(z))$ converges uniformly on compact subsets $\mathbb{U}$ then, the function

$$g(z) = \prod_{n=0}^{\infty} \frac{1}{\varphi(w)} \varphi(\psi_n(z)) \tag{1}$$

is holomorphic on $\mathbb{U}$ and satisfies the equation $\varphi.g \circ \psi = \varphi(w)g$ and is indeed an eigenfunction of $C_{\varphi,\psi}$. That all eigenfunctions of $C_{\varphi,\psi}$ in

$H(\mathbb{U})$, continuous at w, are obtained in this way is the content of the following theorem.

Proposition 2.1. *Let $g \in H(\mathbb{U})$ be a non-zero eigenfunction of $C_{\varphi,\psi}$ which is continuous at w. Then either $g(w) = 0$ or the infinite product $\prod_{n=0}^{\infty} \frac{1}{\varphi(w)}\varphi(\psi_n(z))$ converges uniformly on compact subsets of $\mathbb{U}$ and*

$$g(z) = g(w) \prod_{n=0}^{\infty} \frac{1}{\varphi(w)}\varphi(\psi_n(z)). \tag{2}$$

Proof. Let $g(w) \neq 0$ and $\varphi.g \circ \psi = \lambda g$ for some non-zero scalar λ. Then $\lambda = \varphi(w)$ and for integer $n \geqslant 1$,

$$\left(\prod_{i=0}^{n-1} \varphi(\psi_i(z))\right) g(\psi_n(z)) = \lambda^n g(z) = \varphi(w)^n g(z)$$

and so

$$\left(\prod_{i=0}^{n-1} \frac{1}{\varphi(w)}\varphi(\psi_i(z))\right) g(\psi_n(z)) = g(z) \qquad (z \in \mathbb{U}, n \geqslant 1) \tag{3}$$

where ψ_0 is the identity map on $\mathbb{U}$. Since $g(\psi_n(z)) \to g(w)$, the infinite product $\prod_{i=0}^{+\infty} \frac{1}{\varphi(w)}\varphi(\psi_i(z))$ converges in $H(\mathbb{U})$ to $g(w)^{-1}g(z)$ and (2) is deduced. □

The next proposition shows the iterate sequence of holomorphic self maps can exhibit a stronger form of convergence to the Denjoy-Wolff point.

Proposition 2.2. *The series*

$$\sum_{n=1}^{+\infty} |\psi_n(z) - w|^{\beta} \tag{4}$$

converges uniformly on compact subsets of $\mathbb{U}$ whenever

1. *ψ is not parabolic and $\beta > 0$, or*
2. *ψ is parabolic automorphic and $\beta = 2$.*

Proof. Suppose ψ is not of parabolic type. Then it is either of dilation or hyperbolic type. Let ψ be of dilation type and $w \in \mathbb{U}$. Then zero is the Denjoy-Wolff point of the self map $\alpha_w \circ \psi \circ \alpha_w$ where $\alpha_w(z) = \frac{z-w}{1-\overline{w}z}$. Choose $\delta > 0$ with $|\psi'(w)| < \delta < 1$. So $|\alpha_w \circ \psi \circ \alpha_w(z)| < \delta|z|$ when z is sufficiency near to zero. If K is a compact subset of $\mathbb{U}$, then by the Denjoy-Wolff Theorem, $\alpha_w \circ \psi_n \circ \alpha_w \to 0$ uniformly on K and $|\alpha_w \circ \psi_{n+k} \circ \alpha_w(z)| < \delta^k|\alpha_w \circ \psi_n \circ \alpha_w(z)|$ for sufficiently large n, every positive integer k, and $z \in K$. Upon replacing $\alpha_w(z)$ instead of z in the previous inequality, we get

$$\frac{|\psi_{n+k}(z) - w|}{2} \leqslant |\alpha_w(\psi_{n+k}(z))| \leqslant \delta^k |\alpha_w(\psi_n(z))| \tag{5}$$

Now suppose ψ is hyperbolic and $w \in \partial\mathbb{U}$, then $0 < \psi'(w) < 1$ and by Julia-Caratheodory Inequality ([1], Theorem 3.1) we get

$$\frac{|\psi(z) - w|^2}{1 - |\psi(z)|^2} < \psi'(w)\frac{|z - w|^2}{1 - |z|^2} \qquad (z \in \mathbb{U}).$$

By substituting $\psi_n(z)$ for z, we get

$$\frac{|\psi_n(z) - w|^2}{1 - |\psi_n(z)|^2} < (\psi'(w))^n\frac{|z - w|^2}{1 - |z|^2} \qquad (z \in \mathbb{U}, n \geqslant 0)$$

Now if K is a compact subset of $\mathbb{U}$, then the right hand of the above inequality is bounded on K. Hence it follows that

$$|\psi_n(z) - w| < \text{const.}(\psi'(w))^{\frac{n}{2}} \qquad (z \in K) \tag{6}$$

Thus the inequality (5) and (6) imply that (4) converges uniformly on compact subsets of $\mathbb{U}$ for $\beta > 0$. For the next part let ψ be of parabolic automorphic type. The Linear-Fractional Model Theorem [2, 8, 9] then provides a function σ holomorphic on $\mathbb{U}$ with values in the right half-plane such that $\sigma \circ \psi = \sigma + ib$ for some real $b \neq 0$. Hence more generally $\sigma \circ \psi_n = \sigma + nib$. Let K be an arbitrary compact subset of U. For $n \geqslant 1$, pick $z_n \in K$ such that $|\psi_n(z_n)| \leqslant |\psi_n(z)|$ for all $z \in K$.
The Blaschke condition for a sequence (z_n) in $\mathbb{U}$ is equivalent, via the map $w = \frac{1+z}{1-z}$, to the condition

$$\sum_n \frac{Re w_n}{|1 + w_n|^2} < \infty \tag{7}$$

for sequences (w_n) in the right half-plane. Since the sequence $(\sigma(z_n))$ is bounded, so (7) is to be satisfied by the sequence $w_n = \sigma(z_n) + nib$, which is therefore, the zero-sequence of a bounded holomorphic function F on the right half-plane (see [3] Theorem 11.3, page 191). The function $f = F \circ \sigma$ is then a non-constant bounded holomorphic function on $\mathbb{U}$, and for $n \geqslant 1$:

$$f(\psi_n(z_n)) = F(\sigma(\psi_n(z_n))) = F(\sigma(z_n) + nbi) = 0$$

Thus some nonconstant bounded holomorphic functions on $\mathbb{U}$ vanishes at each point of the sequence $(\psi_n(z_n))$, so that sequence satisfies the Blaschke condition. On the other hand, by the Julia-Caratheodory Inequality,

$$|\psi_n(z) - w|^2 \leqslant \text{const}(1 - |\psi_n(z)|^2) \leqslant \text{const}(1 - |\psi_n(z_n)|^2)$$

on K. Thus (4) uniformly converges on K for $\beta = 2$. $\square$

Recall that for any $w \in \overline{\mathbb{U}}$ and positive real number β, we denote by $Lip_\beta(w)$, the class of holomorphic functions φ satisfying

$$|\varphi(z) - \varphi(w)| = O(|z - w|^\beta) \qquad (z \to w) \tag{8}$$

For example if $\varphi \in H(\mathbb{U})$ is analytic at w, then $\varphi \in Lip_\beta(w)$ for $\beta \in (0, 1]$. Moreover, if $\varphi^{(i)}(w)$ exists and equal to zero for $i = 1, \ldots, n$ then $\varphi \in Lip_\beta(w)$ for $\beta \in (0, n+1]$.

Theorem 2.3. *Let $\varphi \in Lip_\beta(w)$ and $\varphi(w) \neq 0$ then the function $g(z)$ defined by equation (1) is an eigenfunction for $C_{\varphi,\psi}$, whenever*

1. *ψ is of dilation type, or*
2. *ψ is of hyperbolic type and $\beta > 0$, or*
3. *ψ is of parabolic automorphism type and $\beta = 2$.*

Proof. Assume that $\varphi \in Lip_\beta(w)$ for some real number β and K is a compact subset of $\mathbb{U}$. Since $\psi_n \to w$ uniformly on K, by substituting $\psi_n(z)$ instead of z in (8) we get

$$|\varphi(w) - \varphi(\psi_n(z))| = O(|w - \psi_n(z)|^\beta) \qquad (z \in K, n \to \infty)$$

whence

$$|1-\frac{1}{\varphi(w)}\varphi(\psi_n(z)|=O(\frac{1}{|\varphi(w)|}|w-\psi_n(z)|^{\beta})\qquad(z\in K,n\to\infty).$$

Now if ψ is hyperbolic and $\beta>0$ or ψ is parabolic automorphism and $\beta=2$ then by pervious Proposition, $\sum_{n=0}^{\infty}|1-\frac{1}{\varphi(w)}\varphi(\psi_n(z)|$ and consequently $g(z)=\prod_{n=0}^{\infty}\frac{1}{\varphi(w)}\varphi(\psi_n(z))$ converges uniformly on K. Thus (1) is indeed an eigenfunction for $C_{\varphi,\psi}$ and the proof is complete. □

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