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Chebyshev Finite Difference Method for Solving Constrained Quadratic Optimal Control Problems

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Abstract. In this paper the Chebyshev finite difference method is employed for finding the approximate solution of time varying constrained optimal control problems. This approach consists of reducing the optimal control problem to a nonlinear mathematical programming problem. To this end, the collocation points (Chebyshev Gauss-Lobatto nodes) are introduced then the state and control variables are approximated using special Chebyshev series with unknown parameters. The performance index is parameterized and the system dynamics and constraints are then replaced with a set of algebraic equations. Numerical examples are included to demonstrate the validity and applicability of the technique.

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1. Introduction

Orthogonal polynomials and orthogonal functions have been used to solve various problems of dynamical systems and optimal control theory. The key idea of this technique is that it reduces these problems to

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those of solving a system of algebraic equations or a finite dimensional optimization problem, i.e. a mathematical programming problem. This technique is based on parameterizing the rate variables (state and/or control variables), or discretization scheme. Typical examples are the Legendre polynomials ([18]), Chebyshev polynomials ([13,24]), Fourier series ([20,25]) and nonclassical orthogonal polynomials ([16]).

Most of the computing techniques for the solution of optimal control problems successfully solve the unconstrained problems, but the presence of inequality constraints often results in both analytical and computational difficulties. Theoretical aspects of trajectory inequality constraints have been studied in [7,17]. Mehra and Davis ([17]) noted that difficulties arising from handling trajectory inequality constraints are due to the exclusive use of control variables as independent variables, and they presented the so-called generalized gradient technique. In [24] the state and control variables are expanded into Chebyshev series with unknown coefficients. In their method the number of control and state variables are assumed to be equal. The coefficients which evolve from the classical Chebyshev series expansion of the performance index, the system dynamics and the boundary conditions, have to be calculated by some kind of analytical formulation for different problems. A Fourier-based state parametrization approach for solving linear quadratic optimal control problems is proposed in [16]. This approach is based on approximating the state variables by the sum of a third-order polynomial and a finite-term Fourier-type series. Their method requires that the influence matrix multiplied by the control vector $u(t)$ in the system dynamics is invertible; otherwise, a penalty function technique is imposed to produce another invertible influence matrix. Moreover, for time-invariant problems the integral involved in the definition of the performance index has to be evaluated in a closed form. Marzban and Razzaghi ([18]) proposed a method based on hybrid functions approach and Legendre polynomials for the solution of linearly constrained quadratic optimal control problems. In their method the operational matrices of integration and product are utilized to reduce the optimal control problem to the solution of algebraic equations. Moreover, the inequality constraints are first converted to a system of algebraic equations and then converted into another minimization problem.

Chebyshev polynomials are widely used in numerical computation. One of the advantages of using Chebyshev polynomials $T_n(t)$ as a tool for expansion functions is the good representation of smooth functions by finite Chebyshev expansion provided that the function $f(t)$ is infinitely differentiable. The coefficients in Chebyshev expansion, approach zero faster than any inverse power in n as n goes to infinity ([10]). Chebyshev finite difference method (ChFD) has proven to be successful in the numerical solution of various boundary value problems and in the solution of boundary layer equations ([8, 10]). ChFD method can be regarded as a non-uniform finite difference scheme. In this method the derivatives of the function $f(t)$ at a grid point t_j is linear combination of the values of the function f at the Chebyshev Gauss-Lobatto (ChGL) points $t_j = cos(j\pi/N)$, where $j = 0, 1, \ldots, N$, ([8,9,10]). ChFD method is more accurate in comparison to the finite difference ([5,6]) and finite elements methods because the approximation of the derivative is defined over the whole domain. While the finite difference method produces a second order accurate derivative with the error decreasing as $1/m^2$ (m being the number of grid points), the error from the global method decreases exponentially ([8]).

In the present paper we introduce an efficient computational method based on ChFD method for finding approximate solution of constrained optimal control problems. This method requires the definition of collocation points (ChGL nodes) and it is applied to satisfy the system dynamics, initial conditions and the inequality constraints at these grid points. Moreover, in this method both state and control variables are parameterized using special series of Chebyshev polynomials with unknown parameters, hence there is no need for using artificial variables, as in [13]. Also, the performance index is replaced with a nonlinear function. The application of the method to optimal control problems leads to a nonlinear mathematical programming problem.

The paper is organized as follows: In the following section the statement of the linear constrained optimal control problems is described. In Section 3, we describe the basic formulation of ChFD method required for our subsequent development and some error estimates are given.

Section 4 is devoted to description of employing ChFD method to the optimal control problem. In Section 5, we report our numerical finding to demonstrate the accuracy and applicability of the proposed method by considering three examples.

2. Problem Statement

Consider the time-varying system

$$\dot{X}(t) = E(t)X(t) + F(t)U(t), \tag{1}$$

$$X(0) = X_0, \tag{2}$$

subject to the following inequality constraints:

$$G(t)X(t) + H(t)U(t) \leqslant r(t), \tag{3}$$

where $X(t) \in R^m$, $U(t) \in R^n$, $E(t), F(t), G(t)$ and $H(t)$ are matrices of appropriate dimensions, X_0 is a constant special vector and $r(t)$ is an arbitrary known function. The problem is to find the optimal control $U(t)$ and the corresponding state trajectory $X(t)$, $0 \leqslant t \leqslant t_f$, satisfying (1)-(3) while minimizing the following quadratic cost functional

$$J = \frac{1}{2}X^T(t_f)SX(t_f) + \frac{1}{2}\int_0^{t_f} \left[X^T(t)Q(t)X(t) + U^T(t)R(t)U(t)\right] dt, \tag{4}$$

where T denotes transposition, S, $Q(t)$ and $R(t)$ are matrices of appropriate dimensions with S and $Q(t)$ symmetric positive semi-definite matrices and $R(t)$ a symmetric positive definite matrix.
To use the ChFD method, we should transform the time interval $t \in [0, t_f]$ into the interval $\tau \in [-1, 1]$, because Chebyshev polynomials are defined on the interval $[-1, 1]$. This can be achieved by transformation

$$t = \frac{t_f}{2}(\tau + 1).$$

Expressing the optimal control problem in (1)-(4) in terms of τ results the following optimal control problem.

Minimize

$$J = \frac{1}{2}x^T(1)Sx(1) + \frac{t_f}{4}\int_{-1}^{1}\left[x^T(\tau)q(\tau)x(\tau) + u^T(\tau)r(\tau)u(\tau)\right]d\tau, \quad (5)$$

subject to

$$\frac{2}{t_f}\frac{dx(\tau)}{d\tau} = e(\tau)x(\tau) + f(\tau)u(\tau), \quad (6)$$

$$x(-1) = X_0, \quad (7)$$

$$g(\tau)x(\tau) + h(\tau)u(\tau) \leqslant \bar{r}(\tau). \quad (8)$$

3. Chebyshev Finite Difference Method

The well known Chebyshev polynomials of the first kind are defined on the interval $[-1, 1]$ as

$$T_n(\tau) = \cos(n \ \arccos\tau), \qquad n = 0, 1, 2, \ldots,$$

obviously $T_0(\tau) = 1$, $T_1(\tau) = \tau$ and they satisfy the recurrence relation

$$T_{n+1}(\tau) = 2\tau T_n(\tau) - T_{n-1}(\tau), \qquad n = 1, 2, \ldots.$$

The ChGL collocation points (interpolation nodes) are defined to be the extreme

$$\tau_j = \cos\left(\frac{j\pi}{N}\right), \qquad j = 0, 1, \ldots, N,$$

of the Nth-order Chebyshev polynomial $T_N(\tau)$. These collocation points, $\tau_N = -1 < \tau_{N-1} < \ldots < \tau_1 < \tau_0 = 1$ are also viewed as the zeros of $(1-\tau^2)\dot{T}_N(\tau)$, where $\dot{T}_N(\tau) = dT_N(\tau)/d\tau$.

Clenshaw and Curtis ([4]) introduced the following Nth degree interpolating polynomial of a function $f \in L^2[-1, 1]$,

$$f^N(\tau) = \sum_{i=0}^{N}{}'' a_i T_i(\tau), \qquad a_i = \frac{2}{N}\sum_{j=0}^{N}{}'' f(\tau_j)T_i(\tau_j). \quad (9)$$

The summation symbol with double primes denotes a sum with both the first and last terms halved. The first derivative of the function $f^N(\tau)$ in terms of $f(\tau)$ at the ChGL collocation point τ_j are given by [10]

$$\dot{f}^N(\tau_j) = \sum_{i=0}^{N} d_{j,i} f(\tau_i), \tag{10}$$

where

$$d_{j,i} = \frac{4\theta_i}{N} \sum_{n=0}^{N} \sum_{\substack{l=0 \\ (n+l) odd}}^{n-1} \frac{n\theta_n}{c_l} T_n(\tau_i) T_l(\tau_j), \qquad i, j = 0, 1, \ldots, N, \tag{11}$$

with $\theta_0 = \theta_N = 1/2$, $\theta_i = 1$ for $i = 1, 2, \ldots, N-1$, and $c_0 = 2$, $c_l = 1$ for $l \geqslant 1$. As can be seen from (10), the first derivative of the function $f(\tau)$ at any point from ChGL nodes are expanded as linear combination of the values of the function at these points.
The unknown parameters in (9) are $f(\tau_0), f(\tau_1), \ldots, f(\tau_N)$; consequently, (9) can be expressed in terms of these unknown parameters as follows

$$f^N(\tau) = \sum_{j=0}^{N}{}'' f(\tau_j) p_j(\tau), \qquad p_j(\tau) = \frac{2}{N} \sum_{i=0}^{N}{}'' T_i(\tau_j) T_i(\tau). \tag{12}$$

It is readily verified that

$$p_k(\tau_j) = \begin{cases} 1 & k = j, \\ 0 & k \neq j, \end{cases} \tag{13}$$

and therefore $f^N(\tau_j) = f(\tau_j)$ for $j = 0, 1, \ldots, N$.
Now we give some estimates for the interpolation error $f - f^N$. Since f^N in (12) is the interpolating polynomial for function f, it follows that (consult [1,11])

$$\lim_{N \to \infty} \int_{-1}^{1} |f^N(\tau) - f(\tau)|^p (1 - \tau^2)^{-1/2} d\tau = 0, \tag{14}$$

for every $f \in C[-1, 1]$, the Banach space of continuous, real-valued functions on $[-1, 1]$, and for every $p \in (0, \infty)$. In terms of the usual L_p norm (14) may be written as

$$\lim_{N \to \infty} \|f^N(\tau) - f(\tau)\|_{p,w} = 0, \tag{15}$$

from which it follows that $\sup_N \|f^N\|_{p,w} < \infty$.
Furthermore, whenever $f \in H_w^l(-1,1)$ for some $l \geqslant 1$, there is a constant M independent of f and N such that [3]

$$\|f - f^N\|_{L_{2,w}(-1,1)} \leqslant MN^{-l}\|f\|_{H_w^l(-1,1)}. \tag{16}$$

The next Theorem is a generalization of the above error estimate ([3]).

Theorem 3.1. *For all $f \in H_w^l(-1,1), l \geqslant 0$, there exist a constant M independent of $f(\tau)$ and N such that*

$$\|f - f^N\|_{H_w^s(-1,1)} \leqslant MN^{2s-l}\|f\|_{H_w^l(-1,1)}, \tag{17}$$

for all $0 \leqslant s \leqslant l$.
As a consequence, we have

$$\|f' - (f^N)'\|_{L_{2,w}(-1,1)} \leqslant MN^{2-l}\|f\|_{H_w^l(-1,1)}. \tag{18}$$

Thus, for smooth functions f, the rate of convergence of f^N to f is faster than any power of $\frac{1}{N}$. The next Theorem shows uniform convergence ([11]).

Theorem 3.2. *If τ_j, $1 \leqslant j \leqslant N-1$ are the zeros of $\dot{T}_N(\tau)$ adjusted in the interval $(-1,1)$, if $f(z)$ has no singularities except a finite number of poles, and if for some n, $\frac{f(z)}{z^n} \to 0$ as $|z| \to \infty$, then $f^N(\tau) \to f(\tau)$ uniformly on $[-1,1]$.*

4. Employing ChFD Method to Optimal Control Problem

In this section we explain the procedure of converting the optimal control problem in (5)-(8) into a nonlinear mathematical programming problem using ChFD method. To this end, the interpolation (12) and the ChGL collocation points are utilized to approximate the performance index, system dynamics and the inequality constraints.

4.1 The Performance Index Approximation

We approximate the performance index in (5) as follows: Let

$$x^N(\tau) = [x_1^N(\tau), x_2^N(\tau), \ldots, x_m^N(\tau)]^T, \tag{19}$$

$$u^N(\tau) = [u_1^N(\tau), u_2^N(\tau), \ldots, u_n^N(\tau)]^T. \tag{20}$$

where, by (12), we have

$$x_i^N(\tau) = \sum_{k=0}^{N}{}'' x_i(\tau_k) p_k(\tau), \quad i = 1, 2, \ldots, m, \tag{21}$$

$$u_j^N(\tau) = \sum_{k=0}^{N}{}'' u_j(\tau_k) p_k(\tau), \quad j = 1, 2, \ldots, n. \tag{22}$$

Vectors $x^N(\tau)$ and $u^N(\tau)$ in (19)-(20) can be expressed using Kronecker product [2] as follows:

$$x^N(\tau) = (I_m \otimes P^T(\tau))a, \tag{23}$$

$$u^N(\tau) = (I_n \otimes P^T(\tau))b, \tag{24}$$

where $\otimes$ denotes the Kronecker product, I_m and I_n are $m \times m$ and $n \times n$ identity matrices, respectively,

$$a^T = [x_1(\tau_0), x_1(\tau_1), \ldots, x_1(\tau_N), \ldots, x_m(\tau_0), \ldots, x_m(\tau_N)],$$

$$b^T = [u_1(\tau_0), u_1(\tau_1), \ldots, u_1(\tau_N), \ldots, u_n(\tau_0), \ldots, u_n(\tau_N)],$$

are $1 \times m(N+1)$ and $1 \times n(N+1)$ vectors of unknown parameters,

$$P^T(\tau) = [\frac{1}{2}p_0(\tau), p_1(\tau), \ldots, p_{N-1}(\tau), \frac{1}{2}p_N(\tau)], \tag{25}$$

and $p_k(\tau), \quad k = 0, 1, \ldots, N$ are defined in (13).
Now we approximate the performance index by substituting (23)-(24) into (6) to get

$$J^N = \frac{1}{2}a^T(I_m \otimes P(1))S(I_m \otimes P^T(1))a + \frac{t_f}{4}\int_{-1}^{1}[a^T(I_m \otimes P)q(\tau)(I_m \otimes P^T)a$$
$$+ b^T(I_n \otimes P)r(\tau)(I_n \otimes P^T)b\,]d\tau, \tag{26}$$

where J^N is the approximate value of J.

From Kronecker product properties ([2]), it is known that if matrices A, B, D, E and vector z are of appropriate dimensions then $(A\otimes B)(D\otimes E) = AD\otimes BE$and $A(I\otimes z^T) = A\otimes z^T$. Therefore, (27) can be simplified to

$$J^N = \frac{1}{2}a^T(S\otimes P(1)P^T(1))a+\frac{t_f}{4}\int_{-1}^{1}\left[a^T(q(\tau)\otimes PP^T)a + b^T(r(\tau)\otimes PP^T)b\right]d\tau, \tag{27}$$

which can be rewritten as follows

$$J^N = a^T H_1 a + b^T H_2 b, \tag{28}$$

where

$$H_1 = \frac{1}{2}(S\otimes P(1)P^T(1)) + \frac{t_f}{4}\int_{-1}^{1}(q(\tau)\otimes PP^T)d\tau,$$

$$H_2 = \frac{t_f}{4}\int_{-1}^{1}(r(\tau)\otimes PP^T)d\tau.$$

4.2 The System Dynamics and the Inequality Constraints Approximations

The system dynamics and the constraints in (7)-(9), which are infinite dimensional constraints, can be handled by requiring their satisfaction at the ChGL collocation points τ_j. To do so, we first using (11) express the relationship between $\dot{x}_i^N(\tau)$ and $x_i(\tau)$ at the ChGL nodes τ_j for $j = 0, 1, \ldots, N$, as

$$\dot{x}_i^N(\tau_j) = \sum_{k=0}^{N} d_{j,k}x_i(\tau_k), \qquad i = 1, 2, \ldots, m. \tag{29}$$

Consequently, the vectors $\dot{x}^N(\tau_j) = [\dot{x}_1^N(\tau_j), \dot{x}_2^N(\tau_j), \ldots, \dot{x}_m^N(\tau_j)]^T$, $j = 0, 1, \ldots, N$, can be expressed using Kronecker product as follows

$$\dot{x}^N(\tau_j) = (I_m\otimes D_j^T)a, \qquad j = 0, 1, \ldots, N, \tag{30}$$

where $a^T = [x_1(t_0), x_1(t_1), \ldots, x_1(t_N), \ldots, x_m(t_0), \ldots, x_m(t_N)]$ is vector of unknown parameters, $D_j = [d_{j,0}, d_{j,1}, \ldots, d_{j,N}]^T$ is an $(N+1)\times 1$ vector of derivative coefficients defined in (12).

A formal substitution of (23), (24) and (30) into (7)-(9) and collocating at the ChGL nodes τ_j for $j = 0, 1, \ldots, N$, (7)-(9) become the system of algebraic equations:

$$2(I_m \otimes D_j^T)a = t_f(e(\tau_j)(I_m \otimes P^T(\tau_j))a + f(\tau_j)(I_n \otimes P^T(\tau_j))b), \quad (31)$$

$$(I_m \otimes P^T(\tau_N))a = X_0, \quad (32)$$

$$g(\tau_j)(I_m \otimes P^T(\tau_j))a + h(\tau_j)(I_n \otimes P^T(\tau_j))b \leqslant \bar{r}(\tau_j). \quad (33)$$

From the previous reformulation in (28) and (31)-(33), the optimal control problem in (2)-(5) can be approximated by the following nonlinear programming problem:

Minimize

$$J^N = a^T H_1 a + b^T H_2 b, \quad (34)$$

subject to

$$F_1 a - b_1 = 0, \quad (35)$$

$$F_2 a + F_3 b = 0, \quad (36)$$

$$F_4 a + F_5 b - b_2 \leqslant 0, \quad (37)$$

where

$$F_1 = (I_m \otimes P^T(\tau_N)),$$

$$b_1 = X_0,$$

$$F_2 = \tfrac{2}{t_f}(I_m \otimes D_j^T) - e(\tau_j)(I_m \otimes P^T(\tau_j)),$$

$$F_3 = -f(\tau_j)(I_n \otimes P^T(\tau_j)),$$

$$F_4 = g(\tau_j)(I_m \otimes P^T(\tau_j)),$$

$$F_5 = h(\tau_j)(I_n \otimes P^T(\tau_j)),$$

$$b_2 = \bar{r}(\tau_j).$$

Note, by (13), (14), and (25), that

$$P^T(\tau_j) = \begin{cases} \frac{1}{2}e_1 & j = 0, \\ e_{j+1} & j = 1, \ldots, N-1, \\ \frac{1}{2}e_{N+1} & j = N, \end{cases}$$

where e_k is an $1 \times (N+1)$ vector whose kth component is 1 and other components are 0. The nonlinear mathematical programming problem in (34)-(37) can be solved using, for example, active set methods ([23]).

Remark 4.2.1. *As the theorems and error estimates in Section 3. promise us, the truncation error in ChFD method decays as fast as the global smoothness of the underlying solution permits. Specially, since the integrand in the functional (6) is continuously differentiable with respect to $x(\tau)$ and $u(\tau)$, then ${x^N}^T q x^N + {u^N}^T r u^N \to x^T q x + u^T r u$ as $N \to \infty$. From that it follows immediately that as $N \to \infty$, $J^N \to J$.*

Remark 4.2.2. *The selection of N is crucial for the computational efficiency. Clearly as N increases the time needed to solve the problem increases. To avoid selecting large N we propose to solve any problem as follows: Select $N = N_1 = 4$ or 5, then solve the problem and obtain the optimal value J^{N_1}. Increase N to become $N_2 = N_1 + 2$ and solve the problem again to obtain the optimal value J^{N_2}. If the difference between J^{N_1} and J^{N_2} is not too big, then we can keep increasing N by 2 until the reduction in the performance index become negligible. Nevertheless, if the difference is too big, then we may consider increasing N_1 by 3 and proceeding as described above.*

5. Numerical Examples

In this Section, three examples are given to demonstrate the applicability and accuracy of the proposed method. Note that we have computed the numerical results by the well-known symbolic software "Mathematica 5.2".

In order to decide whether or not the computed solution is close enough to the optimal solution, we use, for computational purposes, practical and easy-to-use error estimates [12]: Substituting the calculated control

$u^N(\tau)$ in (7), gives

$$\frac{2}{t_f}\dot{x}(\tau) = e(\tau)x(\tau) + f(\tau)u^N(\tau), \quad -1 \leqslant \tau \leqslant 1. \tag{38}$$

Numerical integration of (38) is possible for a given initial or final conditions. Let $\hat{x}(\tau)$ be the solution obtained from numerical integration of (38), and define a practical easy-to-use error estimate for the dynamical equations

$$\varepsilon_{dyn} = \|x^N - \hat{x}\|_\infty = \max_{-1 \leqslant \tau \leqslant 1} |x^N(\tau) - \hat{x}(\tau)|. \tag{39}$$

Another important error estimate is the SAK

$$SAK = \sum_{j=0}^{N} |\frac{2}{t_f}\dot{x}(\tau_j) - e(\tau_j)x(\tau_j) - f(\tau_j)u(\tau_j)|. \tag{40}$$

5.1 Example 1

This problem is adapted from [21]. Consider the following problem:

Minimize

$$J = \int_0^1 (X_1^2(t) + X_2^2(t) + U^2(t))dt, \tag{41}$$

subject to

$$\begin{bmatrix} \dot{X}_1(t) \\ \dot{X}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} U(t), \tag{42}$$

and the boundary conditions

$$X_1(0) + X_2(0) = 3, \qquad X_2(1) = 1. \tag{43}$$

Here the problem is to find optimal control $U(t)$ which minimizes (41) subject to (42)-(43). After transforming the interval of the problem into the interval $[-1, 1]$, using the transformation $t = \frac{1}{2}(\tau + 1)$, the following problem is achieved.

Minimize

$$J = \frac{1}{2}\int_{-1}^1 (x_1^2(\tau) + x_2^2(\tau) + u^2(\tau))d\tau, \tag{44}$$

subject to

$$2\begin{bmatrix} \dot{x}_1(\tau) \\ \dot{x}_2(\tau) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1(\tau) \\ x_2(\tau) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(\tau), \tag{45}$$

$$x_1(-1) + x_2(-1) = 3, \qquad x_2(1) = 1. \tag{46}$$

In Table 1, the error estimates ε_{dyn} and SAK are given and a comparison is made between the estimated values of J using the present method with $N = 8, 10, 12$ and method in [21] using extended one-step methods.

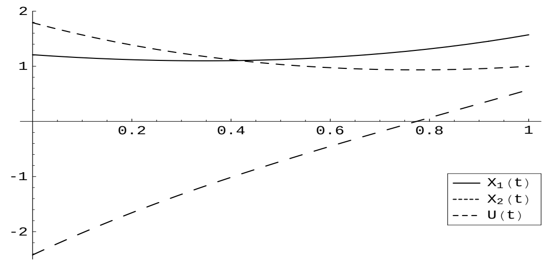

Figure 1: Optimal control and states for $N = 12$ for Example 1.

Note that in [21], methods (2.3), (2.4), and (2.5) are trapezoidal rule, a one-step third order, and a one-step fourth order extended methods for ODEs, respectively.

In Table 2, we list the values of state and control variables obtained using the present method and method in [21]. For $N = 12$, the optimal states and the optimal control are shown in Fig. 1. The boundary conditions in (43) are accurately satisfied and the errors for $N = 12$ are 1.2×10^{-14} and 2.3×10^{-14}, respectively.

Table 1: Estimated values of J and error estimates for Example 1.

Methods	SAK	ε_{dyn}	J
Extended one-step [21]			
$L = 12$ and method (2.3)	—	—	4.20789817
$L = 12$ and method (2.4)	—	—	4.20767792
$L = 12$ and method (2.5)	—	—	4.20767790
Present method with			
$N = 8$	4.3×10^{-12}	8.5×10^{-7}	4.20767795
$N = 10$	2.0×10^{-12}	8.2×10^{-7}	4.20767794
$N = 12$	1.4×10^{-12}	8.1×10^{-7}	4.20767790

Table 2: Values of state and control variables for Example 1.

t	Method (2.5) in [21], $L = 10$			Present method, $N = 10$		
	$X_1(t)$	$X_2(t)$	$U(t)$	$X_1(t)$	$X_2(t)$	$U(t)$
0.0	1.20973	1.79026	-2.42200	1.20971	1.79029	-2.42194
0.2	1.11926	1.38490	-1.65660	1.11925	1.38487	-1.65655
0.4	1.10222	1.11950	-1.01371	1.10220	1.11952	-1.01365
0.6	1.16421	0.97403	-0.45156	1.16419	0.97403	-0.45156
0.8	1.31544	0.93598	0.06658	1.31541	0.93598	0.06657
1.0	1.57137	1.00000	0.57477	1.57134	1.00000	0.57477

5.2 Example 2

Consider the linear system with inequality control constraint ([22])

$$\begin{bmatrix} \dot{X}_1(t) \\ \dot{X}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(t), \tag{47}$$

$$\begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \qquad |U(t)| \leqslant r(t) = 1, \tag{48}$$

with the performance index

$$J = \frac{1}{2}\int_0^1 (X_1^2(t) + U^2(t))dt. \tag{49}$$

The objective is to find optimal control $U(t)$ which minimizes (48) subject to (47)-(48). In the first step of the present method, the transformation $t = \frac{1}{2}(\tau + 1)$ is utilized to get the following problem:

Minimize

$$J = \frac{1}{4}\int_{-1}^1 (x_1^2(\tau) + u^2(\tau))d\tau, \tag{50}$$

subject to

$$2\begin{bmatrix} \dot{x}_1(\tau) \\ \dot{x}_2(\tau) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(\tau) \\ x_2(\tau) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(\tau), \tag{51}$$

$$\begin{bmatrix} x_1(-1) \\ x_2(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \qquad |u(\tau)| \leqslant \bar{r}(\tau) = 1. \tag{52}$$

After approximating each of the state and control variables using (21)-(22), the optimal control problem in (50)-(52) is solved using the proposed method. In Table 3, we give the approximated values of the cost functional J and the error estimates ε_{dyn} and SAK using the present method for $N = 6, 9, 12$, and 15, together with the method outlined in [18] using hybrid of block-pulse and Legendre polynomials and the exact solution.

The computational results for optimal state and control trajectories using the present method for $N = 15$ are given in Figs. 2 and 3, respectively. The maximum violation in the inequality constraint for $N = 15$ is found to be 0.0048.

Table 3: Estimated and exact values of J and error estimates for Example 2.

Methods	SAK	ε_{dyn}	J
Hybrid functions [18]			
$M = 3,\ N = 4$	—	—	8.07059
$M = 4,\ N = 4$	—	—	8.07056
Present method with			
$N = 6$	4.7×10^{-10}	7.0×10^{-6}	8.07110
$N = 9$	9.5×10^{-11}	3.5×10^{-6}	8.07061
$N = 12$	8.6×10^{-11}	5.1×10^{-7}	8.07055
$N = 15$	5.5×10^{-11}	3.8×10^{-7}	8.07056
Exact			8.07054

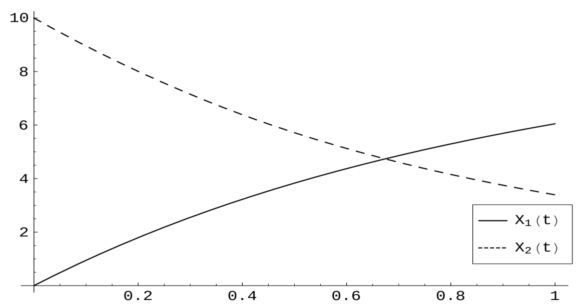

Figure 2: Optimal states for $N = 15$ for Example 2.

5.3 Example 3

This problem is adapted from [15] and also studied by using classical Chebyshev approach ([24]), Fourier-based state parametrization ([25]), spectral Chebyshev ([14]), hybrid functions ([18]), and rationalized Haar functions ([19]). Find the control function $U(t)$ which minimizes the cost functional

$$J = \int_0^1 (X_1^2(t) + X_2^2(t) + 0.005U^2(t))dt, \tag{53}$$

subject to

$$\begin{bmatrix} \dot{X}_1(t) \\ \dot{X}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U(t), \tag{54}$$

$$\begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \tag{55}$$

and the following state variable inequality constraint

$$X_2(t) \leqslant 8(t-0.5)^2 - 0.5. \tag{56}$$

In order to obtain the ChFD method approximations, we first convert the interval $t \in [0, 1]$ into $\tau \in [-1, 1]$. The inequality constraint in (56) can be expressed in terms of τ as follows

$$x_2(\tau) \leqslant 8(\frac{1}{2}(\tau+1) - 0.5)^2 - 0.5 = 2\tau^2 - 0.5.$$

In Table 4, the obtained results for the error estimates ε_{dyn} and SAK and the performance index J using the present method together with other methods in the literature are listed. The computational result for $X_1(t)$, $X_2(t)$ and $U(t)$ using the present method for $N = 13$ are given in Figs. 4 and 5. The maximum violation in the inequality constraint for $N = 13$ is found to be 0.007.

As can be seen from the Table 4, ChFD method can produces more accurate results in comparison with both classical Chebyshev ([24]) and spectral Chebyshev ([14]) methods.

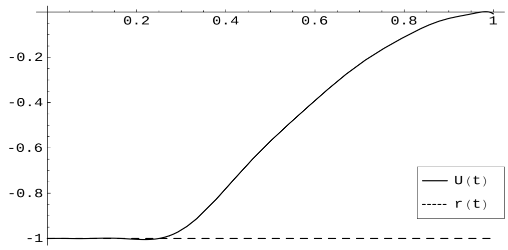

Figure 3: Optimal control for $N = 15$ for Example 2.

Table 4: Results of J and error estimates for Example 4.

Methods	SAK	ε_{dyn}	J
Classical Chebyshev [24]			
$m = 10,\ k = 22$	—	—	0.178800
$m = 12,\ k = 26$	—	—	0.173580
$m = 13,\ k = 28$	—	—	0.171850
Fourier-based [25]			
$k = 7$	—	—	0.17039
$k = 9$	—	—	0.17013
Spectral Chebyshev [14]			
$N = 13$	—	—	0.170785
Hybrid functions [18]			
$w = 15,\ M = 4,\ N = 4$	—	—	0.170136
Haar functions [19]			
$k = 32,\ w = 100$	—	—	0.170185
$k = 64,\ w = 100$	—	—	0.170115
$k = 128,\ w = 100$	—	—	0.170103
Present method with			
$N = 7$	8.8×10^{-10}	4.5×10^{-4}	0.174064
$N = 10$	6.8×10^{-11}	3.2×10^{-5}	0.170875
$N = 13$	3.4×10^{-12}	5.0×10^{-6}	0.169826

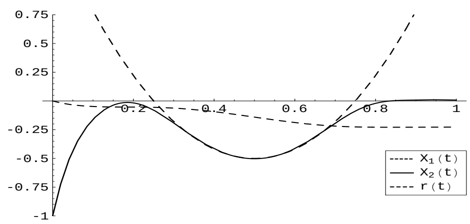

Figure 4: Optimal states for $N = 13$ for Example 3.

6. Conclusion

In this paper, the Chebyshev finite difference method has been used for the numerical solution of constrained quadratic optimal control problems. The problem was reduced to solve a nonlinear programming problem by parameterizing the performance index and satisfaction of the system dynamics and the constraints at the Chebyshev Gauss-Lobatto nodes. This method is not faced with necessity of large computer memory and time and is computationally attractive. Applications was demonstrated through three examples.

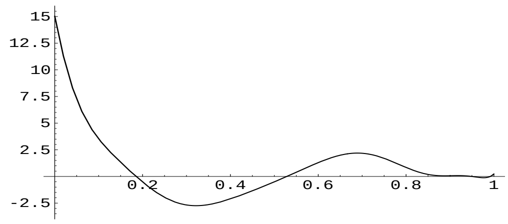

Figure 5: Optimal control for $N = 13$ for Example 3.

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