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Nonlinear Fredholm Integral Equation of the Second Kind with Quadrature Methods

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Abstract. In this paper, a numerical method for solving the nonlinear Fredholm integral equation is presented. We intend to approximate the solution of this equation by quadrature methods and by doing so, we solve the nonlinear Fredholm integral equation more accurately. Several examples are given at the end of this paper.

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1. Introduction

The problem of finding numerical solution for Fredholm integral equations of the second kinds is one of the oldest problems in the applied mathematics literature and many computational methods are introduced in this field ([1,6,7]).
Further more, the solution of an integral equation with quadrature methods is approximated by solving the system:

$$u(x_i) = f(x_i) + \sum_{j=0}^{n} w_j k(x_i, x_j) u(x_j)$$

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see ([1,6,7,9]). These methods discuss about the linear Fredholm integral equation ([6]). In ([12]), Saberi-Nadjafi and Heidari by applying modified trapezoid formula, solved linear integral equation, while finding an approximate solution for the nonlinear kind is difficult. Approaches for solving the nonlinear kind of these equation may be found in [2-4,8,10]. In this paper we use the repeated Simpson quadrature and repeated modified trapezoid formula for solving nonlinear Fredholm integral equations.

2. Quadrature Methods and Nonlinear Integral Equations

Simpson rule and modified trapezoid methods formula for solving a definite integral $\int_a^b f(x)\,dx$ are as follows:
(Simpson's rule):

$$\int_a^b f(x)\,dx = \frac{h}{3}[f(a) + 4f(\frac{a+b}{2}) + f(b)]$$

(modified trapezoid):

$$\int_a^b f(x)\,dx = \frac{h}{2}[f(a) + f(b)] + \frac{h^2}{12}[f'(a) - f'(b)] - \frac{h^5}{720} f^{(4)}(\eta).$$

These methods are explained in [11], and their repeated formulas are as follows:
(Simpson's rule):

$$\begin{aligned}\int_a^b f(x)\,dx &= \sum_{i=1}^{n-1} \int_{x_{i-1}}^{x_{i+1}} f(x)\,dx \\ &= \frac{h}{3} f(a) + \frac{4h}{3} \sum_{i=1}^{\frac{n}{2}} f(x_{2i-1}) + \frac{2h}{3} \sum_{i=1}^{[\frac{n-1}{2}]} f(x_{2i}) \\ &\qquad + \frac{h}{3} f(b) - \frac{(b-a)}{180} h^4 f^{(4)}(\eta)\end{aligned}$$

(modified trapezoid):

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx = \frac{h}{2} f(a) + h \sum_{i=1}^{n-1} f(x_i) + \frac{h}{2} f(b) + \frac{h^2}{12}[f'(a) - f'(b)]$$

The general form of a nonlinear Fredholm integral equation is as follows that F is a nonlinear operator:

$$u(x) = f(x) + \lambda \int_a^b k(x,t) F(u(t))\,dt \qquad a \leqslant x \leqslant b \tag{1}$$

For using Simpson's rule, n must be even and for modified trapezoid formula, $k(x,t)$ and $f(x)$ must be differentiable with respect to their variables, i.e. the functions $\frac{\partial k(x,t)}{\partial x}, \frac{\partial k(x,t)}{\partial t}$ and $f'(x)$ must exist. Before we clarify further, we define the following notation for simplicity :

$$f_i = f(x_i), \quad k_i j = k(x_i, x_j)$$

$$f'_i = f'(x_i), \quad H_i j = H(x_i, x_j)$$

$$u_i = u(x_i), \quad J_i j = J(x_i, x_j)$$

$$u'_i = u'(x_i).$$

3. The Numerical Approach

For solving (1), we approximate the right-hand integral of (1) with the repeated Simpson's rule and modified trapezoid, then we get:
(Simpson's rule):

$$u(x) = f(x) + \tfrac{h}{3}[k(x,x_0)F(u_0) + 4\textstyle\sum_{j=1}^{\frac{n}{2}} k(x, x_{2j-1})F(u_{2j-1})$$

$$+2\textstyle\sum_{j=1}^{[\frac{n-1}{2}]} k(x, x_{2j})F(u_{2j}) + k(x, x_n)F(u_n)] \tag{2}$$

(modified trapezoid):

$$u(x) = f(x) + \tfrac{h}{2}k(x,x_0)F(u_0) + h\textstyle\sum_{j=1}^{n-1} k(x,x_j)F(u_j)$$

$$+\tfrac{h}{2}k(x,x_n)F(u_n) + \tfrac{h^2}{12}[J(x,x_0)F(u_0)$$

$$+k(x,x_0)u_0'F'(u_0) - J(x,x_n)u_n - k(x,x_n)u_n'F'(u_n)], \qquad (3)$$

where $J(x,t) = \frac{\partial k(x,t)}{\partial t}$.

Hence, for $x = x_0, x_1, \cdots, x_n$ in (2) we have:

$$u_i = f_i + \frac{h}{3}[k_{i0}F(u_0) + 4\sum_{j=1}^{\frac{n}{2}} k_{i,2j-1}F(u_{2j-1}) + 2\sum_{j=1}^{[\frac{n-1}{2}]} k_{i,2j}F(u_{2j}) + k_{in}F(u_n)]. \quad (4)$$

This is a nonlinear system of equations and by solving it, we obtain the unknowns u_i for $i = 0, 1, \cdots, n$. Then, with Simpson's rule we can approximate the solution. Now for modified trapezoid formula, by substituting $x = x_i$ in (3) we have:

$$u_i \;= f_i + \tfrac{h}{2}k_{i0}F(u_0) + h\textstyle\sum_{j=1}^{n-1} k_{ij}F(u_j) + \tfrac{h}{2}k_{in}F(u_n)$$

$$+\tfrac{h^2}{12}[J_{i0}F(u_0) + k_{i0}u_0'F'(u_0) - J_{in}u_n - k_{in}u_n'F'(u_n)], \qquad (5)$$

for $i = 0, 1 \ldots n$. This is a system of $(n+1)$ equations and $(n+3)$ unknowns. By taking derivative from the Eq. (1), and setting $H(x,t) = \frac{\partial k(x,t)}{\partial x}$, we obtain:

$$u'(x) = f'(x) + \textstyle\int_a^b H(x,t)F(u(t))\,dt. \qquad (6)$$

Note that if u is a solution of equation (1), then it is also a solution of (6). By using repeated trapezoid quadrature for (6), and placing $x = x_i$ we get:

$$u_i' = f_i' + \tfrac{h}{2}H_{i0}F(u_0) + h\sum_{j=1}^{n-1} H_{ij}F(u_j) + \tfrac{h}{2}H_{in}F(u_n) \qquad (7)$$

for $i = 0, 1 \ldots n$. Note that in the cases $i = 0, n$, from system (7), we obtain two equations. These equations with system (5), make the nonlinear system of equations as follows:

$$
\begin{cases}
u_i = f_i + \left(\dfrac{h}{2}k_{i0} + \dfrac{h^2}{12}J_{i0}\right)F(u_0) + h\displaystyle\sum_{j=1}^{n-1} k_{ij}F(u_j) \\
\qquad + \left(\dfrac{h}{2}k_{in} - \dfrac{h^2}{12}J_{in}\right)F(u_n) + \frac{h^2}{12}\left(k_{i0}u_0'F'(u_0) - k_{in}u_n'F'(u_n)\right) \\
\\
u_0' = f_0' + \frac{h}{2}H_{00}F(u_0) + h\displaystyle\sum_{j=1}^{n-1} H_{0j}F(u_j) + \dfrac{h}{2}H_{0n}F(u_n) \\
u_n' = f_n' + \frac{h}{2}H_{n0}F(u_0) + h\displaystyle\sum_{j=1}^{n-1} H_{nj}F(u_j) + \dfrac{h}{2}H_{nn}F(u_n)
\end{cases} \tag{8}
$$

By solving this system with (n+3) nonlinear equations and (n+3) unknowns, the approximate solution of the Eq. (1), is obtained.

4. Numerical Examples

Example 1. Consider the nonlinear Fredholm integral equation

$$u(x) = \frac{7}{8}x + \frac{1}{2}\int_0^1 xtu^2(t)\,dt$$

with exact solution $u(t) = t$. Simpson's rule and modified trapezoid are exact for this example, because error term in these methods include $f^{(4)}(\eta)$ that is zero for this example. Hence, we have exact solution with least n where $n = 2$. In [8], the authors applied modified homotopy perturbation method for this example and obtained $u(x) = \frac{7}{8}x + \frac{49}{512}x + \frac{343}{16384}x + \cdots \approx x$, but we obtained exact solution.

Example 2. Consider the nonlinear Fredholm integral equation

$$u(x) = (x - \frac{\pi}{8}) + \frac{1}{2}\int_0^1 \frac{1}{(1 + u^2(t))}\,dt$$

In this example, the analytical solution of the integral equation is $u(t) = t$ on $[0, 1]$. Using the procedure in Section 3 for the interval $[0, 1]$ and taking $n = 8, 10, 16, 20$, the u_i's in systems (4) and (8), are computed. Errors of the numerical results are given in Tables 1 and 2.

Table 1: Numerical results of Example 2

N	Simpson's rule	modified trapezoid
8	1.5×10^{-8}	7.5×10^{-10}
10	3.9×10^{-9}	1.9×10^{-10}
16	2.3×10^{-10}	1.1×10^{-11}
20	6.2×10^{-11}	3.1×10^{-12}

This example has been solved in [2], as well and Table 2, shows it's results. As we see the proposed method is more accurate than [2].

Table 2: Numerical results of Example 2 by using [2]

N	E_1:$\phi_1(x) = \frac{1}{1+x^2}$	E_2:$\phi_2(x) = \exp^{\frac{x^2}{4}}$
10	2.1×10^{-5}	1.5×10^{-9}
15	1.5×10^{-8}	2.9×10^{-11}
20	7.3×10^{-11}	7.3×10^{-14}

Example 3. Consider nonlinear integral equation given in [4],

$$u(x) = \exp(1)x + 1 - \int_0^1 (x+t)e^{u(t)}\, dt$$

which has the exact solution $u(t) = t$. Table 3, shows numerical results of this example.

Table 3: Numerical results of Example 3

N	Simpson's rule	modified trapezoid
6	9.0×10^{-4}	8.3×10^{-6}
8	7.0×10^{-5}	2.6×10^{-6}
10	2.9×10^{-6}	1.0×10^{-6}

This example has been solved in [4,5], too and Table 4, shows the obtained results. Again our method is more accurate.

Table 4: Numerical results of Example 2 using [4,5]

nods	Method of[5]	Method of[4]
0.0	2.0×10^{-3}	6.5×10^{-3}
0.2	1.0×10^{-2}	4.9×10^{-3}
0.4	2.0×10^{-2}	2.5×10^{-3}
0.8	0.0×10^{-2}	2.5×10^{-3}
1.0	1.0×10^{-4}	4.9×10^{-3}

5. Conclusion

In this paper, a numerical method based on quadrature methods has been proposed to approximate the solution of nonlinear Fredholm integral equations. In this method, the problem of solving integral equation reduced to a problem of solving a system of algebraic equation. Illustrative examples are given to demonstrate the validity and accurately of the proposed method.

References

[1] K. E. Atkinson, *The Numerical Solution of Integral Equation of the Second Kind*. Cambridge University Press, 1997.

[2] A. Alipanah and M. Dehghan, numerical solution of the nonlinear Fredholm integral equations by positive definite functions. *Appl. Math. Comput.*, 190 (2007), 1754-1761.

[3] E. Babolian, F. Fattahzadeh, and E. Golpar Raboky, A Chebyshev approximation for solving nonlinear integral equations of Hammerstein type. *Appl. Math. Comput.*, 189 (2007), 641-646.

[4] A. H. Borzabadi and O. S. Fard, A numerical scheme for a class of nonlinear Fredholm integral equations of second kind. Comput. *Appl. Math.*, 232 (2009), 449-454.

[5] A. H. Borzabadi, A. N. Kamyad, and H. H. Mehne, A different approach for solving the nonlinear Fredholm integral equations of the second kind. *Appl. Math. Comput.*, 173 (2006), 724-735.

[6] L. M. Delves and J. L. Mohamed, *Computational Methods for Integral Equation*. Cambrindge University Press, 1985.

[7] L. M. Delves and J. Wash, *Numerical Solution of Integral Equation*. Oxford University Press, 1974.

[8] A. Ghorbani and J. Saberi-Nadjafi, Exact solution for nonlinear integral equations by a modified homotopy perturbation method. *Comput. Math.*, 56 (2008), 1032-1039.

[9] R. P. Kenwal, Linear Integral Equation: *Theory and Technique*, Academic Press, New York and London, 1971.

[10] K. Maleknejad, H. Almasieh, and M. Roodaki, Triangular function (TF) method for the solution of nonlinear Volterra-Fredholm integral equations. *Commun Nonlinear Sci Numer Simulat.*, 15 (2010), 3293-3298.

[11] J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis*, second ed, Springer-Verlag, 1993.

[12] J. Saberi-Nadjafi and M. Heidari, Solving linear integral equations of the second kind with repeated modified trapezoid quadrature method. *Appl. Math. Comput.*, 189 (2007), 980-985.

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