

## Mizoguchi-Takahashi's Fixed Point Theorem Concerning $\tau$ -Distance

Z. Mohammadi

Islamic Azad University-Shiraz Branch

**Abstract.** In this paper by using the notion of  $\tau$ -distance, we will prove Mizoguchi-Takahashi's fixed point theorem, which is a generalization of fixed point theorem which has been given by Nadler.

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**Keywords and Phrases:** Set-valued mappings;  $\tau$ -distance; Mizoguchi-Takahashi's fixed point theorem.

### 1. Introduction

In 1922, Banach ([1]) proved the famous fixed point theorem known as the Banach contraction principle which is a very useful tool in nonlinear analysis, control theory, economic theory and global analysis. Later on, the theorem has been generalized in several directions. For example, In 1969, Nadler generalized it to set-valued mappings and proved some fixed point theorems about set-valued contraction mappings ([6]).

A point  $x$  is said to be a fixed point of a single-valued mapping  $f$  (set-valued mapping  $F$ ) provided  $f(x) = x, (x \in F(x))$ . We denote by  $CB(X)$  the class of all nonempty bounded closed subset of  $X$ . Since the mapping  $i : X \rightarrow CB(X)$ , given by  $i(x) = \{x\}$  for each  $x \in X$  is an isometry, the fixed point theorem in this paper for set-valued mapping are generalized of their single-valued analogues.

**Definition 1.1.** Let  $(X, d)$  be a metric space. The Hausdorff metric with respect to  $d$ , denoted by  $H$  and defined by

$$H(A, B) = \max\{\sup_{u \in A} d(u, B), \sup_{v \in B} d(v, A)\}$$

for every  $A, B \in CB(X)$ , where  $d(x, A) = \inf\{d(x, y); y \in A\}$  for every  $A \subset X$ .

**Theorem 1.2.** (Nadler [6]), Let  $(X, d)$  be a complete metric space and  $T$  be a mapping from  $X$  into  $CB(X)$ . Assume that there exists  $r \in [0, 1)$  such that  $H(Tx, Ty) \leq rd(x, y)$  for all  $x, y \in X$ . Then there exists  $z \in X$  such that  $z \in Tz$ .

Mizoguchi and Takahashi ([5]) proved a generalization of the theorem, which is a partial answer of problem 9 in Rich ([7]). See also [3,4,8].

**Theorem 1.3.** ( Mizoguchi and Takahashi ([5])), Let  $(X, d)$  be a complete metric space and  $T$  be a mapping from  $X$  into  $CB(X)$ . Assume  $H(Tx, Ty) \leq \alpha(d(x, y)).d(x, y)$  for all  $x, y \in X$ , where  $\alpha$  is a function from  $[0, \infty)$  into  $[0, 1)$  satisfying  $\limsup_{s \rightarrow t^+} \alpha(s) < 1$  for all  $t \in [0, \infty)$ . Then there exists  $z_0 \in X$  such that  $z_0 \in Tz_0$ .

Recently, Suzuki ([9]) introduced the notion of  $\tau$ -distance on a metric space, which is the generalization of the concept of  $\omega$ -distance and Tatarou's distance. He also improved some fixed point theorems. In this paper, by using the notion of  $\tau$ -distance, we will prove Mizoguchi-Takahashi's fixed point theorem for set-valued mappings which is a real generalization of Nadler's.

## 2. $\tau$ -Distance

Through out of this paper, we denote by  $N$ , the set of all positive integers and by  $R_+$ , the set of all nonnegative real numbers. In this section, we state the definition of  $\tau$ -distance which was first introduced by Suzuki [9]. Then we give some properties which will be connected to  $\tau$ -distance.

**Definition 2.1. ([9]).** Let  $(X, d)$  be a metric space. A function  $p$  from  $X \times X$  into  $[0, \infty)$  is called a  $\tau$ -distance on  $X$ , if there exists a function  $\eta$  from  $X \times [0, \infty)$  into  $[0, \infty)$  and following are satisfied;

( $\tau_1$ )  $p(x, z) \leq p(x, y) + p(y, z)$  for all  $x, y, z \in X$  ;

( $\tau_2$ )  $\eta(x, 0) = 0$  and  $\eta(x, t) \geq t$  for all  $x \in X$  and  $t \in [0, \infty)$ , and  $\eta$  is concave and continuous in its second variable;

( $\tau_3$ )  $\lim_n x_n = x$  and  $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$  imply  $p(w, x) \leq \lim_n \inf p(w, x_n)$  for all  $w \in X$ ;

( $\tau_4$ )  $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$  and  $\lim_n \eta(x_n, t_n) = 0$  imply  $\lim_n \eta(y_n, t_n) = 0$ ;

( $\tau_5$ )  $\lim_n \eta(z_n, p(z_n, x_n)) = 0$  and  $\lim_n \eta(z_n, p(z_n, y_n)) = 0$  imply  $\lim_n d(x_n, y_n) = 0$ .

We may replace ( $\tau_2$ ) by ( $\tau_2$ )' (see [9]):

( $\tau_2$ )'  $\inf\{\eta(x, t) : t > 0\}$  for all  $x \in X$ , and  $\eta$  is nondecreasing in its second variable.

Many useful examples of  $\tau$ -distance are stated in [9,10].

**Definition 2.2. ([9]).** Let  $(X, d)$  be a metric space and  $p$  a  $\tau$ -distance on  $X$ . A sequence  $\{x_n\}$  of  $X$  is called  $p$ -Cauchy if there exist a function  $\eta$  from  $X \times R_+$  into  $R_+$  satisfying ( $\tau_2$ ) – ( $\tau_5$ ) and a sequence  $\{z_n\}$  of  $X$  such that  $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \geq n\} = 0$ .

The following two lemmas are very useful in the proof of fixed point theorem in Section 3.

**Lemma 2.3. ([9]).** Let  $(X, d)$  be a metric space and a  $\tau$ -distance on  $X$ . If  $\{x_n\}$  is a  $p$ -Cauchy sequence, then  $\{x_n\}$  is a Cauchy sequence. Moreover, if  $\{y_n\}$  is a sequence satisfying  $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ , then  $\{y_n\}$  is also a  $p$ -Cauchy sequence and  $\lim_n d(x_n, y_n) = 0$ .

**Lemma 2.4. ([9]).** Let  $(X, d)$  be a metric space and  $p$  a  $\tau$ -distance on  $X$ . If a sequence  $\{x_n\}$  in  $X$  satisfies  $\lim_n \sup\{p(x_n, x_m) : m > n\} = 0$ , then  $\{x_n\}$  is a  $p$ -Cauchy sequence. Moreover, if a sequence  $\{y_n\}$  in  $X$  satisfies  $\lim_n p(x_n, y_n) = 0$ , then  $\{y_n\}$  is also a  $p$ -Cauchy sequence and  $\lim_n d(x_n, y_n) = 0$ .

### 3. Main Result

In this section by using  $\tau$ -distance, we give a proof of Mizoguchi-Takahashi's fixed point theorem which is simpler than previous proofs ([2,5]).

**Theorem 3.1.** *Let  $(X, d)$  be a complete metric space and let  $p$  be a  $\tau$ -distance on  $X$ . Let  $T$  be a mapping from  $X$  into  $CB(X)$ . Assume that  $H(Tx, Ty) \leq \alpha(p(x, y)).p(x, y)$  for all  $x, y \in X$ , where  $\alpha$  is a function from  $[0, \infty)$  into  $[0, 1)$  satisfying  $\limsup_{s \rightarrow t^+} \alpha(s) < 1$  for all  $t \in [0, \infty)$ .*

*Then there exists  $z_0 \in X$  such that  $z_0 \in Tz_0$  and  $p(z_0, z_0) = 0$ .*

**Proof.** Define a function  $\beta$  from  $[0, \infty)$  into  $[0, 1)$  by  $\beta(t) = (\alpha(t) + 1)/2$ . Then the following hold:

$$1) \limsup_{s \rightarrow t^+} \beta(s) < 1.$$

$$2) \forall x, y \in X, u \in Tx, \exists v \in Ty; p(u, v) \leq \beta(p(x, y)).p(x, y).$$

Fix  $u_0 \in X$  and  $u_1 \in Tu_0$ . Then there exists  $u_2 \in Tu_1$  such that  $p(u_1, u_2) \leq \beta(p(u_0, u_1)).p(u_0, u_1)$ . Thus, we have a sequence  $\{u_n\}$  in  $X$  such that  $u_{n+1} \in Tu_n$  and

$$p(u_{n+1}, u_{n+2}) \leq \beta(p(u_n, u_{n+1})).p(u_n, u_{n+1})$$

for all  $n \in N$ . Since  $\beta(t) < 1$  for all  $t \in [0, \infty)$  then  $\{p(u_n, u_{n+1})\}$  is a nonincreasing sequence. Hence  $\{p(u_n, u_{n+1})\}$  converges to some nonnegative real number  $\lambda$ . Since  $\limsup_{s \rightarrow \lambda^+} \beta(s) < 1$  and  $\beta(\lambda) < 1$ , there exists  $r \in [0, 1)$  and  $\varepsilon > 0$  such that  $\beta(s) \leq r$  for all  $s \in [\lambda, \lambda + \varepsilon)$ . We can take  $k \in N$  such that  $\lambda \leq p(u_n, u_{n+1}) < \lambda + \varepsilon$  for all  $n \in N$  with  $n \geq k$ . Since

$$p(u_{n+2}, u_{n+1}) \leq \beta(p(u_n, u_{n+1})).p(u_n, u_{n+1}) < r.p(u_n, u_{n+1}), \quad (1)$$

for any  $n \in N$  with  $n \geq k$ , then we have

$$\begin{aligned} \sum_{n=1}^{\infty} p(u_n, u_{n+1}) &= \sum_{n=1}^k p(u_n, u_{n+1}) + \sum_{n=k+1}^{\infty} p(u_n, u_{n+1}) \\ &\leq \sum_{n=1}^k p(u_n, u_{n+1}) + \sum_{n=1}^{\infty} r^n .p(u_k, u_{k+1}) < \infty, \end{aligned}$$

hence  $\lim_n \sup p(u_n, u_{n+1}) = 0$ . By Lemma 2.4.,  $\{u_n\}$  is a  $p$ -Cauchy sequence and hence, by Lemma 2.3.,  $\{u_n\}$  is a Cauchy sequence. Since

$X$  is complete,  $\{u_n\}$  converges to some point  $\nu_0 \in X$ . From  $(\tau_3)$ , we have

$$p(u_n, \nu_0) \leq \liminf p(u_n, u_m) \leq \frac{r^n}{1-r} p(u_k, u_{k+1}) \quad (2)$$

for  $m > n > k$ . We have also  $w_n \in T\nu$ . Since

$$\limsup \{p(u_n, w_m); m \geq n\} = 0$$

by Lemma 2.3.,  $\{w_n\}$  is a  $p$ -Cauchy sequence and we have  $\lim_n d(u_n, w_n) = 0$ . But

$$d(w_n, \nu_0) \leq d(w_n, u_n) + d(u_n, \nu_0)$$

then  $\{w_n\}$  converges to  $\nu_0$ . By closedness of  $T\nu_0$ , we have  $\nu_0 \in T\nu_0$ .

So we have a sequence  $\{\nu_n\}$  such that  $\nu_{n+1} \in T\nu_n$  and by (1), for all  $n \in N$

$$p(\nu_0, \nu_{n+1}) \leq \beta(p(\nu_0, \nu_n)) \cdot p(\nu_0, \nu_n) \leq r \cdot p(\nu_0, \nu_n).$$

Also

$$p(\nu_0, \nu_n) \leq r \cdot p(\nu_0, \nu_{n-1}) \leq \dots \leq r^n \cdot p(\nu_0, \nu_0).$$

Hence

$$\limsup p(u_n, \nu_n) \leq \lim_n (p(u_n, \nu_0) + p(\nu_0, \nu_n)) = 0.$$

By Lemma 2.4.,  $\{\nu_n\}$  is a  $p$ -Cauchy sequence and converges to  $\nu_0$ . So we have

$$p(\nu_0, \nu_0) \leq \lim_n p(\nu_0, \nu_n) = 0.$$

This completes the proof.  $\square$

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**Zahra Mohammadi**

Department of Mathematics

Instructor of Mathematics

Islamic Azad University-Shiraz Branch

Shiraz, Iran

E-mail: pyramid.122@ gmail.com