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Hahn-Banach Theorem in Vector Spaces

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Abstract. In this paper we introduce a new extension to Hahn-Banach Theorem and consider its relation with the linear operatres. At the end we give some applications of this theorem.

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1. Introduction

Huang and Zhang [2] introduced the notion of cone metric spaces and some fixed point theorems for contractive mappings were proved in these spaces. The results in [2] were generalized by Sh.Rezapour and R. Hamlbarani in [6]. Suppose that $\preceq$ is a partial order on a set S and $A \subseteq S$. The greatest lower bound of A is unique, if it exists. It is denoted by $\inf(A)$. Similarly, the least upper bound of A is unique, if it exists, and is denoted by $\sup(A)$.

Let E be a linear space and P a subset of E. P is called a cone if

(i) P is closed, non-empty and $P \neq \{0\}$.

(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a,b.

(iii) $P \cap -P = \{0\}$.

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For a given cone $P \subseteq E$, we can define a partial ordering $\leqslant$ with respect to P by $x \leqslant y$ if and only if $y - x \in P$. Note that $x < y$ will stand for $x \leqslant y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P.
P is called the normal cone of E, if there is a number $M > 0$ such that for all $x, y \in P$, $0 \leqslant x \leqslant y$ implies $\|x\| \leqslant M\|y\|$.
The least positive number satisfying the above inequality is called the normal constant of P.

2. Main Results

Hahn-Banach Theorem is one of the important theorems in analysis and many authors have investigated on this theorem and its applications ([2-6]).
In the sequel we assume that $(E, \|.\|)$ is a Banach algebra that is ordered by a normal cone P with constant normal M=1, $intP \neq \emptyset$ and $\leqslant$ is partial ordering with respect to P. We recall that a Banach algebra is a pair $(E, \|.\|)$, where E is an algebra and $\|.\|$ is a complete norm such that $\|xy\| \leqslant \|x\| \, \|y\|$.

Definition 2.1. *Let X be a vector space and p be a map from vector space X into E. We call that p is a sublinear map if p(tx)=tp(x) and $p(x + y) \leqslant p(x) + p(y)$ whenever $t > 0$ and $x, y \in X$.*

Theorem 2.2. [**Hahn- Banach Theorem**] *Let Y be a subspace of a vector space X and $p : X \to E$ a sublinear map. If the linear map $T_0 : Y \to E$ satisfies $T_0(y) \leqslant p(y)$ for every $y \in Y$, then there is a linear map $T : X \to E$ such that $T_{|_Y} = T_0$ and $T(x) \leqslant p(x)$ whenever $x \in X$.*

Proof. Let $x_1 \in X \setminus Y$ and $Y_1 = Y \bigoplus \langle\{x_1\}\rangle$. Note that each member of Y_1 can be expressed in the form $y + tx_1$, where $y \in Y$ and t is a scalar, in exactly one way. For $y_1, y_2 \in Y$,

$$\begin{aligned} T_0(y_1) + T_0(y_2) &= T_0(y_1 + y_2) \\ &\leqslant p(y_1 - x_1 + y_2 + x_1) \\ &\leqslant p(y_1 - x_1) + p(y_2 + x_1). \end{aligned}$$

Then

$$sup\{T_0(y) - p(y - x_1) : y \in Y\} \leqslant inf\{p(y + x_1) - T_0(y) : y \in Y\}$$

and so for some $t_1 \in E$

$$sup\{T_0(y) - p(y - x_1) : y \in Y\} \leqslant t_1 \leqslant inf\{p(y + x_1) - T_0(y) : y \in Y\}.$$

For any $y \in Y$ and scalar t, define $T_1(y + tx_1) = T_0(y) + t.t_1$. It is easy to check that T_1 is a linear map whose restriction to Y is T_0. Therefore

$$T_1(y + tx_1) = t(T_0(t^{-1}y) + t_1) \leqslant tp(t^{-1}y + x_1) = p(y + tx_1)$$

and

$$T_1(y - tx_1) = t(T_0(t^{-1}y) - t_1) \leqslant tp(t^{-1}y - x_1) = p(y - tx_1).$$

So $T_1(x) \leqslant p(x)$ whenever $x \in Y_1$.
The second step of the proof is to show that the first step can be repeated until a linear map is obtained. It is dominated by p and its restriction to Y is T_0. Let $\mathcal{U}$ be the collection of all linear maps G such that the domain of G is a subspace of X that includes Y, the restriction of G to Y is T_0, and G dominated by p. Define a preorder $\preceq$ on $\mathcal{U}$ by declaring that $G_1 \preceq G_2$ whenever G_1 is the restriction of G_2 to a subspace of the domain of G_2. It is easy to see that each nonempty chain $\mathcal{C}$ in $\mathcal{U}$ has an upper bound in $\mathcal{U}$. Consider the linear map whose domain is the union Z of the domains of the members of $\mathcal{C}$ and which agrees at each point z of Z with every member of $\mathcal{C}$ that is defined at z. By Zorn's lemma, the preorder set $\mathcal{U}$ has a maximal element T. The domain of T is all of X. On the other hand with by applying the first step there is a T_1 in $\mathcal{U}$ such that $T \preceq T_1$, but $T_1 \npreceq T$. This T satisfies all that is required. □

Proposition 2.3. *Let Y be a closed subspace of a linear normed space X and $T_0 : Y \to E$ be an injective bounded linear map. Then there exists a bounded linear map $T : X \to E$ such that $\|T\| = \|T_0\|$ and $T|_Y = T_0$.*

Proof. For every nonzero element $x \in X$ define $p(x) = \|T_0\| \ \|x\| \frac{T_0(x)}{\|T_0(x)\|}$ and $p(0) = 0$. Since for every nonzero element $x \in X$, we have

$$\|T_0(x)\| \ T_0(x) \leqslant \|T_0\| \ \|x\| \ T_0(x).$$

and so $T_0(x) \leqslant p(x)$. Now by Theorem 2.2., there exists a linear map $T : X \to E$ such that $T|_Y = T_0$ and $T(x) \leqslant p(x)$ whenever $x \in X$. Since P is a normal cone with constant normal 1, $\|T(x)\| \leqslant \|T_0\| \, \|x\|$ and $\|T(x)\| \leqslant \|T_0\|$. Therefore $\|T\| = \|T_0\|$. $\square$

Theorem 2.4. *Let X be a linear normed space and $0 \neq x \in X$. Then for every $e \in S_E$ there is a linear map $T_e : X \to E$ such that $\|T_e\| = 1,\ T_e(x) = \|x\|e$, where $S_E = \{x \in E : \ \|x\| = 1\}$.*

Proof. Define $G_e : \langle x \rangle \to E$ by $G_e(\alpha x) = \alpha\|x\|e$ for every scalar α. Clearly G_e is injective, linear and $G_e(x) = \|x\|e$. Also for $\alpha \neq 0$,

$$\|G_e(\alpha x)\| = |\alpha|\|x\| = \|\alpha x\|.$$

Since E is ordered by a normal cone P with constant normal $M = 1$, then $\|G_e\| \leqslant 1$. Also since,

$$\|G_e\| \, \|x\| \geqslant \|G_e(x)\| = \|x\|,$$

so $\|G_e\| \geqslant 1$. Hence $\|G_e\| = 1$. Let T_e be then Hahn-Banach extension of G_e from proposition 2.3, so the proof is complete. $\square$

In the following we introduce immediate consequence of the above theorem.

Corollary 2.5. *Let X be a linear normed space and $x \neq y \in X$. Then there is a linear map $T : X \to E$ such that $Tx \neq Ty$.*

Corollary 2.6. *Let X be a linear normed space and $x \in X$. Then*

$$\|x\| = \sup_{T \in \mathcal{B}} \|Tx\|,$$

where $\mathcal{B} = \{T : X \to E : T$ is a linear map and $\|T\| = 1\}$.

Proof. By Theorem 2.4., there is a linear map $T : X \to E$ such that $\|T\| = 1,\ \|T(x)\| = \|x\|$. Then $\|x\| = \|T(x)\| \leqslant \sup_{T \in \mathcal{B}} \|Tx\|$. On the other hand since $\|T(x)\| \leqslant \|T\| \, \|x\|$, and so $\sup_{T \in \mathcal{B}} \|Tx\| \leqslant \|x\|$. $\square$

We recall that a point $g_0 \in Y$ is said to be a best approximation for $x \in X$ if and only if $\|x - g_0\| = \|x + Y\| = d(x, Y)$. The set of all best approximations of $x \in X$ in Y is shown by $P_Y(x)$. In the other words,

$$P_Y(x) = \{g_0 \in Y \ : \ \|x - g_0\| = d(x, Y)\},$$

If $P_Y(x)$ is non-empty for every $x \in X$, then Y is called a Proximinal set. The set Y is Chebyshev if $P_Y(x)$ is a singleton set for every $x \in X$ (see [2-6]).
Now we want to present some applications of new extension Hahn-Banach theorem in approximation theory.

Proposition 2.7. *Let Y be a closed subspace of a linear normed space X, and $x \in X \backslash Y$. Then for every $e \in S_E$ there is a linear map $T_e : Y \bigoplus \langle x \rangle \to E$ such that $\|T_e\| = 1,\ T_e x = d(x, Y)e, T_e|_Y = 0$.*

Proof. Define $T_e : Y \bigoplus \langle x \rangle \to E$ by $T_e(y + \alpha x) = \alpha d(x, Y)e$ for every $y \in Y$ and scalar α. It is clear that T_e is linear, $T_e x = d(x, Y)e$ and $T_e|_Y = 0$. For any $y \in Y$ and scalar $\alpha \neq 0$,

$$\|T_e(y + \alpha x)\| = |\alpha| d(x, Y) \leqslant \|y + \alpha x\|,$$

so $\|T_e\| \leqslant 1$. Also since,

$$\|T_e\| \ \|x - y\| \geqslant \|T_e(x - y)\| = d(x, Y) \quad y \in Y,$$

so $\|T_e\| \geqslant 1$. Hence $\|T_e\| = 1$. $\square$

Theorem 2.8. *Let Y be a closed subspace of a cone norm space X. Suppose that $x \in X \backslash Y$ and $g_0 \in Y$. Then $g_0 \in P_Y(x)$ iff for every $e \in S_E$ there is a linear map $T_e : Y \bigoplus \langle x \rangle \to E$ such that*

$$\|T_e\| = 1,\ T_e(x - g_0) = \|x - g_0\| e, T_e|_Y = 0.$$

Proof. Assume $g_0 \in P_Y(x)$. Since $x \in X \backslash Y$, $\|x - g_0\| = d(x, Y)$ and so by Proposition 2.7., there is a linear map $T_e : Y \bigoplus \langle x \rangle \to E$ such that

$$\|T_e\| = 1,\ T_e(x - g_0) = \|x - g_0\| e, T_e|_Y = 0.$$

Conversely suppose there is a linear map $T_e : Y \bigoplus \langle x \rangle \to E$ such that $\|T_e\| = 1,\ T_e(x - g_0) = \|x - g_0\| e, T_e|_Y = 0$. Then

$$\|x - g_0\| = \|T_e(x - g_0)\| = \|T_e(x - g)\| \leqslant \|T_e\|\ \|x - g\| = \|x - g\|$$

and so $g_0 \in P_Y(x)$. $\square$

Corollary 2.9. *Suppose X is a normed linear spaces and $x, y \in X$. Then $x \perp y$ iff for every $e \in S_E$ there is a linear map $T_e : \langle y \rangle \bigoplus \langle x \rangle \to E$ such that $\|T_e\| = 1,\ T_e(x) = \|x\| e, T_e(y) = 0$.*
It is clear that ℓ_∞ is a Banach algebra and $P = \{\{x_n\} \in \ell_\infty : x_n \geqslant 0,\ for\ all\ n\}$ is a normal cone with constant normal $M = 1$. Also in [1] proved that for every linear map $T_0 : Y \to \ell_\infty$ there is a linear map $T : X \to \ell_\infty$ such that $\|T\| = \|T_0\|$ and $T|_Y = T_0$. Consequently we have following result.

Corollary 2.10. *Let Y be a closed subspace of a linear normed space X, and $x \in X \backslash Y$. Then $M \subseteq P_Y(x)$ iff for every $e \in S_{\ell_\infty}$, there is a linear map $T : X \to \ell_\infty$ such that for every $g \in M$*

$$\|T_e\| = 1,\ T_e x = \|x - g\| e, T_e|_Y = 0.$$

References

[1] J. Diestel, *Sequence and Series in Banach spaces*, Springer, 1984.

[2] H. Long-Guang and Z. Xian, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, 332 (2007), 1468-1476.

[3] H. Mazaheri and F. M. Maalek Ghaini, Quasi-orthogonality of the best approximant sets, *Nonlinear Analysis,* 65 (2006) 534-537.

[4] H. Mazaheri and S. M. Vaezpour, orthogonality and ϵ-Orthogonality in Banach spaces, *Aust. J. Math. and Appl.,* 2 (2005), 1-5.

[5] P. L. Papini and I. Singer, Best coapproximation in normed linear spaces, *Mh. Math.,* 88 (1979), 27-44.

[6] Sh. Rezapour and R. Hamlbarani, Some notes on the paper Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, 345 (2008), 719-724.

[7] W. Rudin, Functional Analysis, 2nd edition, Mc Graw-Hill, 1991.

[8] I. Singer, Best approximation in normal linear spaces by elements of linear subspaces, Springer Verlag, New York, 1970.

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