

Hahn-Banach Theorem in Vector Spaces

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Abstract. In this paper we introduce a new extension to Hahn-Banach Theorem and consider its relation with the linear operators. At the end we give some applications of this theorem.

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1. Introduction

Huang and Zhang [2] introduced the notion of cone metric spaces and some fixed point theorems for contractive mappings were proved in these spaces. The results in [2] were generalized by Sh.Rezapour and R. Hambarani in [6]. Suppose that \preceq is a partial order on a set S and $A \subseteq S$. The greatest lower bound of A is unique, if it exists. It is denoted by $\inf(A)$. Similarly, the least upper bound of A is unique, if it exists, and is denoted by $\sup(A)$.

Let E be a linear space and P a subset of E . P is called a cone if

- (i) P is closed, non-empty and $P \neq \{0\}$.
- (ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b .
- (iii) $P \cap -P = \{0\}$.

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For a given cone $P \subseteq E$, we can define a partial ordering \leq with respect to P by $x \leq y$ if and only if $y - x \in P$. Note that $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P .

P is called the normal cone of E , if there is a number $M > 0$ such that for all $x, y \in P$, $0 \leq x \leq y$ implies $\|x\| \leq M\|y\|$.

The least positive number satisfying the above inequality is called the normal constant of P .

2. Main Results

Hahn-Banach Theorem is one of the important theorems in analysis and many authors have investigated on this theorem and its applications ([2-6]).

In the sequel we assume that $(E, \|\cdot\|)$ is a Banach algebra that is ordered by a normal cone P with constant normal $M=1$, $\text{int}P \neq \emptyset$ and \leq is partial ordering with respect to P . We recall that a Banach algebra is a pair $(E, \|\cdot\|)$, where E is an algebra and $\|\cdot\|$ is a complete norm such that $\|xy\| \leq \|x\| \|y\|$.

Definition 2.1. Let X be a vector space and p be a map from vector space X into E . We call that p is a sublinear map if $p(tx) = tp(x)$ and $p(x + y) \leq p(x) + p(y)$ whenever $t > 0$ and $x, y \in X$.

Theorem 2.2. [Hahn- Banach Theorem] Let Y be a subspace of a vector space X and $p : X \rightarrow E$ a sublinear map. If the linear map $T_0 : Y \rightarrow E$ satisfies $T_0(y) \leq p(y)$ for every $y \in Y$, then there is a linear map $T : X \rightarrow E$ such that $T|_Y = T_0$ and $T(x) \leq p(x)$ whenever $x \in X$.

Proof. Let $x_1 \in X \setminus Y$ and $Y_1 = Y \oplus \langle x_1 \rangle$. Note that each member of Y_1 can be expressed in the form $y + tx_1$, where $y \in Y$ and t is a scalar, in exactly one way. For $y_1, y_2 \in Y$,

$$\begin{aligned} T_0(y_1) + T_0(y_2) &= T_0(y_1 + y_2) \\ &\leq p(y_1 - x_1 + y_2 + x_1) \\ &\leq p(y_1 - x_1) + p(y_2 + x_1). \end{aligned}$$

Then

$$\sup\{T_0(y) - p(y - x_1) : y \in Y\} \leq \inf\{p(y + x_1) - T_0(y) : y \in Y\}$$

and so for some $t_1 \in E$

$$\sup\{T_0(y) - p(y - x_1) : y \in Y\} \leq t_1 \leq \inf\{p(y + x_1) - T_0(y) : y \in Y\}.$$

For any $y \in Y$ and scalar t , define $T_1(y + tx_1) = T_0(y) + t.t_1$. It is easy to check that T_1 is a linear map whose restriction to Y is T_0 . Therefore

$$T_1(y + tx_1) = t(T_0(t^{-1}y) + t_1) \leq tp(t^{-1}y + x_1) = p(y + tx_1)$$

and

$$T_1(y - tx_1) = t(T_0(t^{-1}y) - t_1) \leq tp(t^{-1}y - x_1) = p(y - tx_1).$$

So $T_1(x) \leq p(x)$ whenever $x \in Y_1$.

The second step of the proof is to show that the first step can be repeated until a linear map is obtained. It is dominated by p and its restriction to Y is T_0 . Let \mathcal{U} be the collection of all linear maps G such that the domain of G is a subspace of X that includes Y , the restriction of G to Y is T_0 , and G dominated by p . Define a preorder \preceq on \mathcal{U} by declaring that $G_1 \preceq G_2$ whenever G_1 is the restriction of G_2 to a subspace of the domain of G_2 . It is easy to see that each nonempty chain \mathcal{C} in \mathcal{U} has an upper bound in \mathcal{U} . Consider the linear map whose domain is the union Z of the domains of the members of \mathcal{C} and which agrees at each point z of Z with every member of \mathcal{C} that is defined at z . By Zorn's lemma, the preorder set \mathcal{U} has a maximal element T . The domain of T is all of X . On the other hand with by applying the first step there is a T_1 in \mathcal{U} such that $T \preceq T_1$, but $T_1 \not\preceq T$. This T satisfies all that is required. \square

Proposition 2.3. *Let Y be a closed subspace of a linear normed space X and $T_0 : Y \rightarrow E$ be an injective bounded linear map. Then there exists a bounded linear map $T : X \rightarrow E$ such that $\|T\| = \|T_0\|$ and $T|_Y = T_0$.*

Proof. For every nonzero element $x \in X$ define $p(x) = \|T_0\| \|x\| \frac{T_0(x)}{\|T_0(x)\|}$ and $p(0) = 0$. Since for every nonzero element $x \in X$, we have

$$\|T_0(x)\| \|T_0(x)\| \leq \|T_0\| \|x\| \|T_0(x)\|.$$

and so $T_0(x) \leq p(x)$. Now by Theorem 2.2., there exists a linear map $T : X \rightarrow E$ such that $T|_Y = T_0$ and $T(x) \leq p(x)$ whenever $x \in X$. Since P is a normal cone with constant normal 1, $\|T(x)\| \leq \|T_0\| \|x\|$ and $\|T(x)\| \leq \|T_0\|$. Therefore $\|T\| = \|T_0\|$. \square

Theorem 2.4. *Let X be a linear normed space and $0 \neq x \in X$. Then for every $e \in S_E$ there is a linear map $T_e : X \rightarrow E$ such that $\|T_e\| = 1$, $T_e(x) = \|x\|e$, where $S_E = \{x \in E : \|x\| = 1\}$.*

Proof. Define $G_e : \langle x \rangle \rightarrow E$ by $G_e(\alpha x) = \alpha\|x\|e$ for every scalar α . Clearly G_e is injective, linear and $G_e(x) = \|x\|e$. Also for $\alpha \neq 0$,

$$\|G_e(\alpha x)\| = |\alpha|\|x\| = \|\alpha x\|.$$

Since E is ordered by a normal cone P with constant normal $M = 1$, then $\|G_e\| \leq 1$. Also since,

$$\|G_e\| \|x\| \geq \|G_e(x)\| = \|x\|,$$

so $\|G_e\| \geq 1$. Hence $\|G_e\| = 1$. Let T_e be then Hahn-Banach extension of G_e from proposition 2.3, so the proof is complete. \square

In the following we introduce immediate consequence of the above theorem.

Corollary 2.5. *Let X be a linear normed space and $x \neq y \in X$. Then there is a linear map $T : X \rightarrow E$ such that $Tx \neq Ty$.*

Corollary 2.6. *Let X be a linear normed space and $x \in X$. Then*

$$\|x\| = \sup_{T \in \mathcal{B}} \|Tx\|,$$

where $\mathcal{B} = \{T : X \rightarrow E : T \text{ is a linear map and } \|T\| = 1\}$.

Proof. By Theorem 2.4., there is a linear map $T : X \rightarrow E$ such that $\|T\| = 1$, $\|T(x)\| = \|x\|$. Then $\|x\| = \|T(x)\| \leq \sup_{T \in \mathcal{B}} \|Tx\|$. On the other hand since $\|T(x)\| \leq \|T\| \|x\|$, and so $\sup_{T \in \mathcal{B}} \|Tx\| \leq \|x\|$. \square

We recall that a point $g_0 \in Y$ is said to be a best approximation for $x \in X$ if and only if $\|x - g_0\| = \|x - Y\| = d(x, Y)$. The set of all best approximations of $x \in X$ in Y is shown by $P_Y(x)$. In the other words,

$$P_Y(x) = \{g_0 \in Y : \|x - g_0\| = d(x, Y)\},$$

If $P_Y(x)$ is non-empty for every $x \in X$, then Y is called a Proximinal set. The set Y is Chebyshev if $P_Y(x)$ is a singleton set for every $x \in X$ (see [2-6]).

Now we want to present some applications of new extension Hahn-Banach theorem in approximation theory.

Proposition 2.7. *Let Y be a closed subspace of a linear normed space X , and $x \in X \setminus Y$. Then for every $e \in S_E$ there is a linear map $T_e : Y \oplus \langle x \rangle \rightarrow E$ such that $\|T_e\| = 1$, $T_e x = d(x, Y)e$, $T_e|_Y = 0$.*

Proof. Define $T_e : Y \oplus \langle x \rangle \rightarrow E$ by $T_e(y + \alpha x) = \alpha d(x, Y)e$ for every $y \in Y$ and scalar α . It is clear that T_e is linear, $T_e x = d(x, Y)e$ and $T_e|_Y = 0$. For any $y \in Y$ and scalar $\alpha \neq 0$,

$$\|T_e(y + \alpha x)\| = |\alpha|d(x, Y) \leq \|y + \alpha x\|,$$

so $\|T_e\| \leq 1$. Also since,

$$\|T_e\| \|x - y\| \geq \|T_e(x - y)\| = d(x, Y) \quad y \in Y,$$

so $\|T_e\| \geq 1$. Hence $\|T_e\| = 1$. \square

Theorem 2.8. *Let Y be a closed subspace of a cone norm space X . Suppose that $x \in X \setminus Y$ and $g_0 \in Y$. Then $g_0 \in P_Y(x)$ iff for every $e \in S_E$ there is a linear map $T_e : Y \oplus \langle x \rangle \rightarrow E$ such that*

$$\|T_e\| = 1, \quad T_e(x - g_0) = \|x - g_0\|e, \quad T_e|_Y = 0.$$

Proof. Assume $g_0 \in P_Y(x)$. Since $x \in X \setminus Y$, $\|x - g_0\| = d(x, Y)$ and so by Proposition 2.7., there is a linear map $T_e : Y \oplus \langle x \rangle \rightarrow E$ such that

$$\|T_e\| = 1, \quad T_e(x - g_0) = \|x - g_0\|e, \quad T_e|_Y = 0.$$

Conversely suppose there is a linear map $T_e : Y \oplus \langle x \rangle \rightarrow E$ such that $\|T_e\| = 1$, $T_e(x - g_0) = \|x - g_0\|e$, $T_e|_Y = 0$. Then

$$\|x - g_0\| = \|T_e(x - g_0)\| = \|T_e(x - g)\| \leq \|T_e\| \|x - g\| = \|x - g\|$$

and so $g_0 \in P_Y(x)$. \square

Corollary 2.9. *Suppose X is a normed linear spaces and $x, y \in X$. Then $x \perp y$ iff for every $e \in S_E$ there is a linear map $T_e : \langle y \rangle \oplus \langle x \rangle \rightarrow E$ such that $\|T_e\| = 1$, $T_e(x) = \|x\|e$, $T_e(y) = 0$.*

It is clear that ℓ_∞ is a Banach algebra and $P = \{\{x_n\} \in \ell_\infty : x_n \geq 0, \text{ for all } n\}$ is a normal cone with constant normal $M = 1$. Also in [1] proved that for every linear map $T_0 : Y \rightarrow \ell_\infty$ there is a linear map $T : X \rightarrow \ell_\infty$ such that $\|T\| = \|T_0\|$ and $T|_Y = T_0$. Consequently we have following result.

Corollary 2.10. *Let Y be a closed subspace of a linear normed space X , and $x \in X \setminus Y$. Then $M \subseteq P_Y(x)$ iff for every $e \in S_{\ell_\infty}$, there is a linear map $T : X \rightarrow \ell_\infty$ such that for every $g \in M$*

$$\|T_e\| = 1, T_e x = \|x - g\|e, T_e|_Y = 0.$$

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