Journal of Mathematical Extension
Vol. 4, No. 1 (2009), 105-113

On the Stability of a Cubic Functional Equation in Random Normed Spaces

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Abstract. The concept of Hyers-Ulam-Rassias stability has been originated from a stability theorem due to Th. M. Rassias. Recently, the Hyers-Ulam-Rassias stability of the functional equation

$$f(x+2y)+f(x-2y)=2f(x)-f(2x)+4\Big\{f(x+y)+f(x-y)\Big\},$$

has been proved in the case of Banach spaces. In this paper, we will find out the generalized Hyers-Ulam-Rassias stability problem of the above functional equation in random normed spaces.

AMS Subject Classification: 39B72; 47H09.
Keywords and Phrases: Cubic functional equation, stability, fixed point.

1. Introduction

Under what condition is there a homomorphism near an approximately homomorphism between a group and a metric group? This is called the stability problem of functional equations which was first raised by S. M. Ulam [26] in 1940. In next year, D. H. Hyers [8] answers the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately linear mappings was given by Th. M. Rassias [16]. The terminology Hyers-Ulam-Rassias stability originates from this historical background. Since then, a great deal of work has been done by a number of authors (for instance, [2, 4, 5, 6, 7, 8, 9, 10, 12, 15, 17, 18, 19, 20, 21, 22, 23]).

In particular, one of the important functional equations studied is the following functional equation:

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

The quadratic mapping $f(x)=cx^2$ is a solution of this functional equation, and so one is usually said the above functional equation should be quadratic.
A Hyers-Ulam stability problem for the quadratic functional equation was first proved by F. Skof [25] for mappings $f:X\to Y$, where X is a normed space and Y is a Banach space. S. Czerwik [3] generalized the Hyers-Ulam stability of the quadratic functional equation.
The cubic mapping $f(x)=cx^3$ satisfies the functional equation:

$$f(x+2y)+f(x-2y)=2f(x)-f(2x)+4\Big\{f(x+y)+f(x-y)\Big\}. \quad (1)$$

In this note we promise that the equation (1) is called a cubic functional equation and every solution of the cubic functional equation (1) is said to be a cubic mapping.
Our main goal in this note is to investigate the stability problem for the equation (1) in random normed spaces.
In the sequel, we shall adopt the usual terminology, notions and conventions of the theory of random normed spaces as in [24]. Throughout this paper, the spaces of all probability distribution functions is denoted by Λ^+. Elements of Λ^+ are functions $F:R\cup[-\infty,+\infty]\to[0,1]$, such that F is left continuous and nondecreasing on R and $F(0)=0, F(+\infty)=1$. It's clear that the subset

$$D^+=\{F\in\Lambda^+: l^-F(-\infty)=1\},$$

where $l^-f(x)=\lim_{t\to x^-}f(t)$, is a subset of Λ^+. The space Λ^+ is partially ordered by the usual pointwise ordering of functions, that is for all $t\in R$, $F\leqslant G\Leftrightarrow F(t)\leqslant G(t)$. For every $a\geqslant 0$, $H_a(t)$ is the element of D_+, which is defined by

$$H_a(t)=\left\{\begin{array}{lll}0 & \text{if} & t\leqslant a\\ 1 & \text{if} & t>a\end{array}\right. .$$

One can easily show that the maximal element for Λ^+ in this order is the distribution function $H_0(t)$.

Definition 1.1. (see [24]) *A function* $T : [0,1] \times [0,1] \to [0,1]$ *is a continuous triangular norm (briefly a t-norm)if* T *satisfies the following conditions:*
(i) *T is commutative and associative;*
(ii) *T is continuous;*
(iii) $T(x,1) = x$ *for all* $x \in [0,1]$*;*
(iv) $T(x,y) \leqslant T(z,w)$ *whenever* $x \leqslant z$ *and* $y \leqslant w$ *for all* $x,y,z,w \in [0,1]$*. Three typical examples of continuous* $t-$*norms are* $T(x,y) = xy, T(x,y) = max\{a+b-1,0\}$ *and* $T(x,y) = min(a,b)$*. Recall that, if* T *is a* $t-$*norm and* $\{x_n\}$ *are given numbers in* $[0,1]$*, then* $T_{i=1}^{n} x_i$ *is defined recursively by* $T_{i=1}^{1} x_1$ *and* $T_{i=1}^{n} x_i = T(T_{i=1}^{n-1} x_i, x_n)$ *for* $n \geqslant 2$*.*

Definition 1.2. *A random normed space(briefly RN-space) is a triple* (X, Ψ, T)*, where* X *is a vector space,* T *is a continuous t-norm and* $\Psi : X \to D^+$ *is a mapping such that the following conditions hold:*
(i) $\Psi_x(t) = H_0(t)$ *for all* $t > 0$ *if and only if* $x = 0$*;*
(ii) $\Psi_{\alpha x}(t) = \Psi_x(\frac{t}{|\alpha|})$ *for all* $\alpha \in R$*,* $\alpha \neq 0$*,* $x \in X$ *and* $t \geqslant 0$*.*
(iii) $\Psi_{x+y}(t+s) \geqslant T(\Psi_x(t), \Psi_y(s))$*, for all* $x,y \in X$ *and* $t,s \geqslant 0$*. Every normed space* $(X, ||.||)$ *defines a random normed space* (X, Ψ, T_M) *where for every* $t > 0$*,*

$$\Psi_u(t) = \frac{t}{t + ||u||}$$

and T_M *is the minimum t-norm. This space is called the induced random normed space.*

If the t-norm T be $\sup_{0<a<1} T(a,a) = 1$, then every RN-space (X, Ψ, T) be a metrizable linear topological space with the topology τ (called the Ψ-topology or the (ϵ, δ)-topology) induced by the base of neighborhoods of θ, $\{U(\epsilon, \lambda) | \epsilon > 0, \lambda \in (0,1)\}$, where

$$U(\epsilon, \lambda) = \{x \in X | \Psi_x(\epsilon) > 1 - \lambda\}.$$

Definition 1.3. *Let* (X, Ψ, T) *be an RN-space.*
(i) *A sequence* $\{x_n\}$ *in* X *is said to be convergent to* $x \in X$ *if for all*

$t > 0$, $lim_{n\to\infty}\Psi_{x_n - x}(t) = 1$.
(ii) A sequence $\{x_n\}$ in X is said to be Cauchy sequence in X if for all $t > 0$,

$$lim_{n\to\infty}\Psi_{x_n - x_m}(t) = 1.$$

The RN-space (X, Ψ, T) is said to be complete if every Cauchy sequence in X is convergent.

Theorem 1.1. (see [24]) *If (X, Ψ, T) is RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty}\Psi_{x_n}(t) = \Psi_x(t)$.*

2. Stability of Equation (1) in Random Normed Spaces

Remark 2.1. *In the rest of the paper let $M_f(x, y) = f(x+2y) + f(x-2y) - 2f(x) + f(2x) - 4f(x+y) - 4f(x-y)$.*

Theorem 2.1. *Let X be a real linear space,(Z, Ψ, min) be a random space, and $\psi : X^2 \to Z$ be a function such that for some $0 < \alpha < 8$,*

$$\Psi_{\psi(2x,0)}(t) \geqslant \Psi_{\alpha\psi(x,0)}(t) \quad \forall x \in X,\ t > 0 \tag{2}$$

and for all $x \in X$ and $t > 0$

$$\lim_{n\to\infty}\Psi_{(2^n x, 2^n y)}(8^n t) = 1.$$

If (Y, μ, min) be a complete random space and $f : X \to Y$ is a mapping such that for all $x, y \in X$ and $t > 0$

$$\mu_{M_f(x,y)} \geqslant \Psi_{\psi(x,y)}(t), \tag{3}$$

then there is a unique mapping $C : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geqslant \Psi_{\psi(x,0)}((8-\alpha)t). \tag{4}$$

Proof. Existence: Putting $y = 0$ in (3) we see that for all $x \in X$,

$$\mu_{\frac{f(2x)}{8} - f(x)}(t) \geqslant \Psi_{\psi(x,0)}(8t), \tag{5}$$

Replacing x by $8^n x$ in (5), we obtain

$$\mu_{\frac{f(2^{n+1}x)}{8^{n+1}}-\frac{f(2^n x)}{8^n}}(t) \geqslant \Psi_{\psi(2^n x,0)}(8^{n+1}t) \geqslant \Psi_{\psi(x,0)}\Big(\frac{8^{n+1}t}{\alpha^n}\Big). \tag{6}$$

Since

$$\frac{f(2^n x)}{8^n} - f(x) = \sum_{k=0}^{n-1}\Big\{\frac{f(2^{k+1}x)}{8^{k+1}} - \frac{f(2^k x)}{8^k}\Big\}, \tag{7}$$

so by (7) we obtain

$$\mu_{\frac{f(2^n x)}{8^n}-f(x)}\Big(\sum_{k=0}^{n-1}\frac{t\alpha^k}{8^{k+1}}\Big) \geqslant T_{k=0}^{n-1}(\Psi_{\psi(x,0)}(t)) = \Psi_{\psi(x,0)}(t). \tag{8}$$

This implies that

$$\mu_{\frac{f(2^n x)}{8^n}-f(x)}(t) \geqslant \Psi_{\psi(x,0)}\Big(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{8^{k+1}}}\Big). \tag{9}$$

Replacing x by $2^p x$ in (9), we obtain

$$\begin{aligned}
\mu_{\frac{f(2^{n+p}x)}{8^{(n+p)}}-\frac{f(2^p x)}{8^p}}(t) &\geqslant \Psi_{\psi(2^p x,0)}\Big(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{8^{(k+p)+1}}}\Big)\\
&\geqslant \Psi_{\psi(x,x)}\Big(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^{k+p}}{8^{(k+p)+1}}}\Big)\\
&= \Psi_{\psi(x,0)}\Big(\frac{t}{\sum_{k=p}^{n+p-1}\frac{\alpha^k}{8^{k+1}}}\Big).
\end{aligned} \tag{10}$$

As

$$\lim_{p,n\to\infty}\Psi_{\psi(x,0)}\Big(\frac{t}{\sum_{k=p}^{n+p-1}\frac{\alpha^k}{8^{k+1}}}\Big) = 1,$$

then $\{\frac{f(2^n x)}{8^n}\}$ is a Cauchy sequence in complete RN-space (Y, μ, min), so there is some point $C(x) \in Y$ such that $\lim_{n\to\infty}\frac{f(2^n x)}{8^n} = C(x)$. Fix $x \in X$ and put $p = 0$ in (10). Then we obtain

$$\mu_{\frac{f(2^n x)}{8^n}-f(x)}(t) \geqslant \Psi_{\psi(x,0)}\Big(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{8^{k+1}}}\Big). \tag{11}$$

and so, for every $\epsilon > 0$, we have

$$\begin{aligned}\mu_{C(x)-f(x)}(t+\epsilon) &\geqslant T\Big(\mu_{C(x)-\frac{f(2^n x)}{8^n}}(\epsilon), \mu_{\frac{f(2^n x)}{8^n}-f(x)}(t)\Big)\\ &\geqslant T\Big(\mu_{C(x)-\frac{f(2^n x)}{8^n}}(\epsilon), \Psi_{\psi(x,0)}\Big(\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{8^{k+1}}}\Big)\Big). \qquad (12)\end{aligned}$$

Getting the limit as $n \to \infty$ in (12), we get

$$\mu_{C(x)-f(x)}(t+\epsilon) \geqslant \Psi_{\psi(x,0)}((8-\alpha)t). \qquad (13)$$

Since ϵ was arbitrary, by getting $\epsilon \to 0$ in (13), we get

$$\mu_{C(x)-f(x)}(t) \geqslant \Psi_{\psi(x,0)}((8-\alpha)t). \qquad (14)$$

Replacing x and y by $2^n x$ and $2^n y$ in (3), respectively, we get for all $x, y \in X$ and for all $t > 0$,

$$\mu_{\frac{M_f(2^n x, 2^n y)}{8^n}}(t) \geqslant \Psi_{\psi(2^n x, 2^n y)}(8^n t). \qquad (15)$$

Getting the limit as $n \to \infty$ in (15) and using this fact that

$$\lim_{n\to\infty} \Psi_{\psi(2^n x, 2^n y)}(8^n t) = 1,$$

we conclude that

$$C(x+2y) + C(x-2y) = 2C(x) - C(2x) + 4\Big\{C(x+y) + C(x-y)\Big\}.$$

Uniqueness: To prove the uniqueness of the mapping C, assume that there is another mapping $D : X \to Y$ which satisfies (4). Since f is a cubic mapping, C and D are too. Therefore, for all $n \in N$ and every $x \in X$, we can write $C(2^n x) = 8^n C(x)$ and $D(2^n x) = 8^n D(x)$. Therefore, we have

$$\mu_{C(x)-D(x)}(t) = \lim_{n\to\infty} \mu_{\frac{C(2^n x)}{8^n} - \frac{D(2^n x)}{8^n}}(t),$$

so

$$\begin{aligned}\mu_{\frac{C(2^n x)}{8^n} - \frac{D(2^n x)}{8^n}}(t) &\geqslant min\Big\{\mu_{\frac{C(2^n x)}{8^n} - \frac{f(2^n x)}{8^n}}\Big(\frac{t}{2}\Big), \mu_{\frac{D(2^n x)}{8^n} - \frac{f(2^n x)}{8^n}}\Big(\frac{t}{2}\Big)\Big\}\\ &\geqslant \Psi_{\psi(2^n x, 0)}\Big(\frac{8^n(8-\alpha)t}{2}\Big)\\ &\geqslant \Psi_{\psi(x,0)}\Big(\frac{8^n(8-\alpha)t}{2\alpha^n}\Big).\end{aligned}$$

Since $\lim_{n\to\infty} \frac{8^n(8-\alpha)t}{2\alpha^n} = \infty$, then $\lim_{n\to\infty} \Psi_{\psi(x,0)}\left(\frac{8^n(8-\alpha)t}{2\alpha^n}\right) = 1$. Therefore, it follows that , $\mu_{C(x)-D(x)}(t) = 1$ and so $C(x) = D(x)$ for all $x \in X$ and $t > 0$. Which completes the proof. □

Corollary 2.1. *Let X be a real linear space, (Z, Ψ, min) be a random normed space, and (Y, μ, min) a complete random normed space. Let $p \in (0, 1)$ and $z_0 \in Z$. If $f : X \to Y$ is a mapping and for all $x, y \in X$ and $t > 0$*

$$\mu_{M_f(x,y)}(t) \geqslant \Psi_{||x||^p z_0}(t),$$

then there is a unique cubic mapping $C : X \to Z$ such that

$$\mu_{f(x)-C(x)}(t) \geqslant \Psi_{||x||^p z_0}((8 - 8^p)t),$$

Proof. Let $\alpha = 8^p$ and $\psi : X^2 \to Z$ be defined by $\psi(x, y) = ||x||^p z_0$ or $\psi(x, y) = (||x||^p + ||y||^p)z_0$. □

Corollary 2.2. *Let X be a real linear space, (Z, Ψ, min) be a random normed space, and (Y, μ, min) a complete random normed space. Let $p \in (0, 1)$ and $z_0 \in Z$. If $f : X \to Z$ is a mapping and for all $x, y \in X$ and $t > 0$*

$$\mu_{M_f(x,y)}(t) \geqslant \Psi_{(||x||^p+||y||^p+||x||^p.||y||^p)z_0}(t),$$

then there is a unique mapping $C : X \to Z$ such that

$$\mu_{f(x)-C(x)}(t) \geqslant \Psi_{||x||^p z_0}((8 - 8^p)t).$$

Proof. Let $\alpha = 8^p$ and $\psi : X^2 \to Z$ be defined by $\psi(x, y, z) = (||x||^p + ||y||^p + ||x||^p.||y||^p)z_0$. □

Corollary 2.3. *Let X be a real linear space, (Z, Ψ, min) be a RN-space, and (Y, μ, min) a complete RN-spaces. Let $z_0 \in Z$, $f : X \to Y$ be a mapping, and for all $x, y \in X$ and $t > 0$*

$$\mu_{M_f(x,y)}(t) \geqslant \Psi_{\delta z_0}(t).$$

Then there is a unique mapping $C : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geqslant \Psi_{\delta z_0}(7t).$$

Proof. Let $\alpha = 1$ and $\psi : X^2 \to Z$ be defined by $\psi(x, y) = \delta z_0$. $\square$

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