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Admissible and Minimax Estimators of a Lower Bounded Scale Parameter of a Gamma Distribution under the Entropy Loss Function

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Abstract. This paper is concerned with admissible and minimax estimation of scale parameter θ of a gamma distribution under the entropy loss function, when it is known that $\theta \geqslant a$ for some known $a > 0$. An admissible minimax estimator of θ, which is the pointwise limit of a sequence of Bayes estimators, is derived. Also, the admissible estimators and the only minimax estimator of θ in the class of truncated linear estimators are obtained. Finally, the results are extended to a subclass of scale parameter exponential family and the family of transformed chi-square distributions.

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1. Introduction

In some estimation problems, there exists definite prior information on the values of parameter of interest in the form of a bound on it. For example, the average per capita income in a certain country is at least a known amount of money. The problem of estimation of bounded parameters for certain densities have considered by several researchers. These problems were first studied by Brunk ([3]) and van Eeden ([16]) and then grew rapidly. For a classified and extensively reviewed work in this area, as well as a list of references, (see van Eeden [19]).

About the problem of admissible and minimax estimation of a lower-bounded parameter, under Squared Error Loss (SEL) function, Katz ([9]) gives an admissible minimax estimator of a normal mean $\mu \in [a, \infty)$, Berry ([1]) obtains admissible minimax estimators for the exponential distribution with support $\theta \in [a, \infty)$, $\theta \in (-\infty, b]$ or $\theta \in [a, b]$, and Shao and Strawderman ([14,15]) give the class of dominating estimator of Maximum Likelihood Estimator (MLE) for lower bounded normal mean and dominating estimator of truncated linear estimators for lower bounded gamma scale parameter, respectively. Using Scale Invariant Squared Error Loss (SISEL) function, Kaluszka ([7,8]) and van Eeden ([17]) obtain minimax and admissible estimators for lower bounded scale parameter $\theta \geqslant a$ of gamma distribution and van Eeden and Zidek ([22]) obtained a mimimax estimator of a bounded scale parameter. Jafari Jozani et al. ([6]) extend the results of van Eeden ([17]) to a subclass of exponential family, and van Eeden and Zidek ([20,21]) and van Eeden ([18]) consider minimax estimation of a lower bounded scale parameter of a F-distribution.

Let X have a gamma distribution with probability density function (pdf)

$$f_\theta(x) = \frac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\theta}}, \qquad x > 0, \tag{1}$$

where $\alpha > 0$ is a known shape parameter and θ is an unknown scale parameter. Suppose $\theta \geqslant a$ for some known $a > 0$. It is interesting to note that in the literature, estimation of lower bounded parameters are often considered under SEL and SISEL functions which are symmetric about the parameter θ and convex in estimator δ. For example, Jafari Jozani et al. ([6]) found an admissible minimax estimator for θ in a general class of distributions (including gamma distribution) when θ is restricted to $\theta \in [a, \infty)$ under SISEL function.

In some estimation problems, over-estimation maybe more serious than under-estimation. In such cases, the usual methods of estimation, which are based on symmetric loss function may be inappropriate. As an alternative to SISEL, which is appropriate for estimating scale parameter θ, consider the entropy loss function defined by

$$L(\theta, \delta) = \frac{\delta}{\theta} - ln\frac{\delta}{\theta} - 1, \tag{2}$$

which is also known as Stein's loss. This loss is convex in δ and is not symmetric and it penalizes heavily under estimation. For a review of the literature in using entropy loss, (see Parsian and Nematollahi [12]) and references cited therein. Under the loss (2), the best scale invariant and admissible estimator of $\theta > 0$ under the model (1) is $\delta_0(X) = \frac{X}{\alpha}$ (See Dey et al., [4]).

Under the entropy loss function (2) and for a general scale family of distributions, Kubokawa ([10]) showed that the unrestricted Minimun Risk Equivariant (MRE) estimator $\delta_0(X)$ of scale parameter θ is minimax when $\theta \geqslant a$, and also the Generalized Bayes Estimator (GBE) of θ with respect to (w.r.t.) improper prior $\pi(\theta) = \frac{1}{\theta}$, $\theta \geqslant a$, dominates $\delta_0(X)$ and hence is minimax. Also Marchand and Strawderman ([11]) extend the results of Kubokawa ([10]) to a general class of convex loss functions and obtained class of dominating estimator of $\delta_0(X)$.

In this paper we obtain admissible and minimax estimators of θ when $\theta \geqslant a$ in the model (1) under the entropy loss function (2). To this end, in Section 2, an admissible minimax estimator of θ, which is the pointwise limit of a sequence of Bayes estimators, is derived. In Section 3, the admissible estimators and the only minimax estimator in the class of truncated linear estimators are obtained. The results are extended to a subclass of the scale parameter exponential family and the family of transformed chi-square distributions introduced by (Rahman and Gupta [13]).

2. An Admissible Minimax Estimator

Let X have pdf (1) with known $\alpha > 0$ and unknown $\theta \geqslant a$. In this section, we find an admissible minimax estimator of θ, under the entropy loss function (2).

Following van Eeden ([17]), consider the sequence of proper prior

$$\pi_m(\theta) = \frac{a^{\frac{1}{m}}}{m\theta^{1+\frac{1}{m}}}, \quad \theta \geqslant a, \quad a > 0, \ m = 1, 2, \ldots . \tag{3}$$

For $\beta > 0$ and $x > 0$, define

$$g_\beta(x) = x^{\beta-1}e^{-x} \qquad and \qquad G_\beta(x) = \int_0^x t^{\beta-1}e^{-t}dt. \tag{4}$$

Then from (1) and (3), the posterior distribution of θ given $X = x$ is given by

$$\pi_m(\theta|x) = \frac{x^{\alpha_m}e^{-\frac{x}{\theta}}}{\theta^{\alpha_m+1}G_{\alpha_m}(\frac{x}{a})}, \quad \theta \geqslant a, \quad a > 0, \quad m = 1, 2, \ldots, \tag{5}$$

where $\alpha_m = \alpha + \frac{1}{m}$. The Bayes estimator of θ under the loss (2) is $\delta^{\pi_m}(x) = \{E(\frac{1}{\theta}|x)\}^{-1}$, and from (5) we have

$$E(\frac{1}{\theta}|x) = \frac{1}{xG_{\alpha_m}(\frac{x}{a})}\int_a^\infty \frac{x^{\alpha_m+1}e^{-\frac{x}{\theta}}}{\theta^{\alpha_m+1}} = \frac{G_{\alpha_m+1}(\frac{x}{a})}{xG_{\alpha_m(\frac{x}{a})}}.$$

It is easy to verify that

$$G_{\alpha_m}(y) = \frac{1}{\alpha_m}[g_{\alpha_m+1}(y) + G_{\alpha_m+1}(y)],$$

so,

$$\delta^{\pi_m}(x) = \frac{x}{\alpha_m}\left\{\frac{g_{\alpha_m+1}(\frac{x}{a})}{G_{\alpha_m+1}(\frac{x}{a})} + 1\right\}. \tag{6}$$

From (6) we have

$$\begin{aligned}\lim_{m\to\infty} \delta^{\pi_m}(x) &= \frac{x}{\alpha}\left\{\frac{g_{\alpha+1}(\frac{x}{a})}{G_{\alpha+1}(\frac{x}{a})} + 1\right\} \\ &= \delta^\pi(x) \qquad (say). \end{aligned} \tag{7}$$

Notice that $\delta^\pi(X)$ is the generalized Bayes (and limiting Bayes) estimator of $\theta(\geqslant a)$ w.r.t. improper prior

$$\pi(\theta) = \frac{1}{\theta}, \ \theta \geqslant a \tag{8}$$

under the loss (2). In the following theorem we show that $\delta^{\pi}(X)$ is an admissible estimator of $\theta(\geqslant a)$ under the loss (2). Our proof is based on Blyth's ([2]) method.

Theorem 2.1. *Let X have pdf* (1) *with $\theta \geqslant a$ and $\alpha > 1$. Then under the loss* (2)*, the estimator* (7) *is an admissible estimator of θ.*

Proof. Under entropy loss (2), $L(\theta, \delta) = L(\frac{\theta}{a}, \frac{\delta}{a})$. So, without loss of generality we take $a = 1$. Since the risk function of an estimator δ under the loss (2) is continuous in θ, we can use Blyth's ([2]) method. That is, we must show that for all $\eta > 0$ such that $\theta - \eta \geqslant a$,

$$\lim_{m\to\infty} \frac{m\{\Pi_m(\theta+\eta) - \Pi_m(\theta-\eta)\}}{m\{r(\pi_m, \delta^{\pi}) - r(\pi_m, \delta^{\pi_m})\}} = +\infty, \tag{9}$$

where Π_m is the distribution function of θ with pdf (3), δ^{π_m} is Bayes estimator of θ w.r.t. prior (3) and $r(\pi_m, \delta^{\pi})$, $r(\pi_m, \delta^{\pi_m})$ are the Bayes risks of δ^{π} and δ^{π_m} w.r.t to the prior π_m, respectively. From (3) and using L'Hospital's rule, we have

$$\begin{aligned}
\lim_{m\to\infty} m\{\Pi_m(\theta+\eta) - \Pi_m(\theta-\eta)\} &= \lim_{m\to\infty} \int_{\theta-\eta}^{\theta+\eta} \frac{1}{t^{1+\frac{1}{m}}} dt \\
&= \lim_{m\to\infty} \frac{(\theta-\eta)^{-\frac{1}{m}} - (\theta+\eta)^{-\frac{1}{m}}}{\frac{1}{m}} \\
&= \lim_{m\to\infty} \frac{e^{-\frac{1}{m}\ln(\theta-\eta)} - e^{-\frac{1}{m}\ln(\theta+\eta)}}{\frac{1}{m}} \\
&= \ln \frac{\theta+\eta}{\theta-\eta} > 0. \qquad (10)
\end{aligned}$$

Also from (6) and (7) with $a = 1$, we have

$$
\begin{aligned}
m\{r(\pi_m,\delta^{\pi}) &- r(\pi_m,\delta^{\pi_m})\} = \int_1^\infty\int_0^\infty (\frac{x}{\alpha\theta}-\frac{x}{\alpha_m\theta})\frac{1}{\theta^{\frac{1}{m}+1}}f_\theta(x)dxd\theta \\
&+ \int_1^\infty\int_0^\infty \left(\frac{x}{\alpha\theta}\frac{g_{\alpha+1}(x)}{G_{\alpha+1}(x)}-\frac{x}{\alpha_m\theta}\frac{g_{\alpha_m+1}(x)}{G_{\alpha_m+1}(x)}\right)\frac{1}{\theta^{\frac{1}{m}+1}}f_\theta(x)dxd\theta \\
&+ \int_1^\infty\int_0^\infty \left(\ln\frac{x}{\alpha_m\theta}-\ln\frac{x}{\alpha\theta}\right)\frac{1}{\theta^{\frac{1}{m}+1}}f_\theta(x)dxd\theta \\
&+ \int_1^\infty\int_0^\infty \left[\ln\left(\frac{g_{\alpha_m+1}(x)}{G_{\alpha_m+1}(x)}+1\right)-\ln\left(\frac{g_{\alpha+1}(x)}{G_{\alpha+1}(x)}+1\right)\right] \\
&\qquad \times\frac{1}{\theta^{\frac{1}{m}+1}}f_\theta(x)dxd\theta \\
&= J_{1m}+J_{2m}+J_{3m}+J_{4m}. \qquad (11)
\end{aligned}
$$

Note that

$$
J_{1m}=(1-\frac{\alpha}{\alpha_m})\int_1^\infty\frac{1}{\theta^{1+\frac{1}{m}}}d\theta=\frac{1}{\alpha_m},
$$
$$
J_{3m}=\ln\frac{\alpha}{\alpha_m}\int_1^\infty\frac{1}{\theta^{1+\frac{1}{m}}}d\theta=m\ln\frac{\alpha}{\alpha_m},
$$

so, $\lim_{m\to\infty}J_{1m}=\frac{1}{\alpha}$ and using L'Hospital's rule

$$
\lim_{m\to\infty}J_{3m}=\lim_{m\to\infty}\frac{\ln\alpha-\ln(\alpha+1/m)}{1/m}=-\frac{1}{\alpha}.
$$

Hence $\lim_{m\to\infty}(J_{1m}+J_{3m})=\frac{1}{\alpha}-\frac{1}{\alpha}=0$. Since $\frac{g_\beta(x)}{G_\beta(x)}=\frac{x^{\beta-1}e^{-x}}{\int_0^x t^{\beta-1}e^{-t}dt}\leqslant\frac{x^{\beta-1}}{\int_0^x t^{\beta-1}dt}=\frac{\beta}{x}$ and $\ln(x+1)\leqslant x$, $x>0$, for $\alpha>1$ we have

$$
\left|\left\{\frac{x}{\alpha\theta}\frac{g_{\alpha+1}(x)}{G_{\alpha+1}(x)}-\frac{x}{\alpha_m\theta}\frac{g_{\alpha_m+1}(x)}{G_{\alpha_m+1}(x)}\right\}\frac{1}{\theta^{\frac{1}{m}+1}}\right|\leqslant\left(\frac{\alpha+1}{\alpha}+\frac{\alpha_m+1}{\alpha_m}\right)\frac{1}{\theta^2}\leqslant\frac{4}{\theta^2}
$$

and

$$
\begin{aligned}
\left|\left\{\ln\left(\frac{g_{\alpha_m+1}(x)}{G_{\alpha_m+1}(x)}+1\right)-\ln\left(\frac{g_{\alpha+1}(x)}{G_{\alpha+1}(x)}+1\right)\right\}\frac{1}{\theta^{\frac{1}{m}+1}}\right| &\leqslant\left(\frac{\alpha_m+1}{x}+\frac{\alpha+1}{x}\right)\frac{1}{\theta} \\
&\leqslant(2(\alpha+1)+1)\frac{1}{\theta x}.
\end{aligned}
$$

Since $\int_1^\infty \int_0^\infty \frac{4}{\theta^2} f_\theta(x) dx d\theta = 4 < \infty$ and for $\alpha > 1$, $\int_1^\infty \int_0^\infty (2(\alpha + 1) + 1)\frac{1}{\theta x} f_\theta(x) dx d\theta = \frac{(2(\alpha+1)+1}{\alpha - 1} < \infty$, so from Lebesgue dominated convergence theorem, $\lim_{m\to\infty} J_{2m} = \lim_{m\to\infty} J_{4m} = 0$, and hence from (11)

$$\lim_{m\to\infty} m\{r(\pi_m, \delta^\pi) - r(\pi_m, \delta^{\pi_m})\} = 0. \tag{12}$$

From (10) and (12), we conclude (9), which completes the proof. □

Remark 2.1. *From Theorem 2.4. of Kubokawa ([10]) and Corollary 4 of Marchand and Strawderman ([11]), we can conclude that the GBE $\delta^\pi(x)$ in (7) is a minimax estimator of θ when $\theta \geqslant a$, and hence $\delta^\pi(x)$ is an admissible minimax estimator of θ. Note that Kubokawa ([10]) and Marchand and Strawderman (11) showed that the GBE of θ w.r.t. improper prior (8) is minimax and dominate the unrestricted MRE estimator of θ under entropy loss function (2) and under a general class of convex loss functions, respectively. But their results do not address the general and interesting question of admissibility of GBE of $\theta(\geqslant a)$, see section 5 of Marchand and Strawderman ([11]). The admissibility of GBE of $\theta(\geqslant a)$ has been established in some special distributions and under SEL and SISEL function (see for example [5, 6, 17]). We proved in Theorem 2.1. the admissibility of GBE of lower bounded gamma scale parameter under entropy loss function (2).*

Remark 2.2. *Note that van Eeden ([17]) and Jafari Josani et al. ([6]) showed that the estimator*

$$\delta(x) = \frac{x}{\alpha+1}\left\{\frac{g_{\alpha+2}(\frac{x}{a})}{G_{\alpha+2}(\frac{x}{a})} + 1\right\}$$

is admissible for θ when $\theta \in [a, \infty)$ under SISEL function. Our estimator $\delta^\pi(x)$ in (7), which is admissible under entropy loss (2), is similar to their estimator with replacing α by $\alpha + 1$.

3. Admissible and Minimax Truncated Estimators

Let X be a random variable with pdf (1) where $\theta \geqslant 0$. Then $\delta_0(X) = \frac{X}{\alpha}$ is best scale invariant and admissible estimator of θ when θ is unrestricted, i.e., $\theta > 0$. Kubokawa ([10]), Marchand and Strawderman ([11]) showed that $\delta_0(X)$ is minimax in restricted parameter space $\theta \geqslant a$ and any minimax estimator of θ has minimax risk equal to constant risk of δ_0. Since

$$\begin{aligned} Minimax \;\; Value &= R(\theta, \delta_0(X)) = E_\theta\left[\frac{X}{\alpha\theta} - ln\frac{X}{\alpha\theta} - 1\right] \\ &= -E\left[ln\frac{X}{\alpha\theta}\right] = ln\alpha - \Psi(\alpha), \qquad (13) \end{aligned}$$

so, it is easy to see that a (necessary and) sufficient condition for an estimator δ_M to be minimax estimator of $\theta \geqslant a$ under the entropy loss function is given by

$$R(\theta, \delta_M) \leqslant \sup_{\theta \geqslant a} R(\theta, \delta_M) \leqslant \; ln\alpha - \Psi(\alpha). \qquad (14)$$

From (14), or equivalently from Theorem 2.4. of Kubokawa ([10]) and Remark 5 of Marchand and Strawderman ([11]), the truncated version of $\delta_0(X) = \frac{X}{\alpha}$, i.e., $\delta_{\frac{1}{\alpha}}(X) = max(a, \frac{X}{\alpha})$, is minimax estimator of $\theta(\geqslant a)$ under the entropy loss function (2). Note that $\delta_{\frac{1}{\alpha}}(X)$ belongs to the class of truncated linear estimators of $\theta(\geqslant a)$ which is given by

$$C = \{\delta_c | \delta_c(X) = max(a, cX), \quad c > 0, \quad a > 0\}. \qquad (15)$$

This class was studied by van Eeden and Zidek ([20, 21]) and van Eeden ([17]) for the estimation of lower bounded scale parameter of F and gamma distributions, respectively, under SISEL function. In this section, we characterize the admissible and minimax estimators of $\theta(\geqslant a)$ in the class C of truncated linear estimators.

Using the idea of van Eeden ([17]), in the following theorem we show that for the pdf (1) and under the loss (2), exactly one estimator in the class C is minimax estimator.

Theorem 3.1. *Let X be a random variable with pdf* (1) *where* $\theta \geqslant a$. *Then under the loss* (2), *the estimator* $\delta_{\frac{1}{\alpha}}(X) = max(a, \frac{X}{\alpha})$ *in the class C is minimax estimator of* $\theta(\geqslant a)$, *and no other estimator in C is minimax.*

Proof. From (14), it suffices to show that

$$\begin{aligned}
&\sup_{\theta\geqslant a} R(\theta, \delta_{\frac{1}{\alpha}}) = \ln\alpha - \psi(\alpha) \\
&\sup_{\theta\geqslant a} R(\theta, \delta_c) \ > \ln\alpha - \psi(\alpha) \qquad for \ \ c \neq \tfrac{1}{\alpha}.
\end{aligned} \tag{16}$$

The risk function of δ_c under the loss (2) is

$$\begin{aligned}
R(\theta, \delta_c) &= \frac{a}{\theta} - \ln\frac{a}{\theta} - 1 + \int_{\frac{a}{c}}^{\infty} \Big\{ (\frac{cx}{\theta} - \ln\frac{cx}{\theta}) - (\frac{a}{\theta} - \ln\frac{a}{\theta}) \Big\} \\
&\quad \frac{1}{\theta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\theta}} dx \\
&= \frac{a}{\theta} - \ln\frac{a}{\theta} - 1 + \int_{\frac{a}{c\theta}}^{\infty} \Big\{ (cy - \ln cy) - (\frac{a}{\theta} - \ln\frac{a}{\theta}) \Big\} \\
&\quad \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)} dy.
\end{aligned} \tag{17}$$

Hence for all $c > 0$ we have

$$\frac{\partial}{\partial\theta} R(\theta, \delta_c) = \frac{\theta - a}{\theta^2} \left\{ 1 - \frac{1}{\Gamma(\alpha)} \int_{\frac{a}{c\theta}}^{\infty} y^{\alpha-1} e^{-y} dy \right\} > 0,$$

therefore $R(\theta, \delta_c)$ is a strictly increasing function of θ. So,

$$\begin{aligned}
\sup_{\theta\geqslant a} R(\theta, \delta_c) &= \lim_{\theta\to\infty} R(\theta, \delta_c) \\
&= -\ln a - 1 + \int_0^{\infty} (cy - \ln cy + \ln a) \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)} dy \\
&\quad + \lim_{\theta\to\infty} \ln\theta \left\{ 1 - \int_{\frac{a}{c\theta}}^{\infty} \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)} dy \right\} \\
&= -1 + c\alpha - \ln c - \psi(\alpha) \\
&= u_{\alpha}(c), \quad (say).
\end{aligned}$$

Note that $u_\alpha(c)$ is a strictly convex function of c and takes its minimum at $c = \frac{1}{\alpha}$. Therefore, (16) holds true, which completes the proof. $\square$

Remark 3.1. *From Remark* 5. *of Marchand and Strawderman* ([11])*, if an estimator* δ' *satisfy* $\delta' \geqslant \delta^\pi$*, where* δ^π *is GBE given by* (7)*, then* δ' *is not minimax. But we cannot use this result in our problem, because the estimator* δ_c*,* $c \neq \frac{1}{\alpha}$ *in the class C does not satisfy the inequality* $\delta_c \geqslant \delta^\pi$ *in general.*

In the following theorem we characterize the admissible estimators in the class C, which is a result of Lemma 3.2. Theorem 3.1. and 3.2. of van Eeden ([17]) by replacing $\alpha + 1$ with $\alpha(> 1)$ but under the loss (2).

Theorem 3.2. *Let* X *have pdf* (1) *with* $\theta \geqslant a$ *and* $\alpha > 1$*. Then under the entropy loss* (2)*, the estimators* $\delta_c(X)$ *in the class C are admissible if and only if* $c \in (0, \frac{1}{\alpha}]$.

Proof. From (17) we have

$$\frac{\partial}{\partial c} R(\theta, \delta_c) \lesseqqgtr 0 \Longleftrightarrow c \int_{\frac{a}{c\theta}}^{\infty} g_{\alpha+1}(y)dy \lesseqqgtr \int_{\frac{a}{c\theta}}^{\infty} g_\alpha(y)dy.$$

Let $H_\alpha(k) = \{\int_k^\infty g_\alpha(y)dy\}\{\int_k^\infty g_{\alpha+1}(y)dy\}^{-1}$, then from Lemma 3.2. and Theorem 3.1. of van Eeden ([17]), we conclude that there exist a function $c(\theta)$ such that

$$\frac{\partial}{\partial c} R(\theta, \delta_c) \lesseqqgtr 0 \Longleftrightarrow c \lesseqqgtr c(\theta),$$

where $c(\theta)$ satisfies: (i) $0 < c(\theta) < \frac{1}{\alpha}$; (ii) $c(\theta)$ is strictly increasing in θ; (iii) $\lim_{\theta \to a} c(\theta) = 0$; and (iv) $\lim_{\theta \to \infty} c(\theta) = \frac{1}{\alpha}$. Further $R(a, \delta_c)$ is strictly increasing in c and for $c \in (0, \frac{1}{\alpha})$ there exists $\theta(c) > a$ such that $R(\theta(c), \delta_{\frac{1}{\alpha}}) < R(\theta(c), \delta_c)$.

From the above results and by a similar argument given in Theorem 3.2. of van Eeden ([17]), we conclude that for $\frac{1}{\alpha} < c < c'$, the estimator $\delta_c(X)$ dominates $\delta_{c'}(X)$ and $\delta_c(X)$ is admissible in class C if $0 < c \leqslant \frac{1}{\alpha}$. So, the estimators $\delta_c(X)$ in class C are admissible if and only if $0 < c \leqslant \frac{1}{\alpha}$. $\square$

Remark 3.2. *From Theorems* 3.1. *and* 3.2. *the only admissible minimax estimator of* $\theta \geqslant a$ *in the class of truncated linear estimators* C *is* $\delta_{\frac{1}{\alpha}}(X) = max(a, \frac{X}{\alpha})$.

Remark 3.3. *Let* $X_1, ..., X_n$ *be a random sample of size* n *from* $Gamma(\alpha, \theta)$*-distribution with pdf* (1). *Then* $T(\mathbf{X}) = \sum_{i=1}^{n} X_i$ *with* $\mathbf{X} = (X_1, \ldots, X_n)$, *is a complete sufficient statistics for* θ *and* $T(\mathbf{X})$ *has* $Gamma(\alpha n, \theta)$*-distribution. Therefore, the results of Sections* 2 *and* 3 *holds for this case with replacing* α *by* $n\alpha$ *and* X *by* $T(\mathbf{X})$.

Remark 3.4. *The results of Sections* 2 *and* 3 *can be extended to a subclass of exponential family as follow. Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *have the joint pdf*

$$f(\mathbf{x}, \theta) = c(\mathbf{x}, n)\theta^{-\nu} e^{-\frac{T(\mathbf{x})}{\theta}}, \qquad (18)$$

where $\mathbf{x} = (x_1, \ldots, x_n)$, $c(\mathbf{x}, n)$ *is a function of* $\mathbf{x}$ *and* n, ν *is a function of* n *and* $T(\mathbf{X})$ *is a complete sufficient statistics for* θ *with* $Gamma(\nu, \theta)$*-distribution. An admissible linear estimator of* $\theta(> 0)$ *under entropy loss function can be found in Parsian and Nematollahi* ([12]) *and an admissible minimax estimator of* $\theta(\geqslant a)$ *under SISEL function can be found in Jafari Jozani et al.* ([6]). *Since* $T = T(\boldsymbol{X})$ *has a* $Gamma(\nu, \theta)$*-distribution, therefore from Remark* 3.3. *we can extend the results of Sections* 2 *and* 3 *to the subclass of exponential family* (18) *by replacing* α *by* ν *and* X *by* $T(\boldsymbol{X})$.

Remark 3.5. *The results of Sections* 2 *and* 3 *can also be extended to the family of transformed chi-square distributions introduced by Rahman and Gupta* ([13]) *which includes Pareto and beta distributions as a special case. For details see Jafari Jozani et al.* ([6]).

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