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Another Look at the Limit Summability of Real Functions

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Abstract. This paper is a continuation of our recent paper entitled *limit summability of real functions* ([2]). In this work weak, semi, absolutely and uniformly limit summability will be given. Also, we generalize and extend some results of [2].

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1. Weak and Semi Limit Summable Functions

In [2] we have introduced and studied limit summability of real and complex functions. There are some relations between the topic and the Gamma type functions ([1]). Here we state several tests for weak, semi, absolutely and uniformly limit summability of functions.
In general, we assume $f : D_f \to \mathbb{C}$, where $D_f \subseteq \mathbb{C}$. In the real case we take the function $f : D_f \to \mathbb{R}$, where $D_f \subseteq \mathbb{R}$. A positive real function f is a real function such that $R_f \subseteq \mathbb{R}^+$. By $\mathbb{N}^*$, $\mathbb{N}$ we denote the set of positive and non-negative integer numbers, respectively.
For a function with domain D_f, we put

$$\Sigma_f = \{x | x + \mathbb{N}^* \subseteq D_f\}.$$

Let $\mathbb{N}^* \subseteq D_f$ and for any positive integer n and $x \in \Sigma_f$ set

$$R_n(f, x) \ = \ R_n(x) = f(n) - f(x + n),$$

$$f_{\sigma_n}(x) \ = \ xf(n) + \sum_{k=1}^{n} R_k(x).$$

When $x \in D_f$, we may use the notation $\sigma_n(f(x))$ instead of $\sigma_n(f,x)$. The function f is called limit summable at $x_0 \in \Sigma_f$ if the sequence $\{f_{\sigma_n}(x_0)\}$ is convergent. The function f is called limit summable on the set $S \subseteq \Sigma_f$ if it is limit summable at all points of S. Also, we put

$$D_{f_\sigma} = \{x \in \Sigma_f | f \text{ is summable at } x\},$$

and represent the limit function $R_n(f,x)$ as $R(f,x)$ or $R(x)$.

It is easy to see that $\Sigma_f \cap D_f = \Sigma_f + 1 = \{x+1 | x \in \Sigma_f\}$ and

$$0 \in \Sigma_f \Leftrightarrow \mathbb{N}^* \subseteq D_f \Leftrightarrow \mathbb{N} \subseteq \Sigma_f \ .$$

Convention: For brevity we use the term *summable* for *limit summable*, and restrict ourselves to the assumption $\mathbb{N}^* \subseteq D_f$.

As we can see in [2], always $f_\sigma(0) = 0$ and if $0 \in D_f$, then $\{-1, 0\} \subseteq D_{f_\sigma}$ and $f_\sigma(-1) = -f(0)$. But $1 \in D_{f_\sigma}$ if and only if $R_n(1)$ is convergent and $f_\sigma(1) = f(1) + R(1)$. A necessary condition for the summability of f at x is $\lim_{n\to\infty}(R_n(x) - xR_{n-1}(1)) = 0$. Therefore, if $1 \in D_{f_\sigma}$, then the functional sequence $\{R_n(x)\}$ is convergent on D_{f_σ} and $R(x) = R(1)x$ (for all $x \in D_{f_\sigma}$), and

$$(*_1) \quad f_\sigma(x) = f(x) + f_\sigma(x-1) + R(1)x; \quad \forall x \in D_{f_\sigma} + 1.$$

So if $R(1) = 0$, then

$$f_\sigma(m) = f(1) + \cdots + f(m) = \sum_{j=1}^{m} f(j); \quad \forall m \in \mathbb{N}^*.$$

Definition 1.1. *We call the function f weak (limit) summable if $\Sigma_f = D_{f_\sigma}$. Moreover, if $R(1) = 0$, then f is called semi (limit) summable* .

By Lemma 2.1. in [2], f is summable if and only if it is semi summable and $D_f \subseteq D_f - 1$. If f is weak summable, then $1 \in D_{f_\sigma}$ and so $R_n(1)$ is convergent.

Example 1.2. If $0 < |a| < 1$, then $f(x) = a^x + x$ is weak summable but it is not semi summable. The function $g(x) = \frac{1}{x}$ is semi summable but it is not summable.
Now we need a property about complex sequences introduced in the following definition.

Definition 1.3. *We call the complex sequence* $\{a_n\}$ weak convergent *(*semi convergent*) if* $a_{n+1} - a_n$ *is convergent (*$a_{n+1} - a_n$ *converges to* 0*). The sequence* a_n *is called absolutely convergent if the series* $\sum_{n=1}^{+\infty} |a_{n+1} - a_n|$ *is convergent. If* $a_n \to a$ *as* $n \to \infty$ *and* a_n *is absolutely convergent, then we say* a_n *is absolutely convergent to* a*.*
If a_n is absolutely convergent, then it is convergent (and so it is semi convergent). A monotonic sequence is absolutely convergent if and only if it is convergent. Also, if the series $\sum a_n$ is absolutely convergent, then a_n is absolutely convergent to zero.
The sequence $a_n = \sqrt{n}$ ($b_n = n$) is semi convergent (weak convergent) but not convergent (semi convergent). The sequence $c_n = \frac{(-1)^n}{n}$ is convergent but not absolutely convergent. The sequence $d_n = \frac{1}{n}$ is absolutely convergent but $\sum d_n$ is not (absolutely) convergent.

Note: It is interesting to see that a_n is weak convergent (semi convergent) if and only if a_n is weak summable (semi summable or equivalently summable) as a function with domain $\mathbb{N}^*$. Also, for a function f the following are equivalent (see [2]):

The sequence $f_n = f(n)$ is weak convergent, $R_n(f, 1)$ is convergent, $1 \in D_{f_\sigma}$, $\mathbb{N}^* \subseteq D_{f_\sigma}$, $D_{f_\sigma} \cap D_f = D_{f_\sigma} + 1$, $R_n(x)$ is convergent at some $x_0 \in D_{f_\sigma} \setminus \{0\}$, $R_n(x)$ is convergent on D_{f_σ} (in case $D_{f_\sigma} \setminus \{0\} \neq \emptyset$), $x_0, x_0 - 1 \in D_{f_\sigma}$ for some $x_0 \neq 0$.

Therefore, the minimum condition for a function f in the topic of limit summability is $\lim_{n\to\infty} R_n(1) = R(1)$, and this is a necessary condition for weak summability and important properties of f_σ. For example if $f(x) = (2[x]-1)x + [x] - [x]^2$, then $R_n(1)$ is divergent and this function in addition to the evident points $0, -1$, is summable on $(0,1)$ ($D_{f_\sigma} = \{-1\} \cup [0,1)$). But f_σ does not have the useful properties such as $(*_1)$, $D_f \cap D_{f_\sigma} = D_{f_\sigma} + 1$, etc.

As it can be seen in [2], if f is summable then $D_{f_\sigma} = \Sigma_f = D_f - 1$ and

$$(*_2) \quad f_\sigma(x) = f(x) + f_\sigma(x-1); \quad \forall x \in D_f.$$

Now, if f is weak summable then we have

$$(*_3) \quad f_\sigma(x) = f(x) + f_\sigma(x-1) + R(1)x; \quad \forall x \in \Sigma_f + 1,$$

and so

$$(*_4) \quad f_\sigma(x) = f(x) + f_\sigma(x-1); \quad \forall x \in \Sigma_f + 1$$

, if f is semi summable.

For showing the weak summability of a function it is enough to show it is summable on a certain subset of its domain.

Definition 1.4. *Suppose $A \subseteq \mathbb{C}$. The set $S \subseteq A$ is called an integer-trenchant subset of A (or simply trenchant of A) if for every $a \in A$ there exists an integer k such that $a + k \in S$.*

Clearly A is an integer-trenchant of itself. It can be shown that (for any f) if k is a natural number, then the set $\Sigma_f + k$ is a trenchant of Σ_f. For a real function f, the sets $\Sigma_f \cap [M, +\infty)$, $\Sigma_f \cap (M, +\infty)$ are trenchant of Σ_f and $\Sigma_f \cap [M-1, M)$ is a tranchant of $\Sigma_f \cap (-\infty, M)$, for all real numbers M. Moreover, if $\Sigma_f \setminus D_f$ is bounded above, then the set $\Sigma_f \cap [\sigma_f, \sigma_f + 1)$ (center of Σ_f) is a trenchant of Σ_f (with length less than or equal 1).

Theorem 1.5. *Let $R_n(f,1)$ be convergent. If f is summable on a trenchant subset of Σ_f, then f is weak summable.*

Proof. Considering the relations

$$f_{\sigma_n}(x) = f_{\sigma_n}(x-m) + \sum_{j=1}^{m}(f(x-m+j) + R_n(x-m+j)),$$

$$R_n(x-m+j) = R_n(j) + R_{n+j}(x-m),$$

$$f_{\sigma_n}(x) = f_{\sigma_n}(x+m) - \sum_{j=0}^{m-1}(f(x+m-j) + R_n(x+m-j)),$$

(for all natural m and $j = 1, \cdots, m$), we can prove the claim (similar to the proof of Theorem 2.11. in [2]).
In fact the above theorem is a generalization of Theorem 2.11. in [2]. This theorem also says that if $\{1, x\} \subseteq D_{f_\sigma}$, then $x + \mathbb{N} \subseteq D_{f_\sigma}$ and $(x + \mathbb{Z}^-) \cap \Sigma_f \subseteq D_{f_\sigma}$. □

Lemma 1.6. *Suppose that $R_n(1)$ is convergent and $f^*(x) = f(x) + R(1)x$. Then the following are equivalent*
(a) f is weak summable,
(b) The function f^* is semi summable,
(c) The function $g = f^*|_{\Sigma_f+1}$ is summable,
(d) f satisfies the functional equation $(*_3)$,
Also, if f is a real function, then the above properties are equivalent to the following:
(e) f is limit summable on Σ_f^+ (the set of positive elements of Σ_f).

Proof. By considering Theorem 1.5., $(*_1)$ and the relations $f^*_{\sigma_n}(x) = f_{\sigma_n}(x)$, $R(f^*, 1) = R_n(f, 1) - R(f, 1)$, and the equality $\Sigma_{f^*} = \Sigma_f = \Sigma_g = D_g - 1$, it is clear. □

2. Absolutely and Uniformly Limit Summability of Functions

Definition 2.1. *We call the function f absolutely summable at $x \in \Sigma_f$, if $f_{\sigma_n}(x)$ is absolutely convergent. Also, we put*

$$\overline{D}_{f_\sigma} = \{x \in \Sigma_f | f \textit{ is absolutely summable at } x\}.$$

The function f is called uniformly (absolutely) summable on $S \subseteq \Sigma_f$ if $f_{\sigma_n}(x)$ is uniformly convergent on S (if f is absolutely summable at all points of S).
The function f is called absolutely weak summable (absolutely semi summable, absolutely summable) if it is weak summable (semi summable, summable) and $\overline{D}_{f_\sigma} = D_{f_\sigma}$.

Note: It is clear that f is absolutely weak (semi) summable if and only if $\overline{D}_{f_\sigma} = \Sigma_f$ ($\overline{D}_{f_\sigma} = \Sigma_f$ and $R(1) = 0$). For absolutely summability there are some interesting equivalence properties that will be introduced later.

Now put $\overline{R}_n(x) = R_n(x) - xR_{n-1}(1)$, for $n \geqslant 2$ where $x \in \Sigma_f$. If $0 \in D_f$, then $\overline{R}_1(x) = R_1(x) - xR_0(1) = R_1(x) - x(f(0) - f(1))$, so $\overline{R}_n(x)$ is well defined for all n if $0 \in D_f$. A simple calculation shows, that

$$(*_5) \quad f_{\sigma_n}(x) = (x+1)f(1) - f(x+1) + \sum_{k=2}^{n}(R_k(x) - xR_{k-1}(1))$$

$$= f_{\sigma_1}(x) + \sum_{k=2}^{n}\overline{R}_k(x); \quad \forall n > 1.$$

Also, if $0 \in D_f$, then

$$f_{\sigma_n}(x) = xf(0) + \sum_{k=1}^{n}\overline{R}_k(x); \quad \forall n \; .$$

Sometimes, we define $R_0(1) = -f(1)$, if $0 \notin D_f$. Therefore, if $0 \notin D_f$ or $f(0) = 0$, then $\overline{R}_1(x) = f_{\sigma_1}(x)$ so $f_{\sigma_n}(x) = \sum_{k=1}^{n}\overline{R}_k(x)$.

Remark 2.2. *Considering Definition* 2.1. *and* $(*_5)$ *we have:*
a) f is absolutely summable at x if and only if the series $\sum_{n=2}^{\infty}\overline{R}_n(x)$ is absolutely convergent. In particular, $0 \in \overline{D}_{f_\sigma}$ and $1 \in \overline{D}_{f_\sigma}$ if and only if the sequence $R_n(1)$ is absolutely convergent. Also, if f_n is absolutely convergent , then $x \in \overline{D}_{f_\sigma}$ if and only if the series $\sum_{n=1}^{\infty} R_n(x)$ is absolutely convergent.
b) We put $f_{\overline{\sigma}}(x) = |f_{\sigma_1}(x)| + \Sigma_{n=2}^{\infty}|\overline{R}_n(x)|$ for all $x \in \overline{D}_{f_\sigma}$.

Therefore, $f_{\overline{\sigma}}$ is a non-negative valued function, $D_{f_{\overline{\sigma}}} = \overline{D}_{f_\sigma}$ and

$$(*_6) \quad |f_\sigma(x)| \leqslant f_{\overline{\sigma}}(x) \ : \ \forall x \in \overline{D}_{f_\sigma}.$$

c) The complex sequence a_n is absolutely convergent if and only if it is absolutely summable, as a function with domain $\mathbb{N}^*$.

Example 2.3. If $0 < |a| < 1$, then the function $f(x) = a^x$ is absolutely summable, but it is not uniformly summable, because

$$\sup |f_{\sigma_n}(x) - f_\sigma(x)| = |\frac{a^n}{1-a}| \sup |a^{x+1} + (1-a)x - a| = \infty.$$

But it is uniformly summable on every bounded set.

As we know, if $1 \in \Sigma_f$ ($1 \in D_{f_\sigma}$), then $\mathbb{N}^* \subseteq \Sigma_f$ and $\Sigma_f \cap D_f = \Sigma_f + 1$ ($\mathbb{N}^* \subseteq D_{f_\sigma}$ and $D_{f_\sigma} \cap D_f = D_{f_\sigma} + 1$). It is interesting to see that this property for $\overline{D}_{f_\sigma}$ is held too.

Theorem 2.4. *Let $R_n(1)$ be absolutely convergent (equivalently $1 \in \overline{D}_{f_\sigma}$) then:*
a) $\overline{D}_{f_\sigma} \cap D_f = \overline{D}_{f_\sigma} + 1$, $\mathbb{N}^* \subseteq \overline{D}_{f_\sigma}$
b) If $x \in \overline{D}_{f_\sigma}$, then $(x + \mathbb{Z}) \cap \Sigma_f \subseteq \overline{D}_{f_\sigma}$ and $(x + \mathbb{N}) \subseteq \overline{D}_{f_\sigma}$ (and so $\mathbb{Z} \cap \Sigma_f \subseteq \overline{D}_{f_\sigma}$).
c) If f is absolutely summable on a trenchant subset of Σ_f (D_{f_σ}), then f is absolutely weak summable ($\overline{D}_{f_\sigma} = D_{f_\sigma}$).

Proof. (a): If $x - 1 \in \overline{D}_{f_\sigma}$, then $x \in D_f$ and $x - 1, x \in \Sigma_f$. A simple calculation shows that

$$(*_7) \quad \overline{R}_k(x) = \overline{R}_{k+1}(x-1) + \overline{R}_k(1)(x+1); \quad k \geqslant 2,$$

so

$$\sum_{k=2}^{n} |\overline{R}_k(x)| \leqslant \sum_{k=2}^{n} |\overline{R}_{k+1}(x-1)| + |x+1| \sum_{k=2}^{n} |\overline{R}_k(1)|; \quad n \geqslant 2.$$

By virtue of the relation $(*_5)$ and Remark 2.2. we get $x \in \overline{D}_{f_\sigma}$.
Now if $x \in \overline{D}_{f_\sigma} \cap D_f$, then $x - 1, x \in \Sigma_f$ and applying $(*_7)$ we conclude

that $x-1 \in \overline{D}_{f_\sigma}$ (similar to the above case).
(b), (c): The part (a) (with $\Sigma_f \cap D_f = \Sigma_f + 1$) implies that

$$(*_8) \quad \overline{D}_{f_\sigma} \cap (\Sigma_f + m) \subseteq \overline{D}_{f_\sigma} + m \subseteq \overline{D}_{f_\sigma},$$

for all positive integers m. This relation proves (b) and (c). □

Lemma 2.5. *The following are equivalent:*
a) f is absolutely summable,
b) $D_f \subseteq \overline{D}_{f_\sigma}$, $R(1) = 0$,
c) $\overline{D}_{f_\sigma} = \Sigma_f$, $D_f \subseteq D_f - 1$, $R(1) = 0$.

Proof. By Lemma 2.1. in [2], the items $(a) \Rightarrow (b)$ and $(c) \Rightarrow (a)$ are clear.
$(b) \Rightarrow (c)$: Since $1 \in D_f \subseteq \overline{D}_{f_\sigma}$, Theorem 2.4 implies that

$$D_f = D_f \cap \overline{D}_{f_\sigma} = \overline{D}_{f_\sigma} + 1,$$

so $\overline{D}_{f_\sigma} = D_f - 1 = \Sigma_f$.
Therefore, if f is absolutely summable, then $\overline{D}_{f_\sigma} = D_{f_\sigma} = \Sigma_f = D_f - 1$. □

Theorem 2.6. *Let f be a real function that f_n is weak convergent ($R_n(1)$ is convergent), then*
(a) If f is uniformly summable on $\Sigma_f \cap [M, M+1)$ (for a real M), then f is uniformly summable on every bounded subset of $\Sigma_f \cap (-\infty, M+1)$.
(b) If f is uniformly summable on every bounded subset of $\Sigma_f \cap (N, +\infty)$, for some real N, then f is uniformly summable on every bounded subset of Σ_f.
(c) If Σ_f is concentrable and f is uniformly summable on center of Σ_f, then f is uniformly summable on every bounded subset of Σ_f.

Proof. (a) For all $x \in \Sigma_f \cap [M-1, M)$, we have $x+1 \in (\Sigma_f + 1) \cap [M, M+1) \subseteq \Sigma_f \cap [M, M+1)$ and

$$f_{\sigma_n}(x) = f_{\sigma_n}(x+1) - R_n(x+1) - f(x) \quad : \forall x \in \Sigma_f \cap [M-1, M).$$

Therefore, for these x-s, $f_{\sigma_n}(x+1)$ is uniformly convergent. On the other hand, the relation $f_{\sigma_n}(x+1) - f_{\sigma_{n-1}}(x+1) = \overline{R}_n(x+1)$ implies

that $\overline{R}_n(x+1)$ and $R_n(x+1) = \overline{R}_n(x+1) + (x+1)R_{n-1}(1)$ is uniformly convergent on $\Sigma_f \cap [M-1, M)$ (because $R_{n-1}(1)$ is convergent and these x-s are bounded).
Therefore, f is uniformly summable on $\Sigma_f \cap [M-1, M)$. Similarly, f is uniformly summable on $S_i = \Sigma_f \cap [M-i, M+1-i)$ for all positive integers i and on every finite union of S_i-s. Therefore, (a) is proved.

(b): the part (a) implies (b) clearly.

(c): Let Σ_f be concentrable. Then for all positive integers m we have

$$\{\Sigma_f \cap [\sigma_f+m-1, \sigma_f+m)\}+1 = \Sigma_f \cap [\sigma_f+m, \sigma_f+m+1) = \{\Sigma_f \cap [\sigma_f, \sigma_f+1)\}+m,$$

since $x - [x-\sigma_f] \in [\sigma_f, \sigma_f+1)$, for all $x \in \Sigma_f$. Put $S_m = \Sigma_f \cap [\sigma_f + m, \sigma_f + m + 1)$. Therefore, if $x \in S_1$, then $x - 1 \in \Sigma_f \cap [\sigma_f, \sigma_f + 1)$ (the center of Σ_f) and

$$f_{\sigma_n}(x) = f_{\sigma_n}(x-1) + R_n(x) + f(x); \quad \forall x \in S_1,$$

Similar to the part (a), f is uniformly summable on S_1 and so on S_m for all positive integers m (considering the above relation for S_m). Now with due attention to (a) the proof is complete. $\square$

Note: In general, $\{\Sigma_f \cap [M+1, M+2)\} - 1 \subsetneqq \Sigma_f \cap [M, M+1)$, for this reason the part (a) in the above theorem can not be stated for $\Sigma_f \cap [M+1, M+2)$ and for the bounded subsets of $\Sigma_f \cap [M+1, +\infty)$. But in part (c) (when Σ_f is concentrable and $M = \sigma_f$) this problem is removed. Also, note that if in this theorem Σ_f be replaced by D_{f_σ} (in the hypothesis and (a), (b), (c)), then the theorem is valid (for if $R_n(1)$ is convergent, then $D_f \cap D_{f_\sigma} = D_{f_\sigma} + 1$, i.e., there exist similarities between the properties of D_{f_σ} and those of Σ_f).

In the following we introduce a test for (absolutely) summability of the composition of functions.

Theorem 2.7. *Suppose f is a function for which $\sum_{n=1}^{\infty} R_n(x)$ is absolutely convergent on Σ_f and let g be a function that $f(\mathbb{N}^*) \subseteq D_g$, and*

$$|g(s) - g(t)| \leqslant M|s-t| \quad \forall s, t \in D_g.$$

Then gof is absolutely semi summable, moreover

$$|(gof)_\sigma(x)| \leqslant (gof)_{\overline{\sigma}}(x) \leqslant c|x| + M\sum_{n=1}^{\infty}|R_n(x)|,$$

where $c = M\sum_{n=1}^{\infty}|R_n(1)| + |g(f(1))|$.

Proof. First note that ($\mathbb{N}^* \subseteq D_{gof}$ and) $\Sigma_{gof} \subseteq \Sigma_f$. Now if $x \in \Sigma_{gof}$, then one can write

$$|R_n(gof, x)| \leqslant M|R_n(f, x)| + M|x||R_{n-1}(f, 1)| \ : \quad \forall n > 1.$$

Since the series $\sum|R_n(x)|$ and $\sum|R_n(1)|$ are convergent so gof is absolutely summable at x. Also, we have

$$|R_n(gof, 1)| \leqslant M|R_n(f, 1)|,$$

so $R(f, 1) = 0$ implies that $R(gof, 1) = 0$. Therefore, f is absolutely semi summable. Now considering the above inequalities and

$$|(gof)_{\sigma_1}(x)| \leqslant |g(f(1))||x| + M|R_1(x)|$$

with relations $(*_5)$, $(*_6)$, the last part is proved. □

Example 2.8. If $|a| < 1$, then the functions $\sin(a^x)$ and $\cos(a^x)$ are absolutely summable and $\sin(\frac{1}{x}), \cos(\frac{1}{x})$ are absolutely semi summable.

3. Monotonic, Concave and Convex Limit Summable Functions

Let E be a subset of $\mathbb{R}$ (not necessarily an interval) and suppose $E \subseteq D_f$ (f is a real function defined on E). A function f is called convex on E if for every three elements x_1, x_2, x_3 of E with $x_1 < x_2 < x_3$ the following inequalities hold

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leqslant \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leqslant \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

If the above inequalities are reversed, then f is called concave. Therefore, a function f is concave if and only if the function $-f$ is convex. If f is convex on E, then it is so on each subset of E. For example if f' is increasing on (a, b), then f is convex on each subset of (a, b).
The function f is called monotonic (convex or concave) on E from a number on if there exists a real M such that f is non decreasing or non increasing (convex or concave) on $E \cap (M, +\infty)$

Theorem 3.1. *Let f be a real function that the sequence f_n is bounded.*
(a) For a $x_0 \in \Sigma_f$, if f is monotonic on $\mathbb{N}^* \cup (\mathbb{N}^* + x_0)$ from a number on, then f is absolutely summable at x_0.
(b) If f is monotonic on $\Sigma_f + 1$ from a number on, then
i: f is absolutely semi summable.
ii: f is uniformly summable on every bounded subset of Σ_f and the series $\sum_{n=1}^{\infty} R_n(x)$ is uniformly convergent on it.
(c) If f is non-increasing on $\Sigma_f + 1$, then $sgn(x) f_\sigma(x) \geqslant f(\infty)|x|$ (i.e. $f_\sigma(x) \geqslant f(\infty)x$ if $x \geqslant 0$ and $f_\sigma(x) \leqslant f(\infty)x$ if $x \leqslant 0$), where, $f(\infty) = \lim_{n\to\infty} f_n$.
Moreover, if $f(\infty) \geqslant 0$ then f_σ is non-decreasing (on its domain $D_{f_\sigma} = \Sigma_f$) and

$$\frac{f_\sigma(y) - f_\sigma(x)}{y - x} \geqslant f(\infty),$$

for all $x, y \in \Sigma_f$ that $x < y$.
(If f is non-decreasing, then the function $-f$ satisfies the condition (c) and by considering $(-f)_\sigma = -f_\sigma$ one can write similar properties for f, in this case).

Proof. (a): Put $E = \mathbb{N}^* \cup (\mathbb{N}^* + x_0)$ (clearly $E \subseteq \Sigma_f + 1 \subseteq D_f$). Let f be non-increasing on $E \cap (M, +\infty)$, for a positive number M (without loss of generality). Since f_n is non-increasing (from a number on) and convergent, then f_n and so $R_n(1)$ are absolutely convergent. Therefore, $\mathbb{N}^* \subseteq \overline{D}_{f_\sigma}$. Now let m be a positive integer such that $x_0 + m \geqslant 0$. Clearly there exists a N such that

$$(*_9) \quad 0 \leqslant R_k(x_0 + m) \leqslant R_k([x_0] + m + 1) \ : \ \forall k \geqslant N.$$

Since the series $\sum_{k=1}^{\infty} R_k([x_0]+m+1)$ is absolutely convergent, by Remark 2.2. $\sum_{k=1}^{\infty} R_k([x_0]+m)$ is absolutely convergent, so $x_0+m \in \overline{D}_{f_\sigma}$ and $x_0 \in \overline{D}_{f_\sigma}$, by Theorem 2.4.
(b): The part (i) of (b) is a direct result of (a). Suppose f be non-increasing on $\Sigma_f + 1$ from a number on. Similar to ($*_9$), for all positive integers m and each $x \in \Sigma_f \cap [0, m]$, there exists a positive integer N such that if $k \geqslant N$, then $0 \leqslant R_k(x) \leqslant R_k(m)$. So the series $\sum_{k=1}^{\infty} R_k(x)$ is uniformly convergent on $\Sigma_f \cap [0, m]$. Therefore, f is uniformly summable on every bounded subset of Σ_f^+ so f is uniformly summable on every bounded subset of Σ_f, by Theorem 2.6.
(If f is non decreasing, then we can proof the parts (a), (b) similarly.)
(c): If $x, y \in \Sigma_f$ and $x < y$, then $R_k(x) \leqslant R_k(y)$ for all positive integers k (because f is non increasing on all $\Sigma_f + 1$) so $\sum_{k=1}^{n} R_k(x) \leqslant \sum_{k=1}^{n} R_k(y)$ and so

$$\frac{f_{\sigma_n}(y) - f_{\sigma_n}(x)}{y - x} \geqslant f(n),$$

for all n. Applying the above inequality for $x = 0$ and $y = 0$ and putting $f(\infty) = \lim_{n\to\infty} f_n$ with due attention to (a) we get the results. □

Example 3.2. The function $f(x) = \sqrt{x} - \sqrt{x+1}$ is absolutely summable and $\overline{D}_{f_\sigma} = D_{f_\sigma} = D_f - 1 = [-1, +\infty)$.

In ([2]), we prove a main (uniqueness) Theorem. Since it is very important and we use it repeatedly in the sequel, it is introduced here:

Theorem A. *Let f be a real function for which $R_n(f, 1)$ is convergent. Suppose there exists a function λ such that*

$$\lambda(x) = f(x) + \lambda(x-1) \ : \quad \text{for all } x \in \Sigma_f + 1.$$

(a) If $R(1) \geqslant 0$ and λ is convex on $\Sigma_f + 1$ from a number on, then f is weak summable.

(b) If $R(1) \leqslant 0$ and λ is concave on $\Sigma_f + 1$ from a number on, then f is weak summable.

In each of the above cases we have

$$f_\sigma(x) = \lambda(x) + R(1)\frac{x^2+x}{2} - \lambda(0) \ : \quad \textit{for all } x \in \Sigma_f.$$

Proof. See Theorem 3.1. in [2]. □

Theorem 3.3. *Suppose f is a real function for which $R_n(1)$ is bounded.*
(a): If f is concave or convex on $\mathbb{N}^* \cup (\mathbb{N}^* + x_0)$ from a number on, then f is absolutely summable at x_0.
(b) If f is convex or concave on $\Sigma_f + 1$ from a number on, then
i: f is absolutely weak summable.
ii: f is uniformly summable on every bounded subset of Σ_f .
(c) If f is concave on $\Sigma_f^+ + 1$, then

$$f_\sigma(x) \geqslant (x+1)f(1) - f(x+1) \quad : \quad \forall x \in \Sigma_f^+ \cup \{0\}.$$

Moreover, if $f_{\sigma_1}(y) \geqslant f_{\sigma_1}(x)$ for all $x, y \in \Sigma_f^+$ with $x < y$, or $0 \in D_f$ and $f(0) \geqslant 0$, then f_σ is non-decreasing and non negative on $\Sigma_f^+ \cup \{0\}$.
d) If the concavity of f holds on $\Sigma_f + 1$, then the summand function of f (f_σ) is convex (on its domain Σ_f) and f_σ is the only function (with domain Σ_f) that is convex on $\Sigma_f + 1$ (from a number on), $f_\sigma(0) = 0$ and satisfies the functional equation ($*_3$):

$$f_\sigma(x) = f(x) + f_\sigma(x-1) + R(1)x; \quad \forall x \in \Sigma_f + 1.$$

Proof. Put $E = \mathbb{N}^* \cup (\mathbb{N}^* + x_0)$. There exists a positive integer N such that f is concave on $E \cap [N, +\infty)$. First let $x_0 \geqslant N$. Applying the concavity of f for $k < k+1 < k+2$ and $k-1 < k < k+x_0 < k+1+[x_0]$, where $k > N$ is an integer, we infer that

$$(*_{10}) \quad R_k(1) \leqslant \frac{1}{2}R_k(2) \leqslant R_{k+1}(1); \quad \forall k > N,$$

$$(*_{11}) \quad 0 \leqslant \overline{R}_k(x_0) \leqslant \overline{R}_k([x_0]+1); \quad \forall k > N,$$

Considering ($*_{10}$) the sequence $R_k(1)$ is absolutely convergent so $[x_0] + 1 \in \overline{D}_{f_\sigma}$ thus ($*_5$) implies that the series $\sum_{k=2}^{+\infty} \overline{R}_k([x_0]+1)$ is (absolutely)

convergent and so $x_0 \in \overline{D}_{f_\sigma}$. . If $x_0 < N$, then there exists a positive integer m such that $x_0 + m \in E \cap [N, +\infty)$ and we get the result by Theorem 2.4.
(b): The part (i) of (b) is a direct result of (a). Similar to $(*_{11})$, for all positive integers m and each $x \in \Sigma_f \cap [0, m]$, there exists a positive integer N such that if $k > N$, then $0 \leqslant \overline{R}_k(x) \leqslant \overline{R}_k(m)$. So the series $\sum_{k=2}^{\infty} \overline{R}_k(x)$ is uniformly convergent on $\Sigma_f \cap [0, N]$. Therefore, f is uniformly summable on every bounded subset of Σ_f^+ so f is uniformly summable on every bounded subset of Σ_f, by Theorem 2.6.
(c): If $x, y \in \Sigma_f^+$ and $x < y$, then similar to $(*_{11})$ we have $0 \leqslant \overline{R}_k(x) \leqslant \overline{R}_k(y)$ for all positive integers $k \geqslant 2$ so, $0 \leqslant \sum_{k=2}^{n} \overline{R}_k(x) \leqslant \sum_{k=2}^{n} \overline{R}_k(y)$ and so $(*_5)$ implies that

$$(*_{12}) \quad 0 \leqslant f_{\sigma_n}(x) - f_{\sigma_1}(x) \leqslant f_{\sigma_n}(y) - f_{\sigma_1}(y) \ : \ \forall n.$$

If $0 \in D_f$ then the first inequality (in this part) holds for all positive integers k and so

$$(*_{13}) \quad 0 \leqslant f_{\sigma_n}(x) - xf(0) \leqslant f_{\sigma_n}(y) - yf(0).$$

Now we get the results by $(*_{12})$ and $(*_{13})$.
(d): Let $x_1, x_2 \in \Sigma_f, \mu_1, \mu_2 \geqslant 0$, $\mu_1 + \mu_2 = 1$ and $\mu_1 x_1 + \mu_2 x_2 \in \Sigma_f + 1$. Concavity of f on $\Sigma_f + 1$ implies that

$$f(k + \mu_1 x_1 + \mu_2 x_2) = f(\mu_1(k + x_1) + \mu_2(k + x_2)) \geqslant$$
$$\mu_1 f(k + x_1) + \mu_2 f(k + x_2); \quad \forall k \in \mathbb{N}^*,$$

therefore,

$$\sum_{k=1}^{n} R_k(\mu_1 x_1 + \mu_2 x_2) \leqslant \sum_{k=1}^{n} (\mu_1 R_k(x_1) + \mu_2 R_k(x_2)); \quad \forall n \in \mathbb{N},$$

So $f_{\sigma_n}(x)$ is convex on Σ_f for all n, hence $f_\sigma(x)$ is convex on $\Sigma_f = D_{f_\sigma}$. Now if λ is a function that satisfies these conditions, then putting $f^*(x) = f(x) + R(1)x$, f^* (instead of f), λ satisfies the conditions of Theorem A. On the other hand $R(f^*, 1) = 0$ and $f^*_{\sigma_n}(x) = f_{\sigma_n}(x)$ for all positive integers n and $x \in \Sigma_f = \Sigma_{f^*}$ so,

$$f_\sigma(x) = f^*_\sigma(x) = \lambda(x); \quad \forall x \in \Sigma_f. \quad \square$$

Corollary 3.4. *Let f be a real function that is concave (convex) on $\Sigma_f + 1$ and $R(1) = 0$. Then the general form of all convex (concave) solutions of the functional equation*

$$\lambda(x) = f(x) + \lambda(x-1) \quad \textit{for all}\, x \in \Sigma_f + 1,$$

is $\lambda = f_\sigma + c$, for all $c \in \mathbb{R}$.
Moreover, if $D_f \subseteq D_f - 1$ then the functional equation

$$\lambda(x) = f(x) + \lambda(x-1) \quad \textit{for all}\, x \in D_f,$$

has a unique convex (concave) solution with $\lambda(0) = 0$.

Remark 3.5. *Comparing the above corollary and Corollary* 3.4. *in* ([2]) *shows that if $D_f \subseteq D_f - 1$, $R(1) = 0$ and f is concave (convex) on D_f, then the conditions of Corollary* 3.4. (*in*[2]) *are held.*

Corollary 3.6. *Consider the real rational function $f(x) = \frac{p_m(x)}{q_k(x)}$ where $p_m(x) = a_m x^m + \cdots + a_0$, $q_k(x) = b_k x^k + \cdots + b_0$ and $q_k(x)$ has no any positive inreger roots. Clearly, $D_f = \mathbb{R} \setminus \{x_1, \cdots, x_l\}$, $\Sigma_f = \mathbb{R} \setminus (\{x_1, \cdots, x_l\} + \mathbb{Z}^-$, $\Sigma_f + 1 = \mathbb{R} \setminus (\{x_1, \cdots, x_l\} + \mathbb{Z}_0^-$, where $x_1, \cdots, x_l$ are the real roots of $q_k(x)$, and if $q_k(x)$ has no any real roots, then $D_f = \Sigma_f = \Sigma_f + 1 = \mathbb{R}$.*

Since every real rational function is monotonic and convex or concave from a number on, then considering Theorem 3.1, 3.3. we have:
Case 1) $m \leqslant k$: If $q_k(x)$ has no any real roots, then f is absolutely summable and if $q_k(x)$ has some real roots then f is absolutely semi summable.
Case 2) $m > k$: (In this case considering $R_n(f, 1)$) if

$$deg(p_m(x)q_k(x+1) - p_m(x+1)q_k(x)) < deg(q_k(x)q_k(x+1)),$$

then f is absolutely semi summable (if $q_k(x)$ has no any real roots, then f is absolutely summable) and if

$$deg(p_m(x)q_k(x+1) - p_m(x+1)q_k(x)) = deg(q_k(x)q_k(x+1)),$$

then f is absolutely weak summable. Also, if

$$deg(p_m(x)q_k(x+1) - p_m(x+1)q_k(x)) > deg(q_k(x)q_k(x+1)),$$

then $1 \notin D_{f_\sigma}$ and so f is not weak summable.
Moreover, in all of the above cases if f is concave on $\Sigma_f^+ + 1$, then we have:

$$f_\sigma(x) \geqslant \frac{(x+1)(a_0 + \cdots + a_m)q_k(x+1) - (b_0 + \cdots + b_k)p_m(x+1)}{(b_0 + \cdots + b_k)q_k(x+1)},$$

for all $x \in \Sigma_f^+$ (if f is convex, then the above inequality is reversed).

Example 3.7. For any real number r put $p(x) = x^r$ where $D_p = [0, +\infty)$ if $r > 0$, and $D_p = (0, +\infty)$ if $r < 0$. If $r > 1$, then $R_n(p, 1)$ is not convergent, so p is not weak summable (if $r \geqslant 2$, then $\overline{R}_n(x)$ is not convergent for all $x \neq 0$ so p is not summable at any $x \neq 0$) . Now if $r < 1$, then $R_n(p, 1) \to 0$ as $n \to \infty$ and $p(x)$ is concave, if $0 < r < 1$ or convex, if $r < 0$. So if $r < 1$, then x^r with the above cited domain is summable (by Theorem 2.3) and $Dp_\sigma = [-1, +\infty)$ or $(-1, +\infty)$. Also,

$$p_\sigma(x) = \lim_{n\to\infty} [xn^r + \sum_{k=1}^{n} (k^r - (k+x)^r)]$$

$$= \sum_{n=1}^{+\infty} [(1+x)n^r - (n+x)^r - x(n-1)^r],$$

for all $x \in Dp_\sigma$ and

$$p_\sigma(x) \geqslant 1 + x - (1+x)^r; \quad \forall x \geqslant 0.$$

If $r < 0$, then

$$p_\sigma(x) = \sum_{n=1}^{+\infty} [n^r - (n+x)^r].$$

In case $r = 1/2$, we have

$$p_\sigma(x) = x \sum_{n=1}^{+\infty} [\frac{1}{\sqrt{n} + \sqrt{n-1}} - \frac{1}{\sqrt{n} + \sqrt{n+x}}].$$

Finally, p_σ is the only concave (if $r < 0$) or convex (if $0 < r < 1$) function on its domain that $p_\sigma(1) = p(1) = 1$ and

$$p_\sigma(x) = x^r + p_\sigma(x-1); \quad \forall x \in D_p.$$

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