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Existence of Periodic Solution for a Class of Linear Third Order ODE

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Abstract. In this paper, we will consider third order linear differential equation

$$y''' + \alpha y'' + \beta y' + \gamma y + f(t, y) = e(t),$$

where α, β, γ are constant coefficients, $f(t, y)$ is continuous, $e(t)$ is discontinuous, and f and e are periodic functions with respect to t of period w. We will introduce sufficient conditions under which the above equation have at least one non-trivial periodic solution of period w. We will see that under the so called conditions, all the solutions of the equation will be bounded. It must be mentioned that e in this equation is called "controller" in the engineering problems and it was always considered to be continuous to ensure us that periodic solution exists. In this paper, we will show the existence of periodic solution without supposing that e to be continuous.

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1. Introduction

The third order differential equation:

$$y''' + \alpha y'' + \beta y' + \gamma y + f(t, y) = e(t), \tag{1}$$

where α, β, γ are constants f is continuous, e is discontinuous, and f and e are w-periodic functions with respect to t will be discussed.

First of all, we suppose that all the roots of characteristic equation

$$p^3 + \alpha p^2 + \beta p + \gamma = 0 \tag{2}$$

have negative real parts. This will guarantee that all the solutions of the following homogeneous equation are bounded:

$$y''' + \alpha y'' + \beta y' + \gamma y = 0, \tag{3}$$

If so, for each solution ϕ of equation (3), there exists a positive number M that $|\phi(t)| \leqslant Me^{-\alpha t}$ which means that solution of (3) will tend to zero when t tends to infinity ([3]).

For this purpose, the third order polynomial

$$L(p) = p^3 + \alpha p^2 + \beta p + \gamma \tag{4}$$

must satisfy the Hurwitz conditions ([3]). Hurwitz theorem for a general third order polynomial $M(p) = a_0 p^3 + a_1 p^2 + a_2 p + a_3$ says that: All the roots of $M(p) = 0$ have negative real parts if and only if a_0, a_1, a_2, a_3 are positive and in addition $a_1 a_2 > a_0 a_3$. (see [3])

By now, we will assume that the characteristic equation $L(p) = 0$ satisfies the Hurwitz condition, i. e. α, β, γ are positive and $\alpha\beta > \gamma$.

Now for the characteristic equation $L(p) = 0$ two situations may occur:
1) Having 3 real roots (may be indistinct),
2) Having 2 conjugate complex roots and one real root.

For ease, we will consider the second situation and we will find the integral equation for it. All of our next section is based on this integral equation.

2. Integral Equation

Let $r_1 + i\beta, r_1 - i\beta$ and r_2 be the roots of characteristic equation. So a fundamental set of solutions for (3) is

$$\{e^{r_1 x} \cos \beta x, e^{r_1 x} \sin \beta x, e^{r_2 x}\}.$$

Calculating the general solution for the equation (1) by the "variation of parameters method" results in

$$\begin{aligned} y_g(x) &= \alpha_1 e^{r_1 x}\cos\beta x + \alpha_2 e^{r_1 x}\sin\beta x + \alpha_3 e^{r_2 x} + \\ &k_1 \textstyle\int_0^x e^{r_1(x-t)}[k_2 \sin\beta(t-x) - cos\beta(t-x) + \beta][e(t) - f(t,y)]dt, \end{aligned} \tag{5}$$

where $k_1 = \frac{1}{\beta(r_2^2 - r_1^2 + \beta^2)}$ and $k_2 = (r_2 - r_1)$.
Let

$$\phi_0(x) = \alpha_1 e^{r_1 x}\cos\beta x + \alpha_2 e^{r_1 x}\sin\beta x + \alpha_3 e^{r_2 x}$$

and $e(t)dt = dg(t)$ and also

$$K(x,t) = k_1 e^{r_1(x-t)}[k_2 \sin\beta(t-x) - cos\beta(t-x) + \beta].$$

So each solution for equation (1) is in the following form

$$y(x) = \phi_0(x) + \int_0^x K(x,t)dg(t) - \int_0^x K(x,t)f(t,y)dt,$$

where $\int_0^x K(x,t)dg(t)$ is Riemann-Stieltjes integral.
Now, we will try to prove that all the solutions of equation (1), under Hurwitz conditions and under the conditions which we will introduce later, are bounded.

Theorem 2.1. *If f is continuous on $\mathbb{R}$ such that $v_g(t, t+1) < h$ for some constant positive number h and all t, then $\int_0^\alpha f dg$ is bounded for $\alpha \in \mathbb{R}$.*

Proof. For some $N \in \mathbb{N}$ with $N - 1 \leqslant \alpha < N$.
We claim that g is bounded variation on $[0, \alpha]$.

$$\begin{aligned} V_g(0,\alpha) &= V_g(0,1) + V_g(0,2) + \ldots + V_g(N-1,\alpha) \\ &\leqslant V_g(0,1) + V_g(0,2) + \ldots + V_g(N-1,\alpha) + V_g(\alpha, N) \\ &= V_g(0,1) + \ldots + V_g(N-1,N) \\ &< \underbrace{h + \ldots + h}_{Ntimes} = Nh \end{aligned}$$

So we can write g as the difference of two ascending functions g_1 and g_2: $g = g_1 - g_2$ and we have

$$\int_0^\alpha f dg = \int_0^\alpha f d(g_1 - g_2) = \int_0^\alpha f dg_1 - \int_0^\alpha f dg_2 \quad .$$

Then

$$\begin{aligned} |\textstyle\int_0^\alpha f dg| &\leqslant |\textstyle\int_0^\alpha f dg_1| + |\textstyle\int_0^\alpha f dg_2| \\ &\leqslant \textstyle\int_0^\alpha |f| dg_1 + \textstyle\int_0^\alpha |f| dg_2 \\ &\leqslant \max|f|(\textstyle\int_0^\alpha dg_1 + \textstyle\int_0^\alpha dg_2) \\ &= \max|f|(g_1(\alpha) - g_1(0) + g_2(\alpha) - g_2(0)) \\ &\leqslant \max|f|(Vg_1(0,\alpha) + Vg_2(0,\alpha)) < \infty \quad . \end{aligned}$$

It completes the proof. □

Theorem 2.2. *If there exists h, such that* $V_g(t, t+1) < h$ *for all t, (where* $dg(t) = e(t)dt$*) and* $f(t, y) \equiv 0$*, then every solution of equation* (1) *is bounded.*

Proof. When $f(t, y) \equiv 0$, every solution of equation (1) is in the form

$$y(x) = \phi_0(x) + \int_0^x K(x,t) dg(t),$$

where $\phi_0(x)$ is a solution of homogeneous equation (3) (that has already assumed to be bounded) and

$$K(x,t) = k_1[(r_2 - r_1)\sin\beta(t-x) - \cos\beta(t-x) + \beta]e^{r_1(x-t)}.$$

We can see that

$$|K(x,t)| \leqslant k_1[|r_2 - r_1| + 1 + \beta]e^{r_1(x-t)} \leqslant Be^{r_1(x-t)},$$

where $B = (|r_2 - r_1| + 1 + \beta)k_1$.
In addition there exists A; such that for all x, $|\phi_0(x)| \leqslant A$. So we have

$$|y(x)| \leqslant A + B|\int_0^x e^{r_1(x-t)} dg(t)|.$$

For each $x > 0$, there is a $k \in \mathbb{N}$ such that $k \leqslant x < k+1$.
So we have:

$$\begin{aligned} |y(x)| &\leqslant A + B|\int_0^x e^{r_1(x-t)} d(g_1 - g_2)| \\ &\leqslant A + B(|\int_0^x e^{r_1(x-t)} dg_1| + |\int_0^x e^{r_1(x-t)} dg_2| \\ &\leqslant A + B(\int_0^x |e^{r_1(x-t)}| dg_1 + \int_0^x |e^{r_1(x-t)}| dg_2 \\ &\leqslant A + B(\int_0^x e^{r_1(x-t)} dg_1 + \int_x^{k+1} e^{r_1(x-t)} dg_1 + \int_0^x e^{r_1(x-t)} dg_2 \\ &+ \int_x^{k+1} e^{r_1(x-t)} dg_2) \\ &= A + B(\int_0^{k+1} e^{r_1(x-t)} dg_1 + \int_0^{k+1} e^{r_1(x-t)} dg_2). \end{aligned}$$

We can choose g_1 and g_2 such that

$$g_1(x) = \frac{1}{2} V_g(0, x) + \frac{1}{2} g(x),$$

$$g_2(x) = \frac{1}{2} V_g(0, x) - \frac{1}{2} g(x)$$

Then for all $t, i = 1, 2, V_{g_i}(t, t+1) \leqslant V_g(t, t+1) < h$. Now for $i = 1, 2$, we have

$$\begin{aligned} \int_0^{k+1} e^{-r_1 t} dg_i &= \sum_{m=1}^{k+1} \int_{m-1}^m e^{-r_1 t} dg_i \\ &< \sum_{m=1}^{k+1} e^{-r_1 m} \int_{m-1}^m dg_i \\ &< h \sum_{m=1}^{k+1} e^{-r_1 m} = h e^{-r_1} \tfrac{1 - e^{-r_1(k+2)}}{1 - e^{-r_1}}, \end{aligned}$$

So we conclude that

$$\begin{aligned} B e^{r_1 k} \int_0^{k+1} e^{-r_1 t} dg_i &< B e^{r_1 k} h e^{-r_1} \tfrac{1 - e^{-r_1(k+2)}}{1 - e^{-r_1}} \\ &= Bh \tfrac{e^{r_1(k-1)} - e^{r_1(k-1-k-2)}}{1 - e^{-r_1}} \\ &= Bh \tfrac{e^{-r_1(1-k)} - e^{-3r_1}}{1 - e^{-r_1}} \\ &= Bh \tfrac{e^{-3r_1} - e^{-r_1(1-k)}}{e^{-r_1} - 1} \\ &\leqslant Bh \tfrac{e^{-3r_1}}{e^{-r_1} - 1} \end{aligned}$$

and therefore $|y(x)| \leqslant A + B(2h)\frac{e^{-3r_1}}{e^{(-r_1)}-1}$.
The latter bound is independent from k, and then is independent from x. So the proof is complete. □

Theorem 2.3. *Let $L < \frac{-r_1}{B}$ be such that $|f(t,y)| \leqslant L|y|$ for all t, $|y| < H$ and B, r_1 are as in the Theorem 2.2. If there exists h such that $V_g(t,t+1) < h$ for all t whenever $dg = edt$ then every solution of equation (1) is bounded.*

Proof. We have seen that every solution of equation (1) is of the form (5). Also we have seen that there exists A such that $|\phi_0(x)| < A$ for all x. Now we suppose that

$$y(x) = -k_1 \int_0^x e^{r_1(x-t)} K(x,t) f dt + k_1 \int_0^x e^{r_1(x-t)} K(x,t) dg.$$

We will prove that y is bounded.
From the assumption we have

$$|y(x)| \leqslant LB \int_0^x e^{r_1(x-t)} |y| dt + B \int_0^x e^{r_1(x-t)} |dg|,$$

where $|dg| = dg_1 + dg_2$. Let $h(x) = B\int_0^x e^{-r_1 t} dg$ and $\phi(x) = |y(x)|e^{-r_1 x}$ then

$$\phi(x) \leqslant LB \int_0^x \phi(t) dt + h(x). \tag{6}$$

Now let $R(x) = \int_0^x \phi(t)dt$. Then

$$\begin{aligned}
&R'(x) - LBR(x) \leqslant h(x)\\
&e^{-LBx}R'(x) - e^{-LBx}LBR(x) \leqslant e^{-LBx}h(x)\\
&\tfrac{d}{dx}(R(x)e^{-LBx}) \leqslant e^{-LBx}h(x)\\
&R(x)e^{-LBx} \leqslant \textstyle\int_0^x e^{-LBt}h(t)dt\\
&R(x) \leqslant e^{LBx} \textstyle\int_0^x e^{-LBt}h(t)dt\ .
\end{aligned}$$

By (6) and above relations we have

$$\phi(x) \leqslant h(x) + LB \int_0^x e^{LB(x-t)} h(t) dt.$$

By using the integration by parts we have;

$$\phi(x) \leqslant [-e^{LB(x-t)}h(t)]_0^x + \int_0^x e^{LB(x-t)}dh + h(x)$$
$$\phi(x) \leqslant -h(x) + h(0) + \int_0^x e^{LB(x-t)}dh + h(x)$$
$$\phi(x) \leqslant \int_0^x e^{LB(x-t)}dh.$$

Then

$$|y(x)|e^{-r_1 x} \leqslant B\int_0^x e^{LBx}e^{-LBt}e^{-r_1 t}|dg|$$
$$|y(x)| \leqslant Be^{r_1 x}e^{LBt}\int_0^x e^{(-r_1-LB)t}|dg|$$
$$|y(x)| \leqslant Be^{-\lambda x}\int_0^x e^{\lambda t}|dg|,$$

where $-\lambda = r_1 + LB$, and by assumption $-\lambda < 0$. For each $x > 0$ there exists a $k \in \mathbb{N}$ such that $k \leqslant x < k+1$. So we can write

$$\begin{aligned}
|y(x)| &\leqslant Be^{-\lambda k}\int_0^{k+1} e^{\lambda t}|dg| \\
&\leqslant Be^{-\lambda k}\sum_{m=1}^{k+1}\int_{m-1}^{m} e^{\lambda t}|dg| \\
&\leqslant Be^{-\lambda k}\sum_{m=1}^{k+1} e^{m\lambda}\int_{m-1}^{m}|dg| \\
&\leqslant Be^{-\lambda k}(2h)\sum_{m=1}^{k+1} e^{\lambda m} \\
&= 2Bhe^{-\lambda k}e^{\lambda}\left(\tfrac{1-e^{\lambda(k+2)}}{1-e^{\lambda}}\right) \\
&= 2Bh\left(\tfrac{e^{-\lambda(k-1)}-e^{-\lambda(k-1-k-2)}}{1-e^{\lambda}}\right) \\
&= 2Bh\tfrac{e^{-\lambda(k-1)}-e^{3\lambda}}{1-e^{\lambda}} \\
&= 2Bh\tfrac{e^{3\lambda}-e^{\lambda(1-k)}}{e^{\lambda}-1} < 2Bh\tfrac{e^{3\lambda}}{e^{\lambda}-1}.
\end{aligned}$$

Therefore

$$|y_g(x)| \leqslant A + B(2h)\frac{e^{3\lambda}}{e^{\lambda}-1}$$

and this completes the proof. □

3. Existence and Stability of Periodic Solution

We use Banach fixed point theorem to prove that at least one periodic solution exists.

First, we introduce an operator which maps an initial condition for equation (1) to the value of the solution at time T. We know that each initial condition for a differential equation of order 3 is a triple like $(y(t_0), y'(t_0), y''(t_0))$. Here we consider equation (1) as a system of differential equations. Let $y = x_1, y' = x_2$ and $y'' = x_3$. Then equation (1) becomes:

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = -\gamma x_1 - \beta x_2 - \alpha x_3 - f(t, x_1) + e(t) \end{cases} \tag{7}$$

or $X' = A(t)X + F(t, x) + E(t)$ where,

$$X' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \quad , \quad A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \gamma & -\beta & -\alpha \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad , \quad F(t, x) = \begin{bmatrix} 0 \\ 0 \\ -f(t, x_1) \end{bmatrix}$$

and $E(t) = \begin{bmatrix} 0 \\ 0 \\ e(t) \end{bmatrix}$. We consider the space of matrices equipped with the norm $||A|| = \sum_{i,j} |a_{ij}|$ for $A = [a_{ij}]$. Note that the condition $|f(t, y)| \leqslant L|y|$ becomes $||F(t, x)|| \leqslant L||X||$ because we have

$$||F(t, x)|| = |f(t, x_1)| = |f(t, y)| \leqslant L|y| = L|x_1| \leqslant L(|x_1|+|x_2|+|x_3|) = L||X||.$$

Each solution of system (7) is of the following form

$$X(t) = Y(t)X(0) + \int_0^t Y(t-\alpha)(F(\alpha, X(\alpha)) - E(\alpha))d\alpha$$

where $Y(t)$ is a fundamental matrix of solutions with $Y(0) = I$ as initial condition. Hence

$$||X(t)|| \leqslant ||Y(t)||.||X(0)||+\int_0^t ||Y(t-\alpha)||.||F(\alpha, X(\alpha)||d\alpha+\int_0^t ||Y(t-\alpha))||dG,$$

where $dG = Edt$ or $d\begin{bmatrix}0\\0\\g\end{bmatrix} = \begin{bmatrix}0\\0\\e\end{bmatrix} dt$ (we suppose $dg = dg_1 + dg_2$ where g_1 and g_2 are ascending).

From the theory of systems of differential equations, if "a" is a real number such that $a > Re(\lambda_r)$ where $\lambda_r^{,}s$ are roots of characteristic polynomial of the coefficient matrix, then there is a "C" such that for every $x(t), a$ solution of the system $x' = Ax$, we have

$$||x(t)|| \leqslant C||x(0)||exp(at).$$

So in our argument, we can conclude that there exists $B > 0$ and $a > 0$ such that

$$||Y(x)|| \leqslant B\exp(-at).$$

Since $||F(t,x)|| \leqslant L||X||$, then

$$||X(t)|| \leqslant B\exp(-at)||X(0)|| + \textstyle\int_0^t LB\exp(-a(t-\alpha)||X(\alpha)||d\alpha \\ + \textstyle\int_0^t B\exp(-a(t-\alpha))dG.$$

In the proof of Theorem 2.2. we saw that

$$\int_0^t LBe^{-a(t-\alpha)}||X(\alpha)||d\alpha + \int_0^t Be^{-a(t-\alpha)}dG < 2Bh\frac{e^{3\lambda}}{e^{\lambda}-1} = K\ .$$

We will search for δ and K such that $||X(T)|| \leqslant B\delta e^{-at} + K < \delta$ whenever $||X(0)|| < \delta$. We solve this inequality for T:

$$B\delta e^{-at} \leqslant \delta - k$$
$$e^{-at} \leqslant \tfrac{\delta-k}{B\delta}$$
$$-at \leqslant \log(\tfrac{\delta-k}{B\delta})$$
$$at \geqslant \log(\tfrac{B\delta}{\delta-k})$$

or

$$T \geqslant \frac{1}{a}\log(\frac{B\delta}{\delta-k}).$$

It is sufficient to choose $\delta > k$ and then choose $T \geqslant \frac{1}{a}\log(\frac{B\delta}{\delta-k})$. This guarantees that if we consider an operator U, which maps $X(0)$ to $X(T)$,

this operator maps the ball centered at origin in $\mathbb{R}^3$ with radius δ into itself.
Now we will show that U is a contraction, i.e., there exists $0 < B < 1$ such that

$$||X(T) - Y(T)|| \leqslant B||X(0) - Y(0)||).$$

If it is so and if $X(0) = Y(0)$ then $X(T) = Y(T)$ and this means that the solution of (1) with an initial condition is unique. Hence U is well defined.
For U to be contraction, we need another assumption on f, the Lipschitz conditions; i.e.,

$$|f(t, x(t)) - f(t, y(t))| \leqslant k_1|x(t) - y(t)| \quad , |y|, |x| < H,$$

for some $k_1 > o$ and all t. By this condition we have,

$$\begin{aligned} ||F(t, X(t)) - F(t, Y(t))|| &= |f(t, x_1(t)) - f(t, y_1(t))| \\ &\leqslant k_1|x_1(t) - y_1(t)| \\ &\leqslant k_1 \textstyle\sum_{i=1}^{3} |x_i(t) - y_i(t) \\ &= k_1||X||. \end{aligned}$$

Then

$$\begin{aligned} ||U(X(0)) - U(Y(0))|| \quad &= ||X(T) - Y(T)|| \\ &\leqslant B||X(0) - Y(0)|| \exp(-at) \\ &+ \textstyle\int_0^T ||Y(t-\alpha)||.||f(\alpha, x) - f(\alpha, y)||d\alpha. \end{aligned}$$

Therefore

$$||X(T)-Y(T)|| \leqslant B||X(0)-Y(0)|| \exp(-at)+\int_0^T k_1Be^{-a(t-\alpha)}||X(\alpha)-Y(\alpha)||d\alpha.$$

By assuming $||X(T) - Y(T)|| = R(T)$, we have

$$R(T) \leqslant BR(0)e^{-at} + \int_0^T k_1Be^{-a(T-\alpha)}R(\alpha)d\alpha$$

or

$$R(T)e^{at} \leqslant BR(0) + k_1 B \int_0^T k_1 e^{a\alpha} R(\alpha) d\alpha \ .$$

From Gronwall-inequality we have

$$R(T) \leqslant Be^{(k_1B-a)T} R(0) \leqslant BR(0)$$

provided that $k_1 B < a$. So

$$||U(X(0)) - U(Y(0))|| = ||X(T) - Y(T)|| \leqslant B||X(0) - Y(0)||.$$

Then U is a contraction if and only if $B < 1$. B is such that $B = n^2 c = gc$ where $\frac{1}{exp(a-\alpha)T} < c$. By choosing a suitable T we can have c such that $c < \frac{1}{q}$. So U is a contraction. Since closed ball in $\mathbb{R}^3$ is a complete metric space, then from the Banach fixed point theorem, U has a unique fixed point in $B_\delta(0)$. It means that there exists a X_0 such that the solution with initial condition $X(0) = X_0$ satisfies $X(T) = X(0)$, with $T = w$. This solution is the periodic solution.
Now, suppose that X is this periodic solution and Y is another arbitrary solution of (1). As we saw before, for all $t > 0$,

$$||X(t) - Y(t)|| \leqslant B||X(0) - Y(0)||.$$

So for given $\epsilon > 0$, it is sufficient to choose $\delta = \frac{\epsilon}{B}$ with $||X(0)-Y(0)|| < \delta$ then for all $t > 0$

$$||X(t) - Y(t)|| < \epsilon$$

and so X is stable.
The proof is completed. □

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