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Some Concepts of Negative Dependence for Bivariate Distributions with Applications

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Abstract. In this paper, some concepts of negative dependence for bivariate distributions, especially hazard and local negative dependence (HND, LND) are studied. The Clayton-Oakes, φ and γ measures of association and relationship of HND with these measures are obtained. In addition, various examples illustrate the usefulness of these notions in some family of distributions.

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1. Introduction

Let X and Y be absolutely continuous random variables having joint density $f(x, y)$ and survival function $\bar{F}(x, y)$. Basu ([3]) introduced bivariate hazard function by $r(x, y) = f(x, y)/\bar{F}(x, y)$. In the independent case the bivariate hazard function is equal to product of conditional hazard functions, $\frac{\partial}{\partial x}[-\log \bar{F}(x, y)]$ and $\frac{\partial}{\partial y}[-\log \bar{F}(x, y)]$. If equality failed we deal with dependent (positive or negative) random variables. Oluyede ([17, 18]) has obtained some properties and inequalities for positively hazard and local dependence. More details about notions of dependence are in Lehmann ([14]), Karlin ([13]), Esary and Proschan ([5]), Joe ([10]) and Shaked and Shanthikumar ([20]). In this paper we use notions of negatively hazard and local dependence, say HND, LND, and investigate relationship between these concepts with some other concepts of dependence. Finally, we obtain measures of association as Θ-measure (known as Clayton-Oakes measure), φ-measure and γ-measure for some bivariate distributions family, then evaluate the relationship between these measures and $HND(LND)$.

Let (X,Y) be an absolutely continuous random vector with distribution function $F(x,y)$ and survival function$\bar{F}(x,y)$. Next, we need the following definitions.

Definition 1.1. ([17]) *Absolutely continuous random variables X and Y having a joint density function $f(x,y)$ are hazard negative (positive)dependence, HND(HPD), if and only if*

$$\frac{f(x,y)}{\bar{F}(x,y)} \leqslant (\geqslant) \int_x^\infty \frac{f(u,y)du}{\bar{F}(x,y)} \int_y^\infty \frac{f(x,v)dv}{\bar{F}(x,y)} \tag{1}$$

where $\frac{f(x,y)}{\bar{F}(x,y)}$ is the bivariate hazard rate function, and

$$\int_x^\infty \frac{f(u,y)du}{\bar{F}(x,y)} = \frac{\partial}{\partial y}[-\log \bar{F}(x,y)], \quad \textit{and} \quad \int_y^\infty \frac{f(x,v)dv}{\bar{F}(x,y)} = \frac{\partial}{\partial x}[-\log \bar{F}(x,y)]$$

are conditional hazard functions. Note that, equality holds in (1) *if and only if X and Y are independent.*

Definition 1.2. ([18]) *Absolutely continuous random variables X and Y having a joint density function $f(x,y)$ are locally negative (positive) dependence, LND(LPD), if and only if*

$$F(x,y)f(x,y) \leqslant (\geqslant) \int_{-\infty}^x f(u,y)du \int_{-\infty}^y f(x,v)dv, \tag{2}$$

Note that, equality holds in (2) *if and only if X and Y are independent*

Definition 1.3. *A non-negative function h on A^2 , where $A \subseteq \mathbb{R}$, is reverse rule of order* 2 *(RR_2) if for all $x_1 < x_2$ and $y_1 < y_2$, with*

$x_i, y_j \in A \ i = 1, 2 \ j = 1, 2$

$$h(x_1, y_1)h(x_2, y_2) \leqslant h(x_1, y_2)h(x_2, y_1). \tag{3}$$

Definition 1.4. *Let X and Y be continuous random variables. Then;*

- *X and Y are right corner set decreasing, (which we denote $RCSD(X, Y)$), if*
$$P(X > x, Y > y | X > x', Y > y') \tag{4}$$
is decreasing (non-increasing) in x' and in y' , for all x and y .

- *X and Y are left corner set increasing, $LCSI(X, Y)$, if*

$$P(X \leqslant x, Y \leqslant y | X \leqslant x', Y \leqslant y') \tag{5}$$

is increasing (non-decreasing) in x' and in y' , for all x and y .

Definition 1.5. *Let $F_\theta(x)$ be a family of distribution functions. This family is called monotone decreasing likelihood ratio, (MDLR)(monotone increasing likelihood ratio, (MILR)) if for all $\eta > \theta$, $\frac{F_\eta(x)}{F_\theta(x)}$ is decreasing (increasing) in x.*

2. Main Results

In this section, we obtain some useful results about HND and LND which show relation of these concepts with other notions of dependence.

Proposition 2.1. *Let (X, Y) be an absolutely continuous random vector with distribution $F(x, y)$ and survival function $\bar{F}(x, y)$. Then*

i) *$\bar{F}(x, y)$ is RR_2 if and only if for all $x_1 < x_2$ and $y_1 < y_2$,*

$$P(X > x_2, Y > y_2)P(x_1 < X \leqslant x_2, y_1 < Y \leqslant y_2) \leqslant P(x_1 < X \leqslant x_2, Y > y_2)P(X > x_2, y_1 < Y \leqslant y_2). \tag{6}$$

ii) *$F(x, y)$ is RR_2 if and only if for all $x_1 < x_2$ and $y_1 < y_2$,*

$$P(X \leqslant x_1, Y \leqslant y_1)P(x_1 < X \leqslant x_2, y_1 < Y \leqslant y_2) \leqslant P(X \leqslant x_1, y_1 < Y \leqslant y_2)P(x_1 < X \leqslant x_2, Y \leqslant y_1). \tag{7}$$

Proof. We prove part (i). The part of (ii) is similar. Note that $\bar{F}(x,y)$ is RR_2 , *i.e.* for all $x_1 < x_2$ and $y_1 < y_2$

$$\begin{vmatrix} P(X > x_1, Y > y_1) & P(X > x_1, Y > y_2) \\ P(X > x_2, Y > y_1) & P(X > x_2, Y > y_2) \end{vmatrix} \leqslant 0. \tag{8}$$

It is easy to show that (8) is equivalent to

$$\begin{vmatrix} P(x_1 < X \leqslant x_2, y_1 < Y \leqslant y_2) & P(x_1 < X \leqslant x_2, Y > y_2) \\ P(X > x_2, y_1 < Y \leqslant y_2) & P(X > x_2, Y > y_2) \end{vmatrix} \leqslant 0. \tag{9}$$

and (9) is equivalent to (6). This completes the proof. The following proposition gives a relationship between RR_2 and HND(LND). □

Proposition 2.2. *Let (X, Y) be an absolutely continuous random vector with distribution function $F(x,y)$ and survival function $\bar{F}(x,y)$. Then,*

i) *$\bar{F}(x,y)$ is $RR_2 \Rightarrow$ HND(X, Y).*

ii) *$F(x,y)$ is $RR_2 \Rightarrow$ LND(X, Y).*

Proof.

i) Let $x_1 = x$, $x_2 = x + \Delta x$, $y_1 = y$, $y_2 = y + \Delta y$ where $\Delta x, \Delta y > 0$. By using (6) and dividing the result by $\Delta x \Delta y$ and letting $\Delta x \to 0, \Delta y \to 0$, the result follows.

ii) The proof is similar, to (i). □

Theorem 2.3. *Let (X, Y) be an absolutely continuous random vector with distribution function $F(x,y)$ and survival function $\bar{F}(x,y)$. Then,*

i) *$RCSD(X,Y)$ if and only if $\bar{F}(x,y)$ is RR_2.*

ii) *$LCSI(X,Y)$ if and only if $F(x,y)$ is RR_2.*

Proof. The first part is proved, the second is similar.
$RCSD(X,Y) \Rightarrow \bar{F}(x,y)$ is RR_2 : In this case, taking $y = -\infty$, $P(X > x | X > x', Y > y')$ is decreasing in x' and in y', for all $x \in I\!R$. So,

if $x > x'$, then $P(X > x|X > x', Y > y') = \frac{P(X>x,Y>y')}{P(X>x',Y>y')}$ is decreasing in y', consequently for all $y' < y$, we obtain

$$\frac{P(X > x, Y > y)}{P(X > x', Y > y)} \leqslant \frac{P(X > x, Y > y')}{P(X > x', Y > y')}, \tag{10}$$

this implies that $\bar{F}(x, y)$ is RR_2.
$\bar{F}(x, y)$ is $RR_2 \Rightarrow RCSD(X, Y)$: In this case, for all $x > x'$ and $y > y'$, (10) valid and for all $x > x'$, $P(X > x|X > x', Y > y') = \frac{P(X>x,Y>y')}{P(X>x',Y>y')}$ is decreasing in y' . Similarly for all $y > y'$ we have,

$$P(Y > y|X > x', Y > y') \geqslant P(Y > y|X > x, Y > y')$$

i.e. $P(Y > y|X > x', Y > y')$ is decreasing in x' . Now, if $x > x'$, $y < y'$

$$\begin{aligned} P(X > x, Y > y|X > x', Y > y') &= \frac{P(X > x, Y > y')}{P(X > x', Y > y')} \\ &\leqslant \frac{P(X > x, Y > y)}{P(X > x', Y > y)} \\ &= P(X > x, Y > y|X > x', Y > y), \end{aligned}$$

then, $P(X > x, Y > y|X > x', Y > y')$ is decreasing in y'. Similarly for $x \leqslant x'$, $y > y'$, $P(X > x, Y > y|X > x', Y > y')$ is decreasing in x'. Also for $x < x'$, $y < y'$, $P(X > x, Y > y|X > x', Y > y') = 1$. Therefore (X, Y) is $RCSD$. □

Corollary 2.4. *Under the assumptions of Theorem* 2.3. *and Proposition* 2.2.

i) $RCSD(X, Y) \Rightarrow HND(X, Y)$.

ii) $LCSI(X, Y) \Rightarrow LND(X, Y)$.

Theorem 2.5. *Let* $F_\theta(x)$ *and* $G_\theta(y)$ *be two families of distribution functions. For any mixing distribution* K *, consider the distribution*

$$H(x, y) = \int_\Omega F_\theta(x) G_\theta(y) dK(\theta),$$

where Ω *is a Borel set in* $\mathbb{R}^n$ *and* K *is a probability measure on* Ω *.*

(*i*) *If one of the family is $MILR$ and the other is $MDLR$, then $H(x,y)$ is LND.*

(*ii*) *If $F_\theta(x)$ and $G_\theta(y)$ are both $MDLR$ or $MILR$, then $H(x,y)$ is LPD.*

Proof. We prove part (*i*) . The proof of part (*ii*) is similar . Let $F_\theta(x)$ be $MDLR$ and $G_\theta(y)$ be $MILR$, so that for $x < x'$, $y < y'$ and $\eta > \theta$ $(\eta, \theta \in \Omega)$, we have

$$[F_\eta(x)F_\theta(x') - F_\eta(x')F_\theta(x)][G_\eta(y)G_\theta(y') - G_\eta(y')G_\theta(y)] \leqslant 0.$$

After some simple calculation we obtain $H(x,y)H(x',y') \leqslant H(x,y')H(x',y)$. Therefore the distribution function H is RR_2, and hence H is LND. □

3. Example and Measures of Dependence

In this section, we first discuss three local dependence measures, such as γ- measure, the Clayton-Oakes association measure(θ-measure) and ψ- measure and drive the relationship of these measures with hazard negative dependence,then we give some examples.

γ-Measure:

Holland and Wang ([9]) defined, the local dependence function $\gamma_h(x,y)$ as follows;

$$\gamma_h(x,y) = \frac{\partial^2 Log h(x,y)}{\partial x \partial y} = \frac{1}{h(x,y)}\{h^{11}(x,y) - \frac{h^{10}(x,y)h^{01}(x,y)}{h(x,y)}\}, \quad (11)$$

where $h(x,y) \geqslant 0$, $h^{ij} = \frac{\partial^{i+j}h(x,y)}{\partial x^i \partial y^j}$, $i,j = 0,1$, the mixed partial derivative of h(x,y) exists and h is defined on a Cartesian product set. They show that this measure is symmetric and $\gamma = 0$ if and only if X and Y are independent. Also, Jones ([11, 12]) studied dependence properties of this measure and proved that γ is an appropriate index for measuring local likelihood ration dependence.

Remark 3.1. *Let X and Y be continuous random variables with bivariate distribution function $F(x,y)$ and survival function $\bar{F}(x,y)$. Then, it is easy to show that,*

$$\gamma_F(x,y) = \frac{f(x,y)F(x,y) - \int_{-\infty}^{x} f(u,y)du \int_{-\infty}^{y} f(x,v)dv}{F^2(x,y)},$$

and

$$\gamma_{\bar{F}}(x,y) = \frac{f(x,y)\bar{F}(x,y) - \int_x^\infty f(u,y)du \int_y^\infty f(x,v)dv}{\bar{F}^2(x,y)}.$$

Therefore,

- Lemma 4.2. *in Holland and Wang* ([9]) *implies that X and Y are independent if and only if $\gamma_F(x,y) = 0$ ($\gamma_{\bar{F}}(x,y) = 0$) or, equivalently equality occur in* (1) *or* (2).
- *Moreover, it is easy to show that the following implications hold*

$$HND(X,Y)(HPD(X,Y)) \Leftrightarrow \gamma_{\bar{F}}(x,y) \leqslant (\geqslant) 0,$$

and

$$LND(X,Y)(LPD(X,Y)) \Leftrightarrow \gamma_F(x,y) \leqslant (\geqslant) 0.$$

Θ-*Measure*

Clayton([4]) *and Oakes* ([19]) *defined the following associated measure:*

$$\Theta(x,y) = \frac{\bar{F}(x,y)D_{12}\bar{F}(x,y)}{D_1\bar{F}(x,y)D_2\bar{F}(x,y)}, \tag{12}$$

where $D_{12}\bar{F}(x,y) = \frac{\partial^2}{\partial x \partial y}\bar{F}(x,y)$, $D_1\bar{F}(x,y) = \frac{\partial}{\partial x}\bar{F}(x,y)$ *and* $D_2\bar{F}(x,y) = \frac{\partial}{\partial y}\bar{F}(x,y)$. *The function $\Theta(x,y)$ measures the degree of association between X and Y, and has direct relation to local dependence function,* $\gamma_{\bar{F}}(x,y)$.

- $\Theta(x,y) = 1$ *if and only if* $\gamma_{\bar{F}}(x,y) = 0$ *i.e X and Y are independent,*
- $\Theta(x,y) > 1$ *if and only if* $\gamma_{\bar{F}}(x,y) > 0$ *i.e X and Y are positively dependent,*
- $\Theta(x,y) < 1$ *if and only if* $\gamma_{\bar{F}}(x,y) < 0$ *or equivalently X and Y are negatively dependent.*

According to Gupta ([7]) *we have the following quantities to formulate* $\Theta(x,y)$.

$r_1(x,y) := -\frac{\partial}{\partial x}[\log \bar{F}(x,y)] = -\frac{D_1\bar{F}(x,y)}{\bar{F}(x,y)}, r_2(x,y) :=$
$- \frac{\partial}{\partial y}[\log \bar{F}(x,y)] = -\frac{D_2\bar{F}(x,y)}{\bar{F}(x,y)}$

and

$$\frac{\partial^2}{\partial x \partial y} \log \bar{F}(x,y) = r_1(x,y) r_2(x,y)(\Theta(x,y) - 1). \tag{13}$$

So,

$$r(x,y) = r_1(x,y) r_2(x,y) \Theta(x,y), \tag{14}$$

where $r(x,y) = \frac{f(x,y)}{\bar{F}(x,y)}$ *is Basu's failure rate. We observe that,*

$$\Theta(x,y) < 1 \Leftrightarrow \frac{\partial^2}{\partial x \partial y} \log \bar{F}(x,y) < 0 \Leftrightarrow RCSD(X,Y) \Leftrightarrow r(x,y) < r_1(x,y) r_2(x,y).$$

ψ- Measure

The following associated measure (known as ψ- measure) defined by Anderson et al. ([2]);

$$\psi(x,y) = \frac{P(X > x | Y > y)}{P(X > x)} = \frac{\bar{F}(x,y)}{\bar{F}_1(x)\bar{F}_2(y)} \tag{15}$$

Under the some regular conditions, the following statements are valid for ψ- measure in (15);

- $\psi(x,y) = 1 \Leftrightarrow X$ *and* Y *are independent.*
- $\frac{\partial^2}{\partial x \partial y} \psi(x,y) = \gamma_{\bar{F}}(x,y)$.
- *If* $\psi(x,y) > 1$ *then* (X,Y) *is PQD.*
- *If* $\psi(x,y) < 1$ *then* (X,Y) *is NQD.*
- *If* $\Theta(x,y) < (>) 1$ *then* $\psi(x,y) < (>) 1$ *(the converse is not true). For more details, see Gupta* ([7]).

The following proposition gives relationship between the mentioned local dependence measures.

Proposition 3.2. *Let (X,Y) be an absolutely continuous random vector having survival function $\bar{F}(x,y)$. The following statements are equivalent*

- $\Theta(x, y) < 1$,
- $\gamma_{\bar{F}}(x, y) < 0$,
- $\frac{\partial^2}{\partial x \partial y} \psi(x, y) < 0$,
- $r(x, y) < r_1(x, y) r_2(x, y)$,
- $(X, Y) \;\; is \;\; HND.$

Proof. Combining ([6, 11, 12, 13, 14]) the proposition proved immediately. □

Example 3.3. (Farlie-Gumble-Morganstern distribution (FGM) [6]) Consider the family of bivariate distribution functions

$$F(x, y) = F_1(x) F_2(y) [1 + \alpha (1 - F_1(x))(1 - F_2(y))]$$

where $|\alpha| \leqslant 1$ and $F_1(x)$ and $F_2(y)$ are continuous distribution functions. It can be shown that,

$$\gamma_F(x, y) \frac{\alpha f_1(x) f_2(y)}{[1 + \alpha \bar{F}_1(x) \bar{F}_2(y)]^2} \leqslant (\geqslant)\, 0 \quad \Leftrightarrow \quad -1 \leqslant \alpha \leqslant 0 \; (0 \leqslant \alpha \leqslant 1).$$

Therefore, $LND(X, Y)(LPD(X, Y))$ if and only if $-1 \leqslant \alpha \leqslant 0$ $(0 \leqslant \alpha \leqslant 1)$.
In terms of survival functions $\bar{F}(x, y) = P[X > x, Y > y]$; $\bar{F}_i(x_i) = P[X_i > x_i]$; $i = 1, 2$ the FGM family equivalent to

$$\bar{F}(x, y) = \bar{F}_1(x) \bar{F}_2(y) [1 + \alpha\, F_1(x) F_2(y)], \quad |\alpha| \leqslant 1.$$

It follows from simple calculations that

$$\gamma_{\bar{F}}(x, y) = \frac{\alpha\, f_1(x) f_2(y)}{[1 + \alpha\, F_1(x) F_2(y)]^2} \leqslant (\geqslant)\, 0 \quad \Leftrightarrow \quad -1 \leqslant \alpha \leqslant 0 \; (0 \leqslant \alpha \leqslant 1),$$

so $HND(X, Y)(HPD(X, Y))$ if and only if $-1 \leqslant \alpha \leqslant 0$ $(0 \leqslant \alpha \leqslant 1)$.
For more details about FGM family see Mari and Kotz ([15]).

Example 3.4. (Gumbel's bivariate exponential distribution) The survival function of Gumbel's bivariate distribution is

$$\bar{F}(x,y) = \exp\{-\alpha_1 x - \alpha_2 y - \beta xy\}, \quad \alpha_1, \alpha_2 > 0 \text{ and } 0 \leqslant \beta \leqslant \alpha_1\alpha_2.$$

For $x < x'$ and $y < y'$;

$$\begin{aligned} \bar{F}(x,y)\bar{F}(x',y') \ &-\bar{F}(x,y')\bar{F}(x',y) \\ &= \exp\{-\alpha_1(x+x') - \alpha_2(y+y')\} \\ &\times\Big[\exp\{-\beta(xy+x'y')\} - \exp\{-\beta(xy'+x'y)\}\Big] \quad \leqslant 0. \end{aligned}$$

Since $xy + x'y' \geqslant xy' + x'y$, hence $\bar{F}$ is RR_2, and this implies that (X,Y) is HND.

Example 3.5. (Ali-Mikhail-Haq distribution [1]) Consider Ali-Mikhail-Haq family of bivariate distribution functions

$$F(x,y) = \frac{F_1(x)F_2(y)}{1-\beta\,\bar{F}_1(x)\bar{F}_2(y)}, \quad |\beta| \leqslant 1$$

where F_1 and F_2 are continuous distribution functions and $\bar{F}_i = 1 - F_i \quad \mathrm{i} = 1,2.$ by simple calculation, we obtain

$$\gamma_F(x,y) = \frac{\beta\, f_1(x)f_2(y)}{[1-\beta\,\bar{F}_1(x)\bar{F}_2(y)]^2} \leqslant 0\ (\geqslant 0) \quad \Leftrightarrow \quad -1 \leqslant \beta \leqslant 0\ (0 \leqslant \beta \leqslant 1).$$

So, LND(X, Y)(LPD(X, Y)) if and only if $-1 \leqslant \beta \leqslant 0$ $(0 \leqslant \beta \leqslant 1)$.

Remark 3.6. *In the Example* 3.4. *we can use the Proposition* 3.2. *and obtain*

$$\begin{aligned} r_1(x,y) &= -\frac{\partial}{\partial x}[\log\bar{F}(x,y)] = \alpha_1 + \beta y \\ r_2(x,y) &= -\frac{\partial}{\partial y}[\log\bar{F}(x,y)] = \alpha_2 + \beta x \\ r(x,y) &= \frac{f(x,y)}{\bar{F}(x,y)} = (\alpha_1+\beta y)(\alpha_2+\beta x) - \beta \\ \Theta(x,y) &= \frac{r(x,y)}{r_1(x,y)r_2(x,y)} = \frac{(\alpha_1+\beta y)(\alpha_2+\beta x)-\beta}{(\alpha_1+\beta y)(\alpha_2+\beta x)} \end{aligned}$$

since $\alpha_i > 0, \quad i = 1, 2$ *and* $\beta \geqslant 0$, *therefore Proposition* (3.2.) *implies that* (X, Y) *is* HND.

Measure of Dependence Based on Copula

The copula function $C(u, v)$ is a bivariate distribution function with uniform marginals on $[0, 1]$, such that

$$F(x, y) = C_F(F_1(x), F_2(y))$$

By Sklar's Theorem ([21]), this copula exists and is unique if F_1 and F_2 are continuous. Thus we can construct bivariate distributions $F(x, y) = C_F(F_1(x), F_2(y))$ with given univariate marginals F_1 and F_2 by using copula C_F ([16]). Then we have the following properties:

- ([16]) Let X and Y be continuous random variables with joint distribution function $F(x, y)$ and marginals $F_1(x)$ and $F_2(y)$ respectively, then
  i) The copula $C(u, v)$ and survival copula which refer to $\hat{C}(u, v)$ are given by

  $$C_F(u, v) = F(F_1^{-1}(u), F_2^{-1}(v)), \quad \forall u, v \in [0, 1],$$

  and

  $$\hat{C}(u, v) = \bar{F}(\bar{F}_1^{-1}(u), \bar{F}_2^{-1}(v)), \quad \forall u, v \in [0, 1]$$

  Where, F_i^{-1} and $\bar{F}_i^{-1}$ are quasi-inverses of F_i and $\bar{F}_i$, $i = 1, 2$ respectively. Note that;

  $$\hat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v), \quad \forall u, v \in [0, 1]$$

  ii) The partial derivatives $\frac{\partial C_F(u,v)}{\partial u}$ and $\frac{\partial C_F(u,v)}{\partial v}$ exist and $c(u, v) = \frac{\partial^2 C_F(u,v)}{\partial u \partial v}$ is density function of $C_F(u, v)$.

- The Sklar's theorem implies that in FGM family for $-1 \leqslant \alpha \leqslant 1$

$$C(u, v) = \hat{C}(u, v) = uv(1 + \alpha(1 - u)(1 - v)), \tag{16}$$

and

$$c(u,v) = 1 + \alpha(1-2u)(1-2v). \tag{17}$$

Also in Gumbel family for $\alpha_1 = \alpha_2 = 1$, the survival copula is

$$\hat{C}(u,v) = uv.\exp(-\beta\ln(u)\ln(v)), \quad \forall \;\; 0 \leqslant \beta \leqslant 1. \tag{18}$$

Proposition 3.7. *Let (X, Y) be a random vector with FGM distribution function and copula function given in* (16), *then*

i) $\psi(u,v) = \frac{\hat{C}(u,v)}{uv} = 1 + \alpha(1-u)(1-v),$

ii) $\gamma_C(u,v) = \gamma_{\hat{C}}(u,v) = \frac{\partial^2 \log(C(u,v))}{\partial u.\partial v} = \frac{\alpha}{[1+\alpha(1-u)(1-v)]^2},$

iii) $\Theta(u,v) = \frac{\hat{C}(u,v)\frac{\partial^2 \hat{C}(u,v)}{\partial u.\partial v}}{\frac{\partial \hat{C}(u,v)}{\partial u}\frac{\partial \hat{C}(u,v)}{\partial v}} = \frac{(1+\alpha(1-u)(1-v)).(1+\alpha(1-2u)(1-2v)}{(1+\alpha(1-u)(1-2v)1+\alpha(1-v)(1-2u))}.$

• Figure 1 shows the surface of $\gamma_{\alpha_2}(u,v) - \gamma_{\alpha_1}(u,v)$ for some values of α_1, α_2 such that $\alpha_1 < \alpha_2$ in FGM family with uniform marginals on $(0,1)$. These surfaces, show that $\gamma_\alpha(u,v)$ increases in α.

• Figure 2 shows the surface of $\Theta_{\alpha_2}(u,v) - \Theta_{\alpha_1}(u,v)$ for some values of α_1, α_2 such that $\alpha_1 < \alpha_2$ in FGM family with uniform marginals on $(0,1)$. These surfaces, show that $\Theta_\alpha(u,v)$ increases in α.

Proposition 3.8. *Let (X, Y) be a random vector with Gumbel distribution function with $\alpha_1 = \alpha_2 = 1$ and survival copula given in* (18), *then*

i) $\psi(u,v) = \exp(-\beta\ln(u)\ln(v)),$

ii) $\gamma_{\hat{C}}(u,v) = \frac{\beta uv\ln(uv)(1+\beta) - \beta uv - \beta u^2v^2\ln(u) - \beta^2\ln(u)\ln(v)}{u^3v^3},$

iii) $\Theta(u,v) = \frac{u^2v^2 - \beta uv - \beta u^2v^2\ln(u) + \beta^2 uv\ln(u)\ln(v)}{u^2v^2 - \beta uv\ln(u) - \beta uv\ln(v) + \beta^2\ln(u)\ln(v)}.$

•Figure 3 shows the surface of $\gamma_{\beta_2}(u,v) - \gamma_{\beta_1}(u,v)$ for some values of β_1, β_2 such that $\beta_1 < \beta_2$ in Gumbel family. These surfaces, show that $\gamma_\beta(u,v)$ decreases in β.

• Figure 4 shows the surface of $\Theta_{\beta_2}(u,v) - \Theta_{\beta_1}(u,v)$ for some values of β_1, β_2 such that $\beta_1 < \beta_2$ in Gumbel family. These surfaces, show that $\Theta_\beta(u,v)$ is not monotone in β.

Remark 3.9. *It is clear that $\psi_\alpha(u,v)$ in FGM family is increasing in α and $\psi_\beta(u,v)$ in Gumbel family is decreasing in β.*

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References

[1] M. Ali, N. Mikhail, and M. Haq, A class of bivariate distributions including the bivariate logistic, *J. Multivar. Anal.,* 8 (1978), 405-412.

[2] J. E. Anderson, T. A. Louis, N. V. Holm, and B. Harvald, Time dependent association measures for bivariate survival distributions, *J. Amar. Statist. Assoc.*, 87 (1992), 641-650.

[3] A. P. Basu Bivariate failure rate, *J. Amer. Statist. Assoc.* 66, 333 (1971), 103-104.

[4] D. G. Clayton, A model for association in bivariate life tables and its applications in epidemiological studies of familial tendency in chronic disease incidence, *Biometrika,* 65 (1978), 141-151.

[5] J. D. Esary and F. Proschan, Rilationship among some concepts of bivariate dependence, *Ann. Math. Statist,* 43 (1972), 651-655.

[6] D. J. Farlie, The performance of some correlation coefficients for a general bivariate distribution, *Biometrika,* 47 (1960), 307-323.

[7] R. C. Gupta, On some association measures in bivariate distributions and their relationships, *J. Statist. Plann. and Inf.* 117 (2003), 83-98.

[8] R. C. Gupta, Reliability studies of bivariate distribution with pareto conditionals, *J. of Multivar. Anal.* 76 (2001), 214-225.

[9] P. W. Holland and Y. J. Wang, Dependence functions for continous bivariates densities, *Commun. Statist. Theory and Methods,* 16 (3) (1987), 863-876.

[10] H. Joe, *Multivariate Models and Dependence Concepts,* Chapman and Hall, 1997.

[11] M. Jones, The local dependence function, *Biometrika,* 83 (4) (1996), 899-904.

[12] M. Jones, Constant local dependence, *Journal of Multivariate Analysis,* 64 (2) (1998), 148-155.

[13] S. Karlin, *Total Positivity,* Vol. I, Stanford Univercity Press, Stanford, 1968.

[14] E. L. Lehmann, Some concepts of dependence, *Ann. Math. Statist.* 37 (1966), 1137-1153.

[15] D. D. Mari and S. Kotz, *Correlation and dependence,* Imperical College Press, 2004.

[16] R. B. Nelsen, *An Introduction to Copulas,* Springer, New York, 2006.

[17] B. O. Oluyede, On stochastic inequalities and dependence orderings, *Appl. Math. and Compu,* 146 (2003), 601-610.

[18] B. O. Oluyede, On local dependence and stochastic inequalities with applications to contingency tables, *Appl. Math. and Compu,* 151 (2004), 801-813.

[19] D. Oakes, Bivariate survival models induced by frailties, *J. Amer, Statist, Assoc,* 84 (1989), 487-493.

[20] M. Shaked and J. G. Shanthikumar, *Stochastic Order and Their Applications,* Academic Press, New York, 1994.

[21] A. Sklar, Fonctions de repartition a n dimensions et leurs marges, *Publications de l'institue de Statistique de l'Universitte de Paris,* 8 (1959), 229-231.

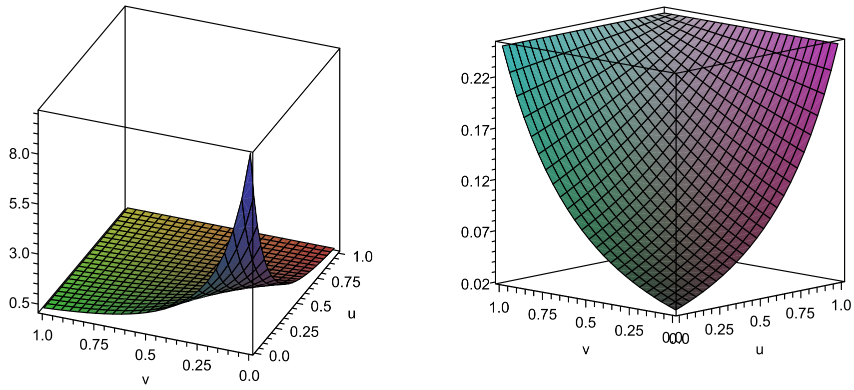

FIGURE 1: Perspective plots of $\gamma_{\alpha_2}(x,y) - \gamma_{\alpha_1}(x,y)$ for FGM family with parameter α_1 and α_2 equal to 0.5 and 0.75(right), -0.75 and -0.5(left).

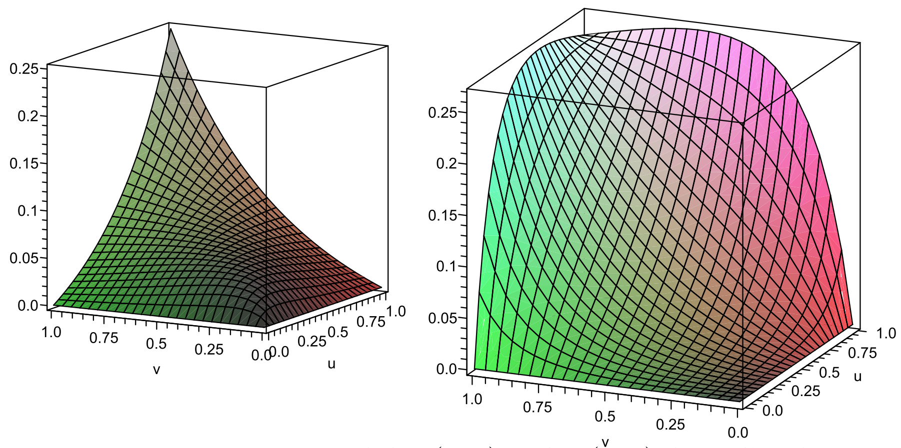

FIGURE 2: Perspective plots of $\Theta_{\alpha_2}(x,y) - \Theta_{\alpha_1}(x,y)$ for FGM family with parameter α_1 and α_2 equal to 0.5 and 0.75(right), -0.75 and -0.5(left).

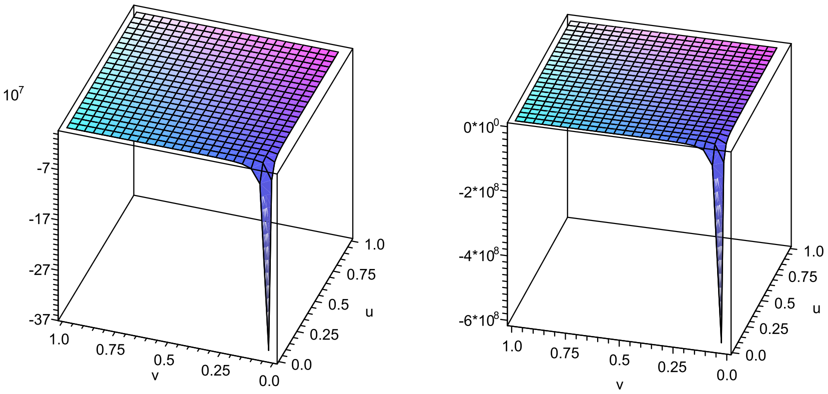

FIGURE 3: Perspective plots of $\gamma_{\beta_2}(x,y) - \gamma_{\beta_1}(x,y)$ for Gumbel family with parameter β_1 and β_2 equal to 0.5 and 0.75(right), 0.5 and 0.25(left).

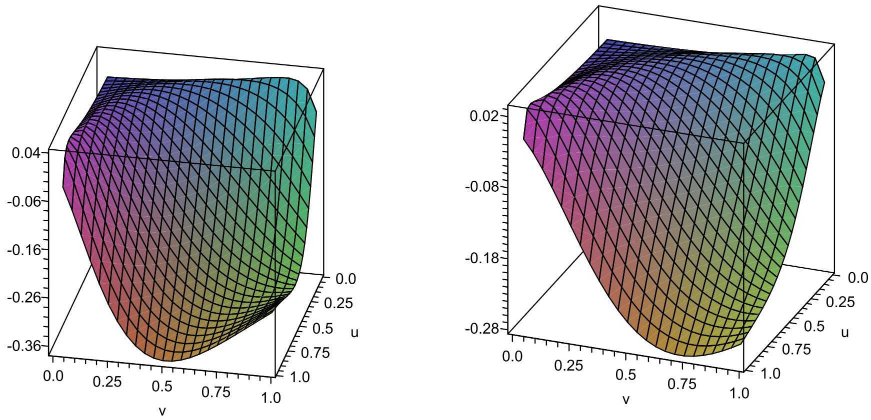

FIGURE 4: Perspective plots of $\Theta_{\alpha_2}(x,y) - \Theta_{\alpha_1}(x,y)$ for FGM family with parameter α_1 and α_2 equal to 0.5 and 0.75(right), -0.75 and -0.5(left).

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