Journal of Mathematical Extension
Vol. 4, No. 1 (2009), 31-41

A Note on the Symmetric Hit Problem

A. S. Janfada
Urmia University

Abstract. The symmetric hit problem was introduced for the first time by the author in his thesis ([5]). The aim of this paper is to solve an important open problem posed in ([7]), in an special case, which is one of the fundamental results in the studies of the symmetric hit problem.

AMS Subject Classification: 55S10.
Keywords and Phrases: Steenrod algebra, hit problem, symmetric hit problem.

1. Introduction

In this article we shall confine our attention to symmetric polynomial algebra $\mathbf{B}(n) = \mathbf{P}(n)^{\Sigma_n}$ over the field of two elements, where $\mathbf{P}(n) = \mathbb{F}_2[x_1, \ldots, x_n]$ is the polynomial algebra in n variables and Σ_n is the symmetric group on n letters acting on the right of $\mathbf{P}(n)$ by matrix substitution ([23]). The Steenrod algebra $\mathcal{A}$ acts on the left of $\mathbf{P}(n)$ and commutes with the action of Σ_n. In particular, $\mathbf{B}(n)$ is a graded $\mathcal{A}$-submodule of the left $\mathcal{A}$-module $\mathbf{P}(n)$. The grading on $\mathbf{P}(n)$ is by degree of homogeneous polynomials, where the variables x_i are in degree 1.

The algebra $\mathbf{P}(n)$ and subalgebra $\mathbf{B}(n)$ of $\mathbf{P}(n)$ realize respectively the cohomology of the product of n copies of infinite real projective space and the cohomology of the classifying space $BO(n)$ of the orthogonal group $O(n)$. Each element of the algebra $\mathbf{B}(n) = \mathbb{F}_2[\sigma_1, \ldots, \sigma_n]$ is a polynomial in the elementary symmetric functions σ_i. The ideal $\mathbf{L}(n)$ in $\mathbf{P}(n)$ generated by $\sigma_n = x_1 \cdots x_n$ can be identified with the cohomology of the n-fold smash product of infinite real projective space in positive dimensions.

We cite ([24, 20]) for the Steenrod algebra. Briefly, the Steenrod algebra $\mathcal{A}$ is a Hopf algebra generated as an algebra by the Steenrod squares Sq^i, for $i \geqslant 0$, subject to the condition $Sq^0 = 1$ and Adem relations ([1]), with coproduct defined by

$$\psi(Sq^n) = \sum_{0 \leqslant i \leqslant n} Sq^i \otimes Sq^{n-i}.$$

Modules $\mathbf{M}$ over the Steenrod algebra which arise from the cohomology of topological spaces or from invariant theory are themselves graded algebras and have a special property called *unstability*, which links the product in $\mathbf{M}$ with the coproduct in $\mathcal{A}$. For more details about unstability condition see ([8]). The following statement summarises the basic properties of these modules ([24, 8]).

Proposition 1.1. *For a module* $\mathbf{M}$ *over the Steenrod algebra and homogeneous elements* $f, g \in \mathbf{M}$ *we have*
1. $Sq^k(f) = 0$ *if* $\deg(f) <= k$*;*
2. $Sq^k(f) = f^2$ *if* $\deg(f) = k$*;*
3. *Cartan formula*

$$Sq^k(fg) = \sum_{0 \leqslant r \leqslant k} Sq^r(f) Sq^{n-r}(g).$$

In particular, the above rules permit the evaluation of a Steenrod square on any polynomial in $\mathbf{P}(n)$ by a reductive process using the Cartan formula. We cite ([19, 24, 28, 27]) for more details about Steenrod squares.

Let $\mathbf{M}$ be a graded $\mathcal{A}$-module. An element $f \in \mathbf{M}$ is *hit* if it satisfies a *hit equation*

$$f = \sum_{i>0} Sq^i(h_i),$$

where each $h_i \in \mathbf{M}$ has a degree strictly less than that of f and the summation is taken over a finite collection of Steenrod squares in positive grading.

For a graded left $\mathcal{A}$-module $\mathbf{M}$, we denote by $\mathbf{Q}(\mathbf{M})$ the quotient of the module $\mathbf{M}$ by the hit elements. Then $\mathbf{Q}(\mathbf{M})$ is a graded vector space over $\mathbb{F}_2$ and a basis for $\mathbf{Q}(\mathbf{M})$ lifts to a minimal generating set for $\mathbf{M}$ as a module over $\mathcal{A}$.

The hit problem is to discover criteria for elements of $\mathbf{M}$ to be hit and find minimal generating sets for $\mathbf{M}$ as an $\mathcal{A}$-module. This problem is generally regarded as difficult for arbitrary number n of variables.

However, there is a statement that can be made about the case $\mathbf{P}(n)$, which we refer to as the Peterson conjecture in honor of Frank Peterson who first formulated the problem and solved it in the case $n = 2$ [17, 18]. The conjecture was proved in a stronger form for the first time by Wood ([28]).

Theorem 1.2. $\dim(\mathbf{Q}^d(\mathbf{P}(n))) = 0$ *if and only if* $\mu(d) > n$.

Here, $\mu(d)$, for a positive integer d, denotes the smallest value of k for which it is possible to write $d = \sum_{i=1}^{k}(2^{\lambda_i} - 1)$, where $\lambda_i > 0$. For example $\mu(17) = 3$ as $17 = 7 + 7 + 3 = 15 + 1 + 1$.

This result then generalized by Singer ([19]) identifying a new class of hit monomials. A symmetric version of the conjecture was proved in [5, 6] for $\mathbf{B}(n)$. This conjecture was also proved ([3]) for algebra of invariants $\mathbf{P}(n)^G$, where G is a permutation group. It would be interesting to find the correct analogue of the Peterson conjecture for arbitrary subgroups of $G \subset \mathrm{GL}(n, \mathbb{F}_2)$. The answer is presumably G-dependent when G is not a permutation group. We cite the comprehensive reference [29] for more information on the hit problem.

A minimal generating set for $\mathbf{Q}^d(\mathbf{P}(3))$ is given in [2, 11, 12]. The author settled [9] a basic criterion for a monomial in $\mathbf{P}(3)$ to be hit.

The symmetric hit problem, which was introduced for the first time by the author in his thesis [5], is the same as the hit problem for $\mathbf{B}(n)$. In particular, a hit equation in $\mathbf{B}(n)$, called a *symmetric hit equation*, is the finite sum

$$f = \sum_i Sq^i(h_i),$$

where now f and the pre-images h_i are symmetric polynomials. In this case we say that f is *symmetrically hit*.

For $n \leqslant 3$, we proved [7] that if a monomial f is hit then $\sigma(f)$, the symmetrization of f, is symmetrically hit. The symmetrization of a monomial is defined in Definition 2.2. This fundamental problem is open for $n \geqslant 4$. The main result of this paper is to solve this problem in some special case.

Theorem 1.3. *Let f be a monomial in $\mathbf{P}(n)$ with distinct exponents. If f is hit, then $\sigma(f)$ is symmetrically hit.*

2. The Symmetric Hit Problem and the Main Result

In this section we discuss the symmetric hit problem in some details. For more details see [5, 6, 7, 9].

Thom was the first one who studied the hit problem for $\mathbf{M}(n)$, the ideal in $\mathbf{B}(n)$ generated by the elementary functions σ_n, and established that $\mathbf{M}(n)$ is a free module over $\mathcal{A}$ up to grading less than $2n$ ([26]). In 1999, in the first page of a preprint manuscript of the paper ([4]) Giambalvo and Peterson announced that not a lot is known about $\mathbf{B}(n)$. That time, the author was working on the symmetric hit problem. The essence of the work was talked, by Wood, in a Satellite Conference ([30]) in Ionnina University on June 2000. The main results of this work were then published in [6, 7]. Alternative proofs for some of results then exhibited in [13]. In a talking ([21]) in Göttingen, in 2007, Singer told that the work on this difficult problem (the symmetric hit problem) has been begun by Janfada and Wood in their papers ([6, 7]). His talk was then published in [22].

We first proved ([6]) a symmetric version of Peterson conjecture for $\mathbf{B}(n)$. Then we exhibited ([7]) a minimal generating set for $\mathbf{Q}^d(\mathbf{B}(3))$. It seems that the 4-variable case, in both the hit problem and the symmetric hit problem, is much more complicated since partitioning the elements to hit and non-hit classes is very hard in four variables. However, Nguyen Sum ([25]) worked a bit on the hit problem for four variables. The recent work of the author ([9]) was to prove a criterion for a monomial in $\mathbf{P}(3)$ to be hit. Finally, we obtained ([10]) some criteria for an element of $\mathbf{B}(n)$ to be non-hit in $\mathbf{B}(n)$.

Study on the symmetric hit problem is continued by Singer ([22]), Pengelly and Williams ([15, 16]), Nam ([14]) and others.

For a monomial $f = x_1^{d_1} \cdots x_n^{d_n}$, let F be the matrix whose row entries are the digits, in reversed binary expansion, of the exponents d_i. Define the *ω-vector* $\omega(f)$ of f to be the column sum of F.

Example 2.1. For the monomial $f = x_1^7 x_2^2$ we have

$$F = \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & \end{matrix}, \ \omega(f) = (1, 2, 1).$$

By ordering vectors lexicographically, we obtain the partial ω-order on the monomials in $\mathbf{P}(n)$. The ω-order will be dominant throughout this work and when, say, we write f is lower than g, we mean lower in the ω-order. This order is extended to $\mathbf{P}(n)$ in the usual way by comparing the highest monomials presented in two polynomials.

Definition 2.2. *An element π in Σ_n acts on a monomial by permuting the variables. The notation $\pi_{(i,j)}$ is used for the element of Σ_n which switches x_i and x_j. The* symmetrization *of a monomial f is the "smallest" symmetric polynomial $\sigma(f) \in \mathbf{B}(n)$ containing f as a term. To be precise, $\sigma(f) = \sum_{i=1}^{t} f\pi_i$, where $\pi_1, \ldots, \pi_t$ run through a set of left coset representatives for the stabilizer of the monomial f in Σ_n.*

Example 2.3. Consider the monomial $f = x_1^7 x_2^2$ in Example 2.1. In $\mathbf{B}(2)$ we have

$$\sigma(f) = f + f\pi_{(1,2)} = x_1^7 x_2^2 + x_1^2 x_2^7.$$

In $\mathbf{B}(3)$, however, $\sigma(f)$ has $|\Sigma_3| = 6$ terms.

$$\begin{aligned} \sigma(f) &= f + f\pi_{(1,2)} + f\pi_{(2,3)} + f\pi_{(1,3)} + f\pi_{(1,2,3)} + f\pi_{(1,3,2)} \\ &= x_1^7 x_2^2 + x_1^2 x_2^7 + x_1^7 x_3^2 + x_2^2 x_3^7 + x_2^7 x_3^2 + x_1^2 x_3^7. \end{aligned}$$

3. Proof of the Main Result

In this section, we prove the main Theorem 1.3. after some preliminary results. Then we obtein some more results.

If $\pi_1, \ldots, \pi_t$ are left coset representatives for a subgroup of the stabilizer of f then $\sum_{j=1}^{t} f\pi_j$ is symmetric but the expression may be zero. For example, the transfer $\tau(f)$ of a monomial f with two equal exponents is zero. It should be emphasized that the meaning of $\sigma(f)$ depends on the set of variables over which symmetrization is taking place. For example $\sigma(x_1)$ means $x_1 + x_2$ in $\mathbf{P}(2)$ but $x_1 + x_2 + x_3$ in $\mathbf{P}(3)$.

The general plan of action in tackling hit problem in $\mathbf{P}(n)$ is to use the ω-order as a potential function and try to manipulate monomials into equivalent polynomials of lower potential with the ultimate aim of achieving some kind of canonical forms. The idea is then to transfer these to $\mathbf{B}(n)$ by symmetrization. As pointed out in the next example, this is not always straightforward business.

Example 3.1. In $\mathbf{P}(2)$ we have the hit equation $x_1^2x_2^2 = Sq^1(x_1x_2^2)$. Symmetrizing the right hand side gives $0 = Sq^1(x_1x_2^2 + x_1^2x_2)$. So we cannot prove this way that $x_1^2x_2^2$ is symmetrically hit. On the other hand, there *is* a symmetric hit equation in $\mathbf{B}(2)$ namely $Sq^2(x_1x_2) = x_1^2x_2^2$. In favorable situations we can manufacture specially adapted hit equations which achieve the goal.

Proposition 3.2. *Let f be a monomial and g a polynomial in $\mathbf{P}^d(n)$. Suppose there is a hit equation $f - g = \sum_{i>0} Sq^i(h_i)$ in $\mathbf{P}^d(n)$, satisfying the condition that the stabilizer of f is a subgroup of the stabilizer of g and a subgroup of the stabilizer of each polynomial h_i. Let $\pi_1, \ldots, \pi_t$ be a collection of left coset representatives for the stabilizer of f in the symmetric group Σ_n. Then*

$$\sigma(f) - \sum_{j=1}^{t} g\pi_j = \sum_{i>0} Sq^i(\sum_{j=1}^{t} h_i\pi_j),$$

is a symmetric hit equation in $\mathbf{B}(n)$. In particular, for a monomial f, we have the equivalence $\sigma(f) \cong \sum_{j=1}^{t} g\pi_j$ in $\mathbf{B}(n)$.

Proof. By Definition 2.2. $\sigma(f) = \sum_{j=1}^{t} f\pi_j$ and the expressions $\sum_{j=1}^{t} g\pi_j$ and $\sum_{j=1}^{t} h_i\pi_j$ are symmetric by our earlier discussion.

The next result is a useful corollary of Proposition 3.2. in the 3-variable case.

Corollary 3.3. *Let f, g be monomials in $\mathbf{P}(3)$ such that f has exactly two equal rows i, j and g has distinct rows. Suppose that $f \cong g + g\pi_{(i,j)}$ in $\mathbf{P}(3)$. Then $\sigma(f) \cong \sigma(g)$ in $\mathbf{B}(3)$.*
To complete the proof of our main Theorem 1.3. we need one more result.

Proposition 3.4. *Let f be a monomial in $\mathbf{P}(n)$ with distinct exponents.*
An equivalence $f \cong g$ in $\mathbf{P}^d(n)$, for any polynomial g, symmetrizes to a symmetric equivalence $\sigma(f) \cong \tau(g)$ in $\mathbf{B}(n)$.

Proof. Since f has distinct exponents, $\sigma(f) = \tau(f)$. The result follows immediately from Proposition 3.2.

Proof of Theorem 1.3. It suffices to take $g = 0$ in Proposition 3.4.
The next result is a helpful tool in the symmetric hit problem. Recall from group theory that an odd (respectively even) permutation is a product of an odd (respectively even) number of transpositions. □

The following result establish a condition for a symmetrized monomial to be symmetrically hit.

Proposition 3.5. *Let f be a monomial in $\mathbf{P}(n)$ with distinct exponents. If $f \cong f\pi'$ in $\mathbf{P}(n)$, for some odd permutation π' in Σ_n, then $\sigma(f)$ is symmetrically hit.*

Proof. We first note that the equivalence $f \cong f\pi'$ implies the hit equation

$$f + f\pi' = \sum_{i>0} Sq^i(h_i).$$

The stabilizer of f is trivial in this case. Now let $\pi_1, \ldots, \pi_r$ be all the even permutations of Σ_n. Then we have

$$\sum_{j=1}^{r}(f + f\pi')\pi_j = \sum_{i>0} Sq^i(\sum_{j=1}^{r} h_i\pi_j).$$

But, as π_j runs over all even permutations of Σ_n, $\pi'\pi_j$ runs over all odd permutations of Σ_n. Hence

$$\sum_{j=1}^{r}(f + f\pi')\pi_j = \sum_{i=1}^{t} f\pi_i = \tau(f) = \sigma(f),$$

The result now follows from Proposition 3.2. by taking $g = 0$.

Remark. *In some circumstances, to prove that a monomial f is not hit in $\mathbf{P}(n)$, which cannot be solved directly, we take the problem to $\mathbf{B}(n)$ and prove that, under suitable conditions, $\sigma(f)$ is not symmetrically hit or, for some $g \in \mathbf{P}(3)$, $\sigma(f) \cong \sigma(g)$ and $\sigma(g)$ is not symmetrically hit. The author himself has used frequently this technique, specially for 3-variable case, in* ([5, 9]). *The basic tool in this situations, of course, is Theorem* 1.3.

The converse of Theorem 1.3. is not true. The following counterexample shows this fact.

Example 3.6. Let F be the following block.

$$\begin{matrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 0 & 1 & - & 1 \end{matrix}$$

Then it can be easily checked that all permutations of F are equivalent to each other and hence $\sigma(F)$ is symmetrically hit, while it is clear from [9, Lemma 2.6] that F is non-hit.

References

[1] J. Adem, The iteration of Steenrod squares in algebraic topology, *Proc. Nat. Acad. Sci. U.S.A.* 38 (1952), 720-726.

[2] J. M. Boardman, Modular representations on the homology of powers of real projective spaces, *Algebraic Topology, Oaxtepec 1991, Contemp. Math.* 146 (1993), 49-70.

[3] H. E. A. Campbell and D. L. Wehlau, *Invariant Theory in All Characters*, CRM Proceedings and Lecture Notes, vol. 35 American Mathematical Society, 2004.

[4] V. Giambalvo and F. P. Peterson, A-generators for ideals in the Dickson algebra, *J. of Pure and Applied Algebra*, 158 (2001), 161-182.

[5] A. S. Janfada, *The hit problem for symmetric polynomials over the Steenrod Algebra*, thesis, Manchester University, 2000.

[6] A. S. Janfada and R. M. W. Wood, The hit problem for symmetric polynomials over the Steenrod Algebra, *Math. Proc. Camb. Phil. Soc.* 133 (2002), 295-303.

[7] A. S. Janfada and R. M. W. Wood, Generating $H^*(BO(3), F_2)$ as a module over the Steenrod algebra, *Math. Proc. Camb. Phil. Soc.* 134 (2003), 239-258.

[8] A. S. Janfada, A note on the unstability condition of the Steenrod squares on the polynomial algebra, *J. Korean Math. Soc.* 46 (5) (2009), 907-918.

[9] A. S. Janfada, A criterion for a monomial in $P(3)$ to be hit, *Math. Proc. Camb. Phil. Soc.* 145(2008), 587-599.

[10] A. S. Janfada, Criteria for a symmetrized monomial in $B(3)$ to be non-hit, *preprint.*

[11] M. Kameko, *Products of projective spacec as Steenrod modules* Thesis, John Hopkin University, 1990.

[12] M. Kameko, Generators of the cohomology of BV_3, *J. Math. Kyoto Univ.,* 38 (1998), 587-593.

[13] D. M. Mayer and L. Smith, *Poincaré duality algebras, Macaulay's dual systems, and Steenrod operations,* Cambridge University Press, 2005.

[14] T. N. Nam, $\mathcal{A} -$ *générateurs génériques l'algèbra* polynomiae, *Adnances in Mathematics*, 186 (2004), 334-362.

[15] D. J. Pengelly and F. Williams, Global structure of the mod two symmetric algebra $H^*(BO; \mathbb{F}_2)$, over the Steenrod algebra, *Algebraic & Geometric Topology,* 3 (2003), 1119-1138.

[16] D. J. Pengelly and F. Williams, The hit problem for symmetric algebras over the Steenrod algebra, *preprint.*

[17] F. P. Peterson, Generators of $H^*(RP^\infty \wedge RP^\infty)$ as a module over the Steenrod algebra, *Abstracts Amer. Math. Soc.* (1987), 833-55-89.

[18] F. P. Peterson, $\mathcal{A}$-generators for certain polynomial algebras, *Math. Proc. Camb. Phil. Soc.* 105 (1989), 311-312.

[19] W. Singer, On the action of Steenrod squares on polynomial algebras, *Proc. Amer. Math. Soc.* 111 (1991), 577-583.

[20] W. M. Singer, *Steenrod squares in spectral sequences*, Mathematical Surveys and Monograph, vol. 129, American Mathematical Society, 2006.

[21] W. M. Singer, Rings of symmetric sunctions as modules over the Steenrod algebra, *Programs of Thirty Years of the Research Seminar, 'Invariantantheorie und Algebraische Topologri', Göttengen, September*, (2007), 27-29.

[22] W. M. Singer, Rings of symmetric sunctions as modules over the Steenrod algebra, *Algebraic & Geometric Topology,* **8** (2008), 541-562.

[23] L. Smith, *polynomial invariants of finite groups*, Res. Notes Math., vol. 6, A. K. Peters Ltd., Wellesley, MA, 1995.

[24] N. E. Steenrod and D. B. A. Epstein, *Cohomology operations*, Annals of Math. Studies 50 (Princeton University Press), 1962.

[25] N. Sum *The hit problem for the polynomial algebra of four variables*, Seoul, November 3, 2007.

[26] R. Thom, Quelque propriétés globales des variétés différentiables, *Comment. Math. Helv.* 28 (1954), 17-86.

[27] R. M. W. Wood, Steenrod squares of Polynomials, Advances in homotopy theory, *London Mathematical Society Lecture Notes, Cambridge University Press,* 139 (1989), 173-177.

[28] R. M. W. Wood, Steenrod squares of polynomials and the Peterson conjecture, *Math. Proc. Camb. Phil. Soc.* 105 (1989), 307-309.

[29] R. M. W. Wood, Problems in the Steenrod algebra, *Bull. London Math. Soc.* 30 (1998), 449-517.

[30] R. M. W. Wood, Hit problems and the Steenrod algebras, *Proceeding of the Summer School 'Interactions between Algebraic Topology and Invariant Theory', Ioninna University, Greec* (2000), 65-103.

Ali Sarbaz Janfada
Department of Mathematics
Urmia University
P.O.Box 165
Urmia, Iran
E-mail: a.sjanfada@urmia.ac.ir