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$\mathbb{A}$-Best Approximation in Pre-Hilbert C^*-Modules

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Abstract. While there have been many number of studies about best approximation in some spaces, there has been little work on pre-Hilbert C^*-modules. Here we provide such a study that leads to a number of approximation theorems. In particular, some results about existence and uniqueness of best approximation of submodules on Hilbert C^*-modules are also presented. This will be done by considering the C^*-algebra valued map $x \to |x|$ where $|x| = \langle x, x\rangle^{\frac{1}{2}}$. Also we show that when K is a convex subset of a pre- Hilbert C^*-module X; it is a Chebyshev set with respect to C^*-valued norm which is defined on X. In the end, we study various properties of an $\mathbb{A}$-valued metric projection onto a convex set and a submodule.

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1. Introduction

As we know, the best approximation problem in the normed linear spaces, has many important applications in mathematics and some other sciences. In general, this problem has been generalized in two directions. On the one side, the property sets that element best approximation obtained from it, for example

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convexity and non-convexity property. On the other side, the action spaces are replaced by various metric and normed spaces.
There are many researches about best approximation in inner product spaces(see, e.g., [8]) and Banach spaces (see, e.g., [4, 14]). Recently, in [2, 15, 16] the authors introduced the notion of cone metric and cone normed spaces. After cone metric spaces over topological vector spaces, in [13] the authors introduce the concept of tvs-cone b-metric spaces over a solid cone. In the following section another state of metric spaces is a cone metric spaces with Banach algebras. According to our records, we expand the theory of best approximation to inner-product C^*-modules. The paper is organized as follows. In Section 2., we give some preliminary results and facts about module spaces, and properties of C^*-algebras. In Section 3., in particular, we get some results about existence and uniqueness of best approximation of submodules on inner-product C^*-modules. Also various properties of an $\mathbb{A}$-valued metric projection onto a convex set (or a submodule), when $\mathbb{A}$ is a commutative C^*-algebra, has also been studied.

2. Preliminaries

It is well known that an algebra $\mathbb{A}$, together with a conjugate linear involution map $* : a \to a^*$ is called a $*$-algebra if $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(\lambda a + b)^* = \overline{\lambda}a^* + b^*$ for all $a, b \in \mathbb{A}, \lambda \in \mathbb{C}$. Moreover, the pair $(\mathbb{A}, *)$ is called a unital $*$-algebra if $\mathbb{A}$ contains the identity element. By a Banach $*$-algebra we mean a complete normed unital $*$-algebra $(\mathbb{A}, *)$ such that the norm on $\mathbb{A}$ is submultiplicative and satisfied $\|a^*\| = \|a\|$ for $a \in \mathbb{A}$.
Furthermore in a Banach $*$-algebra $(\mathbb{A}, *)$, if we have $\|a^*a\| = \|a\|^2$ for all $a \in \mathbb{A}$, then $\mathbb{A}$ is known as a C^*-algebra. $\mathbb{A}$ positive element a of a C^*-algebra $\mathbb{A}$ is a self-adjoint element whose spectrum $\sigma(a)$ is contained in $[0, \infty)$. If $a \in \mathbb{A}$ is positive, we write $a \geqslant 0$ and denote by $\mathbb{A}^+$ the set of all positive elements of $\mathbb{A}$. By [Theorem 4.2.2 [10]] $\mathbb{A}^+$ is a pointed, closed and convex cone i.e. $\mathbb{A}^+$ is closed and

(a) $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta \geqslant 0$, $a, b \in \mathbb{A}^+$ imply that $\alpha a + \beta b \in \mathbb{A}^+$;

(b) $\mathbb{A}^+ \cap (-\mathbb{A}^+) = \{0\}$.

Using positive elements, one can define a partial ordering on the set of self-adjoint elements of a C^*-algebra $\mathbb{A}$ as follows: $a \geqslant b$ if and only if $a - b \in \mathbb{A}^+$.
Let $\mathbb{A}$ be a C^*-algebra. A pre-Hilbert $\mathbb{A}$-module X which is a left $\mathbb{A}$-module, together with an $\mathbb{A}$-valued mapping $\langle . \rangle : X \times X \to \mathbb{A}$ with the following properties:

a) $\langle x, x \rangle \geqslant 0_{\mathbb{A}}$ for every x and $\langle x, x \rangle = 0_{\mathbb{A}}$ if and only if $x = 0_X$.

b) $\langle x, y\rangle^* = \langle y, x\rangle$ for every $x, y \in X$.

c) $\langle \alpha x + \beta z, y\rangle = \alpha\langle x, y\rangle + \beta\langle z, y\rangle$ for $\alpha, \beta \in \mathbb{C}$.

d) $\langle ax, y\rangle = a\langle x, y\rangle$ for $a \in \mathbb{A}$.

The map $\langle ., .\rangle$ is called the $\mathbb{A}$-valued inner product on X. A pre-Hilbert $\mathbb{A}$-module $(X; \langle ., .\rangle)$ is called Hilbert $\mathbb{A}$-module if it is complete with respect to the norm $\|.\| = \|\langle ., .\rangle\|^{\frac{1}{2}}$.
In particular any C^*-algebra is a Hilbert module over itself. On the other hand any Hilbert module over the field of complex numbers $\mathbb{C}$ is a Hilbert space. Thus Hilbert C^*-modules generalize both C^*-algebras and Hilbert spaces. There are some similarities between Hilbert C^*-modules and Hilbert spaces, but there is a fundamental way in which Hilbert C^*-modules differ from Hilbert spaces. More information about Hilbert C^*-modules can be found in [11].
Let H be Hilbert space and K a closed convex subset of H, then there exists a unique element $\mathbf{P}_K(x)$ of K such that

$$\|x - \mathbf{P}_K(x)\| = \inf_{k\in K} \|x - k\|.$$

But Cheney and Wulber [5] have studied subspaces of Hilbert $C(X)$-module $C(X)$ which neither existence nor uniqueness of projection holds. When K is a convex subset of a pre- Hilbert C^*-module X, the uniqueness condition is satisfed with respect to C^*- valued norm [8].
Let us start with some basic definitions, which will be used later. We will present the definition of Banach cone algebras and some properties related to this concept.

Definition 2.1. [1, 2] *Let X be a space and P be a cone subset of a Banach space $\mathbb{A}$. Consider the mapping $||.||_{\mathbb{A}} : X \to P$ satisfies*

(1) $||x||_{\mathbb{A}} \geqslant 0_{\mathbb{A}}$ for all $x \in X$ and $||x||_{\mathbb{A}} = 0_{\mathbb{A}} \Leftrightarrow x = 0_X$;

(2) $||\alpha x||_{\mathbb{A}} = |\alpha|\ ||x||_{\mathbb{A}}$ for all $x \in X$ and $\alpha \in \mathbb{R}$;

(3) $||x + y||_{\mathbb{A}} \leqslant ||x||_{\mathbb{A}} + ||y||_{\mathbb{A}}$ for all $x, y \in X$.

Then $||.||_{\mathbb{A}}$ is called a cone norm on X with respect to $\mathbb{A}$ and $(X, ||.||_{\mathbb{A}})$ is called a cone normed space.

Example 2.2. [2] Let $X = \mathbb{R}^2$, $P = \{(x, y) : x \geqslant 0, y \geqslant 0\} \subset \mathbb{R}^2$ and $\|(x, y)\| = (\alpha|x|, \beta|y|)$ where $\alpha, \beta > 0$ are fixed. Then $(X, ||.||)$ is a cone normed space over $\mathbb{R}^2$.

Example 2.3. [6] Let $X = C^1_{\mathbb{R}}[0, 1]$ with $||x|| = ||x||_\infty + ||x'||_\infty$ and $P = \{x \in X | x(t) \geqslant 0\}$ is cone normed space.

Definition 2.4. *Let $(X, ||.||)$ be a cone normed space, $\{x_n\}$ be a sequence in X and $x \in X$. The sequence $\{x_n\}$ is said to be convergent to x, if for any $c > 0$ there is $N_0 \in \mathbb{N}$ such that for all $n > N_0$, $||x_n - x|| \leqslant c$. We denote it by* $\lim x_n = x$. *Likewise, $\{x_n\}$ is called a Cauchy sequence in X if for any $c > 0$ there is $N_0 \in \mathbb{N}$ such that for all $n, m \geqslant N_0$, $||x_n - x_m|| \leqslant c$. A cone norm space X is said to be complete if every Cauchy sequence in X is convergent in X. Complete cone normed spaces are called cone Banach spaces.*

A set K in a cone normed space X is compact if for every sequence $\{x_n\}$ in K there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a point $x \in K$ such that $x_{n_k} \to x$.

Definition 2.5. *Let K be a nonempty subset of a cone normed space X, $x \in X$ and $\mathbb{A}$ be a Banach space. An element $k_0 \in K$ is called an $\mathbb{A}$-best approximation from K to x if*

$$||x - k_0||_{\mathbb{A}} = d(x, K) = \inf_{k \in K} ||x - k||_{\mathbb{A}}. \tag{1}$$

The set of all $\mathbb{A}$-best approximation points to x from K is denoted by $\mathbf{P}_K^A(x)$. Thus

$$\mathbf{P}_K^A(x) := \{k_0 \in K : \ ||x - k_0||_{\mathbb{A}} = d(x, K)\}. \tag{2}$$

If for each $x \in X$ corresponds at least(respectively exactly) one $\mathbb{A}$-best approximation in K then K is called a proximinal(respectively Chebyshev)set.

3. $\mathbb{A}$-best approximation in pre-Hilbert C^*-modules

We now establish some interesting results of $\mathbb{A}$-best approximation in pre-Hilbert C^*-modules. Let X be an inner-product $\mathbb{A}$-module such that $\langle X; X\rangle$ is commutative, where $\langle X; X\rangle = cl\ span\{\langle x, y\rangle : \ x, y \in X\}$. Now define

$$\|x\|_{\mathbb{A}} := |x| = \langle x, x\rangle^{\frac{1}{2}}, \forall x \in X.$$

We show that $\|.\|_{\mathbb{A}}$ is a cone norm on X. In fact since $\langle x, x\rangle \geqslant 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$, therefore $\|x\|_{\mathbb{A}} \geqslant 0$ and $\|x\|_{\mathbb{A}} = 0$ if and only if $x = 0$.
For $\alpha \in \mathbb{R}$, $||\alpha x||_{\mathbb{A}}^2 = \langle \alpha x, \alpha x\rangle = |\alpha|^2 \langle x, x\rangle$, then $||\alpha x||_{\mathbb{A}} = |\alpha|\ ||x||_{\mathbb{A}}$.
More precisely triangle inequality is satisfied for $\mathbb{A}$-valued norms if and only if $\langle X; X\rangle$ is commutative(see e.g., [3, 9]).
It is clear that when $\mathbb{A}$ is commutative, $(X, |.|)$ is cone normed space. In general $|.|$ is not actually a norm, because it might not satisfies $|x + y| \leqslant |x| + |y|$ (see e.g., [11]).
In the following, we shall prove some theorems of generalized best approximation in the setting of cone normed pre-Hilbert C^*-modules X over a commutative C^*-algebra $\mathbb{A}$.

Theorem 3.1. *Every complete convex set in a pre-Hilbert C^*-module X over a commutative C^*-algebra $\mathbb{A}$ is proximinal.*

Proof. Let K be a complete convex set in X and fix any $x \in X$. Suppose that $\{y_n\}$ be a sequence in K such that $\lim_{n\to\infty} |x - y_n| = d(x, K)$. Then by the parallelogram law we have

$$\begin{aligned} |y_n - y_m|^2 &= |(x - y_m) - (x - y_n)|^2 \\ &= 2(|x - y_m|^2 + |x - y_n|^2) - |2x - (y_n + y_m)|^2 \\ &= 2(|x - y_m|^2 + |x - y_n|^2) - 4|x - (\frac{y_n + y_m}{2})|^2. \end{aligned}$$

Since K is convex, $\frac{y_n+y_m}{2} \in K$, and hence

$$|y_n - y_m|^2 \leqslant 2|x - y_m|^2 + 2|x - y_n|^2 - 4d(x, K)^2.$$

It follows that the right side tends to zero as n and m tend to infinity. Thus $\{y_n\}$ is a Cauchy sequence. Since K is complete set, y_n converges to some point $y \in K$. This proves that K is proximinal. □

Example 3.2. Let $X = \{a, b\}$, $\mathbb{A} = C(X)$ and $E = \{f \in C(X) : f(a) = 0\}$. It is obvious that E is a maximal ideal of the C^*-algebra $\mathbb{A}$ and so can be regarded as a Hilbert $\mathbb{A}$-module. Assume that $K = con(f_1) \subseteq E$ where $f_1(b) = 1$. The set K is proximinal, since for each $f \in E$ a straightforward verification shows that if $f(b) \leqslant 0$ then $\mathbf{P}_K(f) = 0$, if $f(b) \geqslant 1$ then $\mathbf{P}_K(f) = f_1$ and if $0 \leqslant f(b) \leqslant 1$, $\mathbf{P}_K(f) = f$.

Theorem 3.3. *Let K be a convex subset of X. Then for each $x \in X$ has at most one best approximation in K.*

Proof. Let $x \in X$ and suppose $y_1, y_2 \in \mathbf{P}_K^{\mathbb{A}}(x)$. Then $\dfrac{y_1 + y_2}{2} \in K$ by convexity. By the parallelogram law, we have

$$\begin{aligned} 0 \leqslant |y_1 - y_2|^2 &= |(x - y_2) - (x - y_1)|^2 \\ &= 2(|x - y_2|^2 + |x - y_1|^2) - |2x - (y_1 + y_2)|^2 \\ &= 2(|x - y_2|^2 + |x - y_1|^2) - 4|x - (\frac{y_1 + y_2}{2})|^2 \\ &\leqslant 4d(x, K)^2 - 4d(x, K)^2 = 0. \end{aligned}$$

This follows that $y_1 = y_2$. □

Definition 3.4. *Let S be any nonempty subset of the inner product space X. The dual cone (or negative polar) of S is the set*

$$S^\circ := \{x \in X \mid Re\langle x, y\rangle \leqslant 0 \; for \; y \in S\}. \tag{3}$$

The orthogonal complement of S is the set

$$S^{\perp} := \{x \in X \mid \langle x, y\rangle = 0 \ for \ y \in S\}. \tag{4}$$

Corollary 3.5. *Let K be a convex subset of X, $x \in X$ and $y_0 \in K$. Then $y_0 \in \mathbf{P}_K^{\mathbb{A}}(x)$ if and only if $x - y_0 \in (K - y_0)^{\circ}$.*

Proof. Suppose $y_0 \in \mathbf{P}_K^{\mathbb{A}}(x)$ and $y \in K$. For each $0 < \lambda < 1$ the element $y_\lambda = \lambda y + (1-\lambda)y_0$ is in K by convexity and

$$\begin{aligned} 0 &\geqslant |x - y_0|^2 - |x - y_\lambda|^2 \\ &= |x - y_0|^2 - |x - y_0 - \lambda(y - y_0)|^2 \\ &= -2Re\lambda\langle x - y_0, y_0 - y\rangle - \lambda^2|y - y_0|^2, \end{aligned}$$

Thus

$$2Re\langle x - y_0, y_0 - y\rangle + \lambda|y - y_0|^2 \geqslant 0.$$

Now if $\lambda \to 0$ then $Re\langle x - y_0, y - y_0\rangle \leqslant 0$.
Conversely, if $x - y_0 \in (K - y_0)^{\circ}$ by (3) we have $Re\langle x - y_0, y - y_0\rangle \leqslant 0$ for $y \in K$. Therefore

$$\begin{aligned} |x - y_0|^2 - |x - y|^2 &= |x - y_0|^2 - |x - y_0 + y_0 - y|^2 \\ &= -|y - y_0)|^2 - 2Re\lambda\langle x - y_0, y_0 - y\rangle \leqslant 0. \end{aligned}$$

Thus $|x - y_0| \leqslant |x - y|$. So $y_0 \in \mathbf{P}_K^{\mathbb{A}}(x)$. □

Corollary 3.6. *Let K be a convex cone in X, $x \in X$ and $y_0 \in K$. Then $y_0 \in \mathbf{P}_K^{\mathbb{A}}(x)$ if and only if $x - y_0 \in K^{\circ} \cap y_0^{\perp}$.*

Proof. By Corollary 3.5, $z \in \mathbf{P}_K^{\mathbb{A}}(x)$ if and only if $x - z \in (K - y_0)^{\circ}$. From (3), Re $\langle z, k - y_0\rangle \leqslant 0$ for all $k \in K$. Taking $k = 0$ and $k = 2y_0$ it follows that the last statement is equivalent to $\langle z, y_0\rangle = 0$ and Re $\langle z, k\rangle \leqslant 0$ for all $k \in K$. That is, $z \in K^{\circ} \cap y_0^{\perp}$. □

Theorem 3.7. *Let M be a submodule of X, $x \in X$ and $y_0 \in M$. Then $y_0 \in \mathbf{P}_M^{\mathbb{A}}(x)$ if and only if $x - y_0 \in M^{\perp}$.*

Proof. Since M is a submodule, then $(M - y_0)^{\circ} = M^{\circ}$ and $-M = M$ implies $M^{\perp} = M^{\circ} \cap (-M)^{\circ} = M^{\circ}$. Now by Corollary 3.5, $y_0 \in \mathbf{P}_M^{\mathbb{A}}(x)$ if and only if $x - y_0 \in M^{\perp}$. □

Theorem 3.8. *Let K be a closed convex subset of X. Then*

(1) For $x \in X$,

$$x = \mathbf{P}_K^{\mathbb{A}}(x) + \mathbf{P}_{K^\circ}^{\mathbb{A}}(x), \qquad \mathbf{P}_K^{\mathbb{A}}(x) \perp \mathbf{P}_{K^\circ}^{\mathbb{A}}(x).$$

(2) $|x|^2 = |\mathbf{P}_K^{\mathbb{A}}(x)|^2 + |\mathbf{P}_{K^\circ}^{\mathbb{A}}(x)|^2$.

(3) $|\mathbf{P}_K^{\mathbb{A}}(x)| \leqslant |x|$ *for $x \in X$.*

Proof.

1) Let $x \in X$ and $k_0 = x - \mathbf{P}_K^{\mathbb{A}}(x)$. By Corollary 3.5, $k_0 \in K^\circ$ and $k_0 \perp (x - k_0)$. For every $y \in K^\circ$,

$$Re\langle x - k_0, y\rangle = Re\langle \mathbf{P}_K^{\mathbb{A}}(x), y\rangle \leqslant 0.$$

Hence $x - k_0 \in (K^\circ)^\circ$. By Theorem 3.5 (applied to K° rather than K), we get that $k_0 = \mathbf{P}_{K^\circ}^{\mathbb{A}}(x)$. This proves that, $x = \mathbf{P}_K^{\mathbb{A}}(x) + \mathbf{P}_{K^\circ}^{\mathbb{A}}(x)$, and $\mathbf{P}_K^{\mathbb{A}}(x) \perp \mathbf{P}_{K^\circ}^{\mathbb{A}}(x)$. To complete the proof, we must verify the uniqueness of this representation. Let $x = y + z$, where $y \in K$, $z \in K^\circ$ and $y \perp z$. For each $k \in K$,

$$Re\langle x - y, k\rangle = Re\langle z, k\rangle \leqslant 0,$$

and

$$\langle x - y, y\rangle = \langle z, y\rangle = 0.$$

By Theorem 3.5, $y = \mathbf{P}_K^{\mathbb{A}}(x)$. Similarly, $z = \mathbf{P}_{K^\circ}^{\mathbb{A}}(x)$.

2) The first statement follows from (1) and the Pythagorean theorem.

3) This follows using (2). □

Corollary 3.9. *Let M be a closed complemented submodule of X. Then the statements are true :*

(1) $Id = \mathbf{P}_M^{\mathbb{A}} + \mathbf{P}_{M^\perp}^{\mathbb{A}}$.

(2) For $x \in X$, $|x|^2 = |\mathbf{P}_M^{\mathbb{A}}(x)|^2 + |\mathbf{P}_{M^\perp}^{\mathbb{A}}(x)|^2$.

(3) $|\mathbf{P}_M^{\mathbb{A}}(x)| \leqslant |x|$ *for $x \in X$.*

Proof. It is a direct consequently of Theorems 3.7 and 3.8. □

Theorem 3.10. *Let M be a Chebyshev submodule of X. Then*

(1) $\mathbf{P}_M^{\mathbb{A}}$ *is a bounded linear operator and* $\|\mathbf{P}_M^{\mathbb{A}}\| = 1$ *(unless $M = \{0\}$).*

(2) $\mathbf{P}_M^{A}$ *is idempotent:* $\mathbf{P}_M^{A^2} = \mathbf{P}_M^{\mathbb{A}}$.

(3) $\mathbf{P}_M^{\mathbb{A}}$ *is nonnegative:* $\langle \mathbf{P}_M^{\mathbb{A}}(x), x\rangle \geqslant 0$.

Proof.

(1) Let $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$. By Theorem 3.7, $x - \mathbf{P}_M^{\mathbb{A}}(x)$ and $y - \mathbf{P}_M^{\mathbb{A}}(y)$ are in $M^\perp$. Since $\alpha \mathbf{P}_M^{\mathbb{A}}(x) + \beta \mathbf{P}_M^{\mathbb{A}}(y) \in M$ and $M^\perp$ is a subspace, Theorem 3.7 implies that $\alpha \mathbf{P}_M^{\mathbb{A}}(x) + \beta \mathbf{P}_M^{\mathbb{A}}(y) = \mathbf{P}_M^{\mathbb{A}}(\alpha x + \beta y)$. Thus $\mathbf{P}_M^{\mathbb{A}}$ is linear. By using part (3) of Theorem 3.8, we get $\|\mathbf{P}_M^{\mathbb{A}}(x)\| \leqslant \|x\|$ for all x. Thus $\mathbf{P}_M^{\mathbb{A}}$ is bounded and $\|\mathbf{P}_M^{\mathbb{A}}\| \leqslant 1$. Since $\mathbf{P}_M^{\mathbb{A}}(y) = y$ for every $y \in M$ and hence $\|\mathbf{P}_M^{\mathbb{A}}\| = 1$.

(2) This follows because each $m \in M$ is its own best approximation in M.

(3) For any x in X, Theorem 3.7 implies that $\langle \mathbf{P}_M^{\mathbb{A}}(x), x - \mathbf{P}_M^{\mathbb{A}}(x) \rangle = 0$, hence $\langle \mathbf{P}_M^{\mathbb{A}}(x), x \rangle = \langle \mathbf{P}_M^{\mathbb{A}}(x), \mathbf{P}_M^{\mathbb{A}}(x) \rangle \geqslant 0$. □

Theorem 3.11. *Let K be a convex subset of X and M be any Chebyshev submodule of X that contains K. Then:*

(1) $\mathbf{P}_M^{\mathbb{A}} \mathbf{P}_K^{\mathbb{A}} = \mathbf{P}_K^{\mathbb{A}}$.

(2) $d(x, K)^2 = d(x, M)^2 + d(\mathbf{P}_M^{\mathbb{A}}, K)^2$ *for every* $x \in X$.

Proof. The statement $\mathbf{P}_K^{\mathbb{A}} \subseteq \mathbf{P}_M^{\mathbb{A}} \mathbf{P}_K^{\mathbb{A}}$ is obvious, since $K \subseteq M$. For any $y \in K$, we have $y \in M$ and $x - \mathbf{P}_M^{\mathbb{A}}(x) \in M^\perp$ by Theorem 3.7, so that

$$|x - y|^2 = |x - \mathbf{P}_M^{\mathbb{A}}(x)|^2 + |\mathbf{P}_M^{\mathbb{A}}(x) - y|^2 \quad (*).$$

Since the first term on the right of $(*)$ is independent of y, it follows that $y \in K$ minimizes $|x - y|^2$ if and only if it minimizes $|\mathbf{P}_M^{\mathbb{A}}(x) - y|^2$. This proves that $\mathbf{P}_K^{\mathbb{A}}(x)$ exists if and only if $\mathbf{P}_K^{\mathbb{A}}(\mathbf{P}_M^{\mathbb{A}}(x))$ exists and $\mathbf{P}_K^{\mathbb{A}}(x) = \mathbf{P}_K^{\mathbb{A}}(\mathbf{P}_M^{\mathbb{A}}(x))$. □

Definition 3.12. *Let M be a closed submodule of X and $x \in X$. We consider the set*

$$\mathbf{R}_M^{\mathbb{A}}(x) := \{m_0 \in M : |m_0 - m| \leqslant |x - m|, for\ m \in M\}.$$

Theorem 3.13. *Let M be a closed submodule of X, $m_0 \in M$ and $x \in X$. Then $m_0 \in \mathbf{R}_M^{\mathbb{A}}(x)$ if and only if $M \subseteq (x - m_0)^\perp$.*

Proof. Let $m_0 \in \mathbf{R}_M^{\mathbb{A}}(x)$ and $m \in M$, then by the submodule of M, $m_1 = m_0 - \frac{1}{\lambda} m \in M$. By Definition 3.12 we have

$$|m| = |m_0 - m_1| \leqslant |\lambda x - \lambda m_1| = |m + \lambda(x - m_0)|.$$

Therefore $|m|^2 \leqslant |m + \lambda(x - m_0)|^2$. In case $\lambda \in \mathbb{R}$ we get

$$\langle m, x - m_0 \rangle + \langle x - m_0, m \rangle + \lambda \langle x - m_0, x - m_0 \rangle \geqslant 0, (\lambda \geqslant 0). \tag{5}$$

And

$$\langle m, x - m_0\rangle + \langle x - m_0, m\rangle + \lambda\langle x - m_0, x - m_0\rangle \leqslant 0, (\lambda \leqslant 0). \tag{6}$$

Taking $\lim_{\lambda\to 0^+}$ in (5) and $\lim_{\lambda\to 0^-}$ in (6) we obtain

$$\langle m, x - m_0\rangle + \langle x - m_0, m\rangle = 0. \tag{7}$$

And for $i\lambda$, we get

$$\langle m, x - m_0\rangle - \langle x - m_0, m\rangle = 0. \tag{8}$$

(7) and (8) implies that $\langle m, x - m_0\rangle = 0$.
Conversely, If $\langle m, x - m_0\rangle = 0$ we have

$$\begin{aligned} |m - m_0|^2 - |x - m|^2 &= |m - m_0|^2 - |x - m_0 + m_0 - m|^2 \\ &= -|x - m_0)|^2 - 2Re\lambda\langle x - m_0, m_0 - m\rangle \leqslant 0. \end{aligned}$$

Thus $|m - m_0| \leqslant |x - m|$. So $m_0 \in \mathbf{R}^{\mathbb{A}}_M(x)$. ☐

Corollary 3.14. *Let M be a closed complemented submodule of X. Then $\mathbf{R}^{\mathbb{A}}_M(x) = \mathbf{P}^{\mathbb{A}}_M(x)$ for $x \in X$.*

Proof. Let $m_0 \in \mathbf{R}^{\mathbb{A}}_M(x)$ by Theorem 3.13 we have $M \perp x - m_0$. Then for $m \in M$, $\langle x - m_0, m\rangle^* = \langle m, x - m_0\rangle = 0$. Therefore $\langle x - m_0, m\rangle = 0$. By Theorem 3.7, $m_0 \in \mathbf{P}^{\mathbb{A}}_M(x)$ thus $\mathbf{R}^{\mathbb{A}}_M(x) \subseteq \mathbf{P}^{\mathbb{A}}_M(x)$. Similarly we have $\mathbf{P}^{\mathbb{A}}_M(x) \subseteq \mathbf{R}^{\mathbb{A}}_M(x)$. Then $\mathbf{R}^{\mathbb{A}}_M(x) = \mathbf{P}^{\mathbb{A}}_M(x)$. ☐

Theorem 3.15. *Let M be a Chebyshev submodule of X and $x \in X$. Then*

(i) $\|\mathbf{R}^{\mathbb{A}}_M(x)\| \leqslant \|x\|$.

(ii) $\|x - \mathbf{R}^{\mathbb{A}}_M(x)\| \leqslant 2\|x - \mathbf{P}^{\mathbb{A}}_M(x)\|$.

Proof. (i) By Theorem 3.13 we have

$$\langle \mathbf{R}^{\mathbb{A}}_M(x), x - \mathbf{R}^{\mathbb{A}}_M(x)\rangle = 0.$$

Hence $|\mathbf{R}^{\mathbb{A}}_M(x)| \leqslant |\mathbf{R}^{\mathbb{A}}_M(x) + \lambda(x - \mathbf{R}^{\mathbb{A}}_M(x))|$ for $\lambda \in \mathbb{R}$. Putting $\lambda = 1$ we get

$$|\mathbf{R}^{\mathbb{A}}_M(x)| \leqslant |x|.$$

So $\|\mathbf{R}^{\mathbb{A}}_M(x)\| \leqslant \|x\|$.
(ii) Again from Theorem 3.13 we have

$$\langle \mathbf{R}^{\mathbb{A}}_M(x) - \mathbf{P}^{\mathbb{A}}_M(x), x - \mathbf{R}^{\mathbb{A}}_M(x)\rangle = 0,$$

hence
$$|\mathbf{R}_M^{\mathbb{A}}(x) - \mathbf{P}_M^{\mathbb{A}}(x)| \leqslant |\mathbf{R}_M^{\mathbb{A}}(x) - \mathbf{P}_M^{\mathbb{A}}(x) + \lambda(x - \mathbf{R}_M^{\mathbb{A}}(x))|,$$
and for $\lambda = 1$ we get
$$|\mathbf{R}_M^{\mathbb{A}}(x) - \mathbf{P}_M^{\mathbb{A}}(x)| \leqslant |x - \mathbf{P}_M^{\mathbb{A}}(x)|.$$
So $\|\mathbf{R}_M^{\mathbb{A}}(x) - \mathbf{P}_M^{\mathbb{A}}(x)\| \leqslant \|x - \mathbf{P}_M^{\mathbb{A}}(x)\|$. Therefore
$$\begin{aligned} \|x - \mathbf{R}_M^{\mathbb{A}}(x)\| &\leqslant \|x - \mathbf{P}_M^{\mathbb{A}}(x) + \mathbf{P}_M^{\mathbb{A}}(x) - \mathbf{R}_M^{\mathbb{A}}(x)\| \\ &\leqslant \|x - \mathbf{P}_M^{\mathbb{A}}(x)\| + \|\mathbf{P}_M^{\mathbb{A}}(x) - \mathbf{R}_M^{\mathbb{A}}(x)\| \\ &= 2\|x - \mathbf{P}_M^{\mathbb{A}}(x)\|. \quad \square \end{aligned}$$

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