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# Tensor Product of Operator-Valued Frames in Hilbert C*-Modules

**K. Musazadeh**
Islamic Azad University, Mahabad-Branch

**A. Khosravi**
Tarbiat Moallem University

**Abstract.** We show that the tensor product of two operator-valued frames for two Hilbert C*-modules is an operator-valued frame for the tensor product of these Hilbert C*-modules.

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## 1. Introduction

Frames on Hilbert C*-modules have been defined for unital C*-algebras by Frank and Larson [1] and investigated by many authors, see [2, 6, 11]. Recently, some generalizations of frames are proposed, for example, fusion frames, g-frames ([7]), operator-valued frames on Hilbert C*-modules for a unital C*-algebra ([4]). Furthermore, frames and bases in tensor products of Hilbert C*-modules have been studied in [6]. For more details about the tensor product of Hilbert spaces and C*-algebras we refer to [9]. We note that Hilbert C*-modules are used in the study of locally compact quantum groups, completely positive maps between C*-algebras, noncommutative geometry and K-theory. Also tensor product is useful in the approximation of multi-variate functions of combinations of univariate ones. In this section we recall some of the essential definitions and results which are needed in the sequel. For more details see [4].

Let $\mathcal{A}$ be a C*-algebra. We denote the Hilbert (right) C*-$\mathcal{A}$-module by $\mathcal{H}_{\mathcal{A}}$. The classic example of Hilbert (right) $\mathcal{A}$-module and the only one we will consider in this paper is the standard module $\mathcal{H}_{\mathcal{A}} := \ell^2(\mathcal{A})$, the space of all sequences $\{a_i\}_{i\in I} \subset \mathcal{A}$ such that $\sum_{i\in I} a_i^* a_i$ converges in norm to a positive element of $\mathcal{A}$. $\ell^2(\mathcal{A})$ is endowed with the natural linear structure and right $\mathcal{A}$-multiplication, and with the $\mathcal{A}$-valued inner product defined by $< \{a_i\}, \{b_i\} >= \sum_{i\in I} a_i^* b_i$, where the sum converges in norm by the Schwartz Inequality ([5]).

A map $T$ from $\mathcal{H}_{\mathcal{A}}$ to $\mathcal{H}_{\mathcal{A}}$ is adjointable if there is a map $T^* : \mathcal{H}_{\mathcal{A}} \to \mathcal{H}_{\mathcal{A}}$ such that $< T^*\xi, \eta >=< \xi, T\eta >$ for all $\xi, \eta \in \mathcal{H}_{\mathcal{A}}$, ([4]). The collection of adjointable operators is denoted by $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$. Then $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$ is a C*-algebra ([5]). For each pair of elements $\xi, \eta \in \mathcal{H}_{\mathcal{A}}$, a bounded rank-one operator is defined by $\theta_{\xi,\eta}(\zeta) = \xi < \eta, \zeta >$, for all $\zeta \in \mathcal{H}_{\mathcal{A}}$. The closed submodule of $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$ generated by rank-one operators is denoted by $\mathcal{K}(\mathcal{H}_{\mathcal{A}})$. When $\mathcal{A} = \mathbb{C}$, then $\mathcal{H}_{\mathcal{A}} = \ell^2$, $\mathcal{B}(\mathcal{H}_{\mathcal{A}}) = \mathcal{B}(\ell^2)$, and $\mathcal{K}(\mathcal{H}_{\mathcal{A}})$ coincides with the ideal $\mathcal{K}$ of all compact operators on $\ell^2$. $\mathcal{K}(\mathcal{H}_{\mathcal{A}})$ is always a closed ideal of $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$. The analog of the strong operator topology on $\mathcal{B}(\ell^2)$ is the *strict topology* on $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$ defined by

$$\mathcal{B}(\mathcal{H}_{\mathcal{A}}) \ni T_\lambda \to T \ \textit{strictly if} \ \|(T_\lambda - T)S\| \to 0 \ \textit{and} \ \|S(T_\lambda - T)\| \to 0, \forall S \in \mathcal{K}(\mathcal{H}_{\mathcal{A}}).$$

There is an alternate view of the objects $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$ and $\mathcal{K}(\mathcal{H}_{\mathcal{A}})$. Embed the tensor product $\mathcal{A} \otimes \mathcal{K}$ into its Banach space second dual $(\mathcal{A} \otimes \mathcal{K})^{**}$, which, as is well known, is a W*-algebra ([10]). The multiplier algebra of $\mathcal{A} \otimes \mathcal{K}$, is defined as the collection

$$\{T \in (\mathcal{A} \otimes \mathcal{K})^{**} : TS, ST \in \mathcal{A} \otimes \mathcal{K} \ \ \forall S \in \mathcal{A} \otimes \mathcal{K}\}.$$

Equipped with the norm of $(\mathcal{A}\otimes\mathcal{K})^{**}$, $M(\mathcal{A}\otimes\mathcal{K})$ is a C*-algebra. Assuming that $\mathcal{A}$ is unital we apply the following two *-isomorphisms without further references:

$$\mathcal{B}(\mathcal{H}_{\mathcal{A}}) \cong M(\mathcal{A} \otimes \mathcal{K}) \qquad \textit{and} \qquad \mathcal{K}(\mathcal{H}_{\mathcal{A}}) \cong \mathcal{A} \otimes \mathcal{K}.$$

The algebra $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$ is technically hard to work with, while $M(\mathcal{A} \otimes \mathcal{K})$ is more accessible due to many established results. More details on the subject can be found in ([6, 8]). We denote the tensor product of two Hilbert C*-modules $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_{\mathcal{B}}$ by $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$ which is a Hilbert $\mathcal{A} \otimes \mathcal{B}$-module, where $\mathcal{A}$ and $\mathcal{B}$ are C*-algebras. See ([6, 8]).

## 2. Operator-Valued Frames on Hilbert C*-Modules

We generalize an important result about the tensor product of frames on Hilbert C*-modules to operator-valued frames. First, we recall some definitions.

**Definition 2.1.** *According to* ([1]), *a (vector) frame on the Hilbert C*-module $\mathcal{H}_\mathcal{A}$ of a unital C*-algebra $\mathcal{A}$ is a collection of elements $\{\xi_i\}_{i\in I}$ in $\mathcal{H}_\mathcal{A}$ for which there are two positive scalars $a$ and $b$ such that for all $\xi \in \mathcal{H}_\mathcal{A}$,*

$$a < \xi, \xi > \leqslant \sum_{i\in I} < \xi, \xi_i >< \xi_i, \xi > \leqslant b < \xi, \xi >,$$

*where the convergence is in the norm of the C*-algebra $\mathcal{A}$.*

Let $\eta \in \mathcal{H}_\mathcal{A}$ be an arbitrary unital vector, i.e., $< \eta, \eta >= Id$, then by a result in ([4]), $E_0 := \theta_{\eta,\eta} \in \mathcal{A} \otimes \mathcal{K}$ is a projection. Then $\mathcal{H}_0 := E_0\mathcal{H}_\mathcal{A}$ is a submodule of $\mathcal{H}_\mathcal{A}$ and we can identify $E_0 M(\mathcal{A} \otimes \mathcal{K})$ with $B(\mathcal{H}_\mathcal{A}, \mathcal{H}_0)$, the set of linear bounded adjointable operators from $\mathcal{H}_\mathcal{A}$ to the submodule $\mathcal{H}_0$.

**Definition 2.2.** *Let $\mathcal{A}$ be a unital C*-algebra and $I$ be a countable index set. Let $E_0$ be a projection in $M(\mathcal{A} \otimes \mathcal{K})$. Denote by $\mathcal{H}_0$ the submodule $E_0\mathcal{H}_\mathcal{A}$ and identify $B(\mathcal{H}_\mathcal{A}, \mathcal{H}_0)$ with $E_0 M(\mathcal{A} \otimes \mathcal{K})$. A collection $\{\Lambda_i \in B(\mathcal{H}_\mathcal{A}, \mathcal{H}_0) : i \in I\}$ is called an operator-valued frame on $\mathcal{H}_\mathcal{A}$ with range in $\mathcal{H}_0$ if the sum $\sum_{i\in I} \Lambda_i^* \Lambda_i$ converges in the strict topology to a bounded invertible operator on $\mathcal{H}_\mathcal{A}$ denoted by $D_\Lambda$. $\{\Lambda_i\}_{i\in I}$ is called a tight operator-valued (resp., Parseval operator-valued frame) if $D_\Lambda = \lambda Id_{\mathcal{H}_\mathcal{A}}$ for a positive number $\lambda$ (resp., $D_\Lambda = Id_{\mathcal{H}_\mathcal{A}}$),* ([4]).

Notice that if $\{\Lambda_i\}_{i\in I}$ is an operator-valued frame with range in $\mathcal{H}_0$, then it is also an operator-valued frame with range in any larger submodule. For more details see ([4]). Most properties of frames on Hilbert spaces hold also for Hilbert C*-modules. Also Kaftal, Larson and Zhang ([4]) showed that most results for operator-valued frames on Hilbert spaces are also true for operator-valued frames on Hilbert C*-modules. In this section we have another result about this subject which is a generalization of a result in ([6]).

**Lemma 2.3.** *Let* $\{T_\alpha\}_\alpha \subset B(\mathcal{H}_\mathcal{A})$ *converges strictly to* $T$ *and* $\{S_\beta\}_\beta \subset B(\mathcal{H}_\mathcal{A})$ *converges strictly to* $S$. *Then* $\{T_\alpha \otimes S_\beta\}_{\alpha,\beta} \subset B(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{A})$ *converges strictly to* $T \otimes S$.

**Proof.** We first show that $\mathcal{K}(\mathcal{H}_\mathcal{A}) \otimes \mathcal{K}(\mathcal{H}_\mathcal{A})$ and $\mathcal{K}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{A})$ are at least algebraically equivalent. For this, let $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathcal{H}_\mathcal{A}$. Then for all $\zeta_1, \zeta_2 \in \mathcal{H}_\mathcal{A}$ we have

$$(\theta_{\xi_1,\eta_1} \otimes \theta_{\xi_2,\eta_2})(\zeta_1 \otimes \zeta_2) = \theta_{\xi_1,\eta_1}(\zeta_1) \otimes \theta_{\xi_2,\eta_2}(\zeta_2) = \xi_1 < \eta_1, \zeta_1 > \otimes \xi_2 < \eta_2, \zeta_2 >$$

$$= (\xi_1 \otimes \xi_2)(< \eta_1, \zeta_1 > \otimes < \eta_2, \zeta_2 >) = (\xi_1 \otimes \xi_2) < \eta_1 \otimes \eta_2, \zeta_1 \otimes \zeta_2 >$$
$$= \theta_{\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2}(\zeta_1 \otimes \zeta_2).$$

Therefore, $\theta_{\xi_1,\eta_1} \otimes \theta_{\xi_2,\eta_2} = \theta_{\xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2}$. Since $\mathcal{K}(\mathcal{H}_\mathcal{A}) \otimes \mathcal{K}(\mathcal{H}_\mathcal{A})$ and $\mathcal{K}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{A})$ are closed linear spans of these rank-one operators, the result follows. Now let $U, V \in \mathcal{K}(\mathcal{H}_\mathcal{A})$. Then $U \otimes V \in \mathcal{K}(\mathcal{H}_\mathcal{A}) \otimes \mathcal{K}(\mathcal{H}_\mathcal{A}) = \mathcal{K}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{A})$. So we have

$$\|(T_\alpha \otimes S_\beta - T \otimes S)(U \otimes V)\|$$
$$= \|(T_\alpha \otimes S_\beta - T_\alpha \otimes S + T_\alpha \otimes S - T \otimes S)(U \otimes V)\|$$
$$= \|[T_\alpha \otimes (S_\beta - S) + (T_\alpha - T) \otimes S](U \otimes V)\|$$
$$= \|[T_\alpha \otimes (S_\beta - S)](U \otimes V) + [(T_\alpha - T) \otimes S](U \otimes V)\|$$
$$\leqslant \|T_\alpha U \otimes (S_\beta - S)V\| + \|(T_\alpha - T)U \otimes SV\|$$
$$\leqslant \|T_\alpha U\| \|(S_\beta - S)V\| + \|(T_\alpha - T)U\| \|SV\|.$$

Since $\|(S_\beta - S)V\| \to 0$ and $\|(T_\alpha - T)U\| \to 0$ by the hypothesis,

$$\|(T_\alpha \otimes S_\beta - T \otimes S)(U \otimes V)\| \to 0.$$

Similarly, we can show that $\|(U \otimes V)(T_\alpha \otimes S_\beta - T \otimes S)\| \to 0$, Hence the result follows. □

**Proposition 2.4.** *Let* $\{\Lambda_i\}_{i\in I}$ *and* $\{\Gamma_i\}_{i\in I}$ *be operator-valued frames in* $B(\mathcal{H}_\mathcal{A}, \mathcal{H}_0)$. *Then* $\{\Lambda_i \otimes \Gamma_j\}_{i,j}$ *is an operator-valued frame in* $B(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{A}, \mathcal{H}_0 \otimes \mathcal{H}_0)$.

**Proof.** Suppose that $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$ are operator-valued frames in $B(\mathcal{H}_\mathcal{A}, \mathcal{H}_0)$. Then we have

$$\sum_{i,j}(\Lambda_i \otimes \Gamma_j)^*(\Lambda_i \otimes \Gamma_j) = \sum_{i,j}(\Lambda_i^* \otimes \Gamma_j^*)(\Lambda_i \otimes \Gamma_j) = \sum_{i,j}(\Lambda_i^*\Lambda_i \otimes \Gamma_j^*\Gamma_j)$$

$$= \sum_{i\in I}\Lambda_i^*\Lambda_i \otimes \sum_{j\in I}\Gamma_j^*\Gamma_j.$$

But $\sum_{i\in I}\Lambda_i^*\Lambda_i$ converges strictly to $S_\Lambda$ and $\sum_{j\in I}\Gamma_j^*\Gamma_j$ converges strictly to $S_\Gamma$ by the definition. Hence by the above lemma $\sum_{i\in I}\Lambda_i^*\Lambda_i \otimes \sum_{j\in I}\Gamma_j^*\Gamma_j$ converges strictly to $S_\Lambda \otimes S_\Gamma$, which is a bounded invertible operator on $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{A}$. □

Since by Theorem 1.4. of ([4]) a collection $\{\xi_i\}_{i\in I}$ in $\mathcal{H}_\mathcal{A}$ is a frame if and only if $\{A_i \in B(\mathcal{H}_\mathcal{A}, \mathcal{H}_0) : i \in I\}$ is an operator-valued frame in $\mathcal{H}_\mathcal{A}$, where $A_i(\xi) = < \xi, \xi_i >$, as a particular case of Proposition 2.4. we get a result proved in ([6]).

**Corollary 2.5.** *Let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be standard frames for $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$, respectively. Then $\{f_i \otimes g_j\}_{i,j\in I}$ is a standard frame for $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$.*

**Proof.** See ([6]). □

Generalized frames in Hilbert C*-modules is another aspect of operator-valued frames in Hilbert C*-modules which is introduced in ([7]).

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**Kamran Musazadeh**
Department of Mathematics
Islamic Azad University, Mahabad Branch
Mahabad, Iran.
E-mail: kamran_ms2004@yahoo.com

**Amir Khosravi**
Faculty of Mathematical Sciences and Computer
Tarbiat Moallem University
Tehran, Iran.
E-mail: khosravi_amir@yahoo.com