Journal of Mathematical Extension
Vol. 4, No. 1 (2009), 7-21

# Degree of Approximation and Green Potential *

**M. Simkani**
University of Michigan-Flint

**Abstract.** We will relate the degree of rational approximation of a meromorphic function $f$ to the minimum value, on the natural boundary of $f$, of Green potential of the weak* limit of the normalized pole-counting measures.

**AMS Subject Classification:** 30E10; 31A05.
**Keywords and Phrases:** Rational approximation, degree of approximation, Green potential.

## 1. Introduction

This paper is about the quantity

$$d = d(S, f, E) := \inf_{\{s_n\}\in S} \limsup_{n\to\infty} \|f - s_n\|_E^{\frac{1}{n}} \tag{1}$$

which we call *the degree of approximation from $S$ to $f$ on $E$*. Here, $E$ is a subset of $\overline{\mathbb{C}}$, the extended complex plane; $f$ is a function defined on $E$; $S$ is a class of sequences of functions; and $\|.\|_E$ denotes the sup-norm on $E$.

One of the earliest ideas for approximation came from Sangamagrama Madhava (1350-1425). He described–all in words–how to represent the length of a circular arc as an infinite series, in terms of sine and cosine of the corresponding central angle,

$$L = \frac{r\sin\theta}{\cos\theta} - \frac{r\sin^3\theta}{3\cos^3\theta} + \cdots$$

*Invited paper

for $0 \leqslant \theta \leqslant 45°$, which is equivalent to what we now call the Taylor/Maclaurin series for arc tangent.
To relate this historic example to our discussion here, let $f(x) = \arctan x$, and see whether the degree of approximation from the class $S$ of all sequences of polynomials $\{s_n\}$ (with real or complex coefficients) of respective degrees $\leqslant n$ is achieved by Madhava's series on a certain set $E$.
First, consider the case where $E = [-1, 1]$. If we let

$$s_n(x) = \sum_{k=1}^{n} \frac{\sin \frac{k\pi}{2}}{k} x^k \tag{2}$$

for $n = 1, 2, \ldots$, then for this sequence the lim-sup in (1) would equal 1, so $d \leqslant 1$. But from two classical results of Bernstein ([2,16]) we can see that $d = \sqrt{2} - 1 < 1$. Bernstein's theorems show that when $f$ is holomorphic inside an ellipse with foci $\pm 1$, and has at least one singularity on the ellipse, letting $\{s_n\}$ be the sequence of polynomials of best uniform approximation on $[-1, 1]$, the lim-sup in (1) equals the difference in the lengths of the semi-axes of the ellipse. In our case, the principle branch of $f(z) = \arctan(z)$ is holomorphic in $\overline{\mathbb{C}}$, minus its branch cut, which starts at $-i$, goes to $\infty$ along the negative imaginary axis, and then comes back along the positive imaginary axis to end at $i$. Therefore the largest ellipse within which $f$ is holomorphic has a minor semi-axis of length 1, and a major semi-axis of length $\sqrt{2}$, leading to $d = \sqrt{2} - 1 < 1$, since the inf in (1) is reached through the polynomials of best approximation.
Note that when we enlarge the interval $E$, $d$ moves up closer to 1, and when we shrink it, $d$ gets smaller. It is zero when $E = \{0\}$. In any case where $E$ is a real interval, $d$ is strictly less than the lim-sup in (1) with $s_n$ as in (2). For $E = [-a, a]$ with $0 < a \leqslant 1$, we have $d = (\sqrt{1 + a^2} - 1)/a$.

Next, let $E = \{z \in \mathbb{C} : |z| \leqslant a\}$ where $0 < a < 1$. From a more general result of Walsh ([18, §4.6]) we can see that if the lim-sup in (1) was smaller than $a$ for some sequence of polynomials $s_n$ of respective degree $\leqslant n$, then $f$, the arc tangent, would have to be holomorphic inside a disc larger than the unit disc, which we know it is not the case because the arc tangent has branch points at $\pm i$. On the other hand the sequence (2) obviously produces $a$ as the lim-sup in (1). Therefore $d = a$ in this case.

The above situations are all about polynomials, which can be considered as rational functions with poles at $\infty$. We shall see that in a more general situation we can allow the poles to converge (in some sense) to some arbitrary Borel measure whose support is not necessarily the point at $\infty$. This paper is an extension of what Sergei Natanovich Bernstein, Franciszek Leja, and Joseph Leonard Walsh studied. Numerous other mathematicians have worked on this and related areas. For just a few of those results see [1,3,4,5,6,7,8,10,11,12,14,15,16].

Let $f$ be meromorphic in $D$, a domain on the extended complex plane $\overline{\mathbb{C}}$. Suppose that $D \neq \overline{\mathbb{C}}$, and that the $\partial D$ is the natural boundary of $f$. Let $E$ be a closed subset of $D$ such that $f$ is holomorphic on $E$, and let $G := \overline{\mathbb{C}} \backslash E$. We assume that $G$ has finitely many connected components, each intersecting the $\partial D$. Let $\mu$ be a unit Borel measure with support $Supp(\mu) \subset G$, such that it intersects every connected component of $G$. Let $h_G^\mu$ be the Green potential of $\mu$ in $G$ ([9, §I.3], [13, §II.5], [16]). These are some of the properties of $h_G^\mu$:

(1) $h_G^\mu > 0$ in $G$.

(2) $h_G^\mu$ is lower semi-continuous on $\overline{\mathbb{C}}$, superharmonic in $G$, and harmonic in $G \backslash Supp(\mu)$.

(3) For quasi-every $y \in \partial G$ (for every $y$ except a subset of logarithmic capacity 0) we have

$$\lim_{x \to y, x \in G} h_G^\mu(x) = 0. \tag{3}$$

Let the $\partial G$ be regular, in the sense that (3) holds for all $y \in \partial G$. Let $H$ be the union of all those connected components of $\overline{\mathbb{C}} \backslash Supp(\mu)$ that intersect the $\partial G$. Then we define

$$E_\sigma^\mu := \{x \in H : h_G^\mu(x) < \log \sigma\}.$$

Given rational functions $r$ and $q$, we define the *pole-counting measures* $\pi^r$ and $\kappa_q^r$ as

$$\pi^r(A) := \text{the number of poles of } r \text{ in } A \text{ (counting multiplicities)}$$

and

$$\kappa_q^r(A) \quad : \quad = \text{the number of those poles of } r \text{ in } A$$
$$\text{that are not poles of } q \text{ (counting multiplicities)}$$

for every Borel set $A \subset \overline{\mathbb{C}}$.
We define the class $S^\mu$ as the collection of all infinite sequences $\{s_n\}$ of rational functions satisfying the following conditions:

1. $$\limsup_{n\to\infty} \frac{\pi^{s_n}(\overline{\mathbb{C}})}{n} \leqslant 1.$$
2. $$\kappa_{s_n}^{s_{n+1}}(\overline{\mathbb{C}}) = o\left(\frac{n}{\log n}\right).$$
3. $$\frac{1}{\pi^{s_n}(\overline{\mathbb{C}})}\pi^{s_n} \overset{\text{weak}^*}{\to} \mu.$$
4. The sequence $\{\pi^{s_n}(K)\}$ is bounded for every closed set $K \subset \overline{\mathbb{C}} \backslash Supp(\mu)$.

## 2. Main Results

The theorems of this section are corollaries to the main theorems of [16]. Let

$$-\log\delta = \min_{x\in\partial D} h_G^\mu(x) \leqslant \min_{x\in Supp(\mu)} h_G^\mu(x).$$

Then it follows from [16, Theorem 4] that we can find $\{t_n\} \in S^\mu$ such that

$$\limsup_{n\to\infty} \|f - t_n\|_E^{\frac{1}{n}} \leqslant \delta.$$

This proves

**Theorem 2.1.** *If*

$$\min_{x\in\partial D} h_G^\mu(x) \leqslant \min_{x\in Supp(\mu)} h_G^\mu(x),$$

*then*

$$d(S^\mu, f, E) \leqslant e^{-\min_{x\in\partial D} h_G^\mu(x)}.$$

Let

$$\min_{x\in\partial D} h_G^\mu(x) < -\log\delta < \min_{x\in Supp(\mu)} h_G^\mu(x).$$

Now suppose that

$$\limsup_{n\to\infty} \|f - t_n\|_E^{\frac{1}{n}} \leqslant \delta$$

for some sequence $\{t_n\} \in S^\mu$. Then by [16, Theorem 3] $f$ is meromorphic in

$$E_{1/\delta}^\mu = \{x \in H : h_G^\mu(x) < -\log\delta\}$$

which is a contradiction, since this set includes some points of the natural boundary of $f$. This proves:

**Theorem 2.2.** *If*

$$\min_{x\in\partial D} h_G^\mu(x) < \min_{x\in Supp(\mu)} h_G^\mu(x),$$

*then*

$$d(S^\mu, f, E) = e^{-\min_{x\in\partial D} h_G^\mu(x)}.$$

## 3. Some Special Cases

In this section we consider some special cases:

**3.1.** Let $f$ be an entire function but not a polynomial. Therefore the natural boundary of $f$ is $\{\infty\}$. Let $E$ be the unit disc, and $\mu$ be the unit mass at $\infty$. Then

$$h_G^\mu(x) = \max\{0, \log|x|\}.$$

The class $S^\mu$ includes the sequence of partial sums of the Taylor/Maclaurin series for $f$. In this case

$$d(S^\mu, f, E) = 0 = e^{-\min_{x\in\partial D} h_G^\mu(x)}.$$

**3.2.** Let $f$ be holomorphic in $\mathbb{C}\backslash\{0\}$, with essential singularities at 0 and $\infty$. Then the natural boundary of $f$ is $\{0, \infty\}$. Let $E$ be the unit circle. Let $\mu$ be a unit mass with mass $m > 0$ at 0, and mass $1 - m > 0$ at $\infty$. Then

$$h_G^\mu(x) = \max\{0, \log|x|\} - m\log|x|.$$

The class $S^\mu$ includes a sequence of partial sums of the Laurent series for $f$, rearranged so that the poles at 0 and $\infty$ remain in the same proportion as the mass of $\mu$ at 0 and $\infty$ asymptotically. In this case

$$d(S^\mu, f, E) = 0 = e^{-\min_{x\in\partial D} h_G^\mu(x)}.$$

**3.3.** Let $f$ be holomorphic in an annulus centered at the origin with radii $a$ and $b$ where $0 < a < 1 < b < \infty$. Let the boundary of this annulus be the natural boundary of $f$. Let $E$ and $\mu$ be as in the previous case. Then $h_G^\mu$ and $S^\mu$ are as in the previous case. In this case, by Theorem 2.2.

$$d(S^\mu, f, E) = e^{-\min_{x\in\partial D} h_G^\mu(x)} = \max\left\{a^m, b^{m-1}\right\}.$$

**3.4.** In the next section we will construct a special function as a rational series, that is holomorphic inside the unit disc, with the unit circle as its natural boundary. Let $f$ be that function. Let $E$ be the closed disc of radius $a < 1$, centered at the origin, and $\mu$ be the uniform distribution of a unit mass on the unit circle. Then the class $S^\mu$ includes the sequence of partial sums of the rational series (4) which gives

$$d(S^\mu, f, E) = 0.$$

(See Property 4 of $f$ in the next section.) On the other hand

$$h_G^\mu(x) = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{a^2 - xe^{-i\theta}}{ax - ae^{i\theta}}\right| d\theta.$$

On the natural boundary of $f$, the unit circle, $h_G^\mu$ is constant, positive, and finite. Therefore, in this case

$$d(S^\mu, f, E) = 0 < e^{-\min_{x\in\partial D} h_G^\mu(x)}.$$

Note that in cases 3.1. and 3.2. as well as in this case, the support of $\mu$ lies on the natural boundary of $f$, in which Theorem 2.1. applies. This case shows that in the conclusion of the theorem strict inequality is possible in some cases.

**3.5.** Up to this point, we have made the assumption that $f$ is holomorphic on $E$. In this special case we are replacing that condition with continuity. Let $E$ be the closed unit disc. As in the previous case, let $f$ be the function to be constructed in the next section, which is holomorphic in the $int(E)$ and continuous on $E$, with the $\partial E$ as its natural boundary. And let $\mu$ be as in the previous case. Then, in this case

$$d(S^{\mu}, f, E) = 0 < 1 = e^{-\min_{x \in \partial D} h_G^{\mu}(x)}.$$

## 4. A Special Rational Series

Here, we will construct a function $f$ in the form

$$f(z) = \sum_{k=1}^{\infty} \frac{b_k}{z_k - z} \tag{4}$$

for $z \in \Delta : |z| \leqslant 1$, with the properties that:

1. $f$ is continuous on $\Delta$,
2. $f$ is holomorphic in the $int(\Delta)$,
3. $f$ has the $\partial\Delta$ as its natural boundary,
4. $$\left\| \limsup_{n\to\infty} \frac{b_n}{z_n - \cdot} \right\|_{\Delta}^{\frac{1}{n}} = 0,$$
5. and every point of the $\partial\Delta$ is a limit point of the poles, in a way that
$$\frac{1}{\pi^{s_n}(\overline{\mathbb{C}})} \pi^{s_n} \overset{\text{weak}^*}{\longrightarrow} \mu,$$

where $s_n$ is the $n$-th partial sum of (4) and $\mu$ is the uniform distribution of a unit mass on the $\partial\Delta$.
(We used this special rational series for cases 3.4 and 3.5, but it has interesting properties for its own sake.)
Let us begin the construction:
Every positive integer $k$ can be uniquely written as

$$k = 2^{q_k} + r_k$$

where $q_k$ and $r_k$ are nonnegative integers and $0 \leqslant r_k < 2^{q_k}$. We define $\delta_k$, $\theta_k$, $z_k$, and $b_k$ respectively as

$$\begin{aligned}
\delta_k &= 2^{-2q_k}\pi^2 \\
\theta_k &= 2^{-q_k}\left(r_k + \frac{1}{2}\right)2\pi \\
z_k &= (1+\delta_k)e^{i\theta_k} \\
b_k &= \frac{1}{k^k}
\end{aligned}$$

for each positive integer $k$. Clearly $f$ is continuous on $\Delta$ and analytic in the $int(\Delta)$, so it remains to prove that the $\partial\Delta$ is the natural boundary of $f$.
For the following lemmas, we define $\varsigma_j$, $\lambda_{j,k}$, and $\tau_{j,k}$ respectively as

$$\begin{aligned}
\varsigma_j &= \frac{1}{2}e^{i\theta_j} \\
\lambda_{j,k} &= |z_k - \varsigma_j| \\
\tau_{j,k} &= \arg(z_k - \varsigma_j) - \theta_j
\end{aligned}$$

for $j, k \in \mathbb{N}$, where arg is chosen so that $-\pi < \tau_{j,k} \leqslant \pi$.
For each $j \in \mathbb{N}$, we define $M_j \subset \mathbb{N}$ as the set of all positive integers $m$ with the following properties:

1. $m > \max\{j, 110\}$
2. $\theta_m - \theta_j = 2^{-q_m}\pi$
3. $q_{m-2} = q_{m-1} = q_m = q_{m+1}$

Clearly $M_j$ is an infinite set. For convenience we state the following lemma, which is a consequence of the 1st property:

**Lemma 4.1.** *We have*

$$\left(\frac{1}{15}\right)\left(\frac{1}{44}\right)2^{2q_m}m\log m > 2^{q_m+1}$$

*for all* $m \in M_j$.

Since $\tau_{j,m} > \theta_m - \theta_j = 2^{-q_m}\pi$ or $2^{q_m+1}\tau_{j,m} > 2\pi$, and $\tau_{j,m-1} = -\tau_{j,m}$, for $m \in M_j$, the following key lemma is an easy consequence of Lemma 4.1.

**Lemma 4.2.** *For each* $j \in \mathbb{N}$ *and* $m \in M_j$*, there exists* $n_{j,m} \in \mathbb{N}$*, such that*

$$\left(\frac{1}{44}\right)2^{2q_m}m\log m < n_{j,m} < \left(\frac{16}{15}\right)\left(\frac{1}{44}\right)2^{2q_m}m\log m \tag{5}$$

*and*

$$\cos n_{j,m}\tau_{j,m} = \cos n_{j,m}\tau_{j,m-1} > \frac{1}{2}$$

*hold.*

Here is a sufficient condition for the main result of this section:

**Lemma 4.3.** *The function* $f$ *defined by* (4) *has the* $\partial\Delta$ *as its natural boundary if*

$$\limsup_{n\to\infty}\left|\sum_{k=1}^{\infty}\frac{b_k\cos n\tau_{j,k}}{\lambda_{j,k}^n}\right|^{\frac{1}{n}} \geqslant 2 \tag{6}$$

*for all* $j \in \mathbb{N}$.

**Proof.** Since the series (4) is uniformly convergent on $\Delta$ we have

$$\begin{aligned}\left|\frac{f^{(n-1)}(\varsigma_j)}{(n-1)!}\right| &= \left|\sum_{k=1}^{\infty}\frac{b_k}{(z_k-\varsigma_j)^n}\right| \\ &= \left|\sum_{k=1}^{\infty}\frac{b_k e^{-i(\tau_{j,k}+\theta_j)n}}{\lambda_{j,k}^n}\right| \\ &\geqslant \left|\sum_{k=1}^{\infty}\frac{b_k\cos n\tau_{j,k}}{\lambda_{j,k}^n}\right|\end{aligned}$$

for all $j, n \in \mathbb{N}$. This completes the proof, since the set $\{\varsigma_j : j \in \mathbb{N}\}$ is dense on the circle $|z| = \frac{1}{2}$, and that already $f$ is analytic in the $int(\Delta)$. $\square$

As a simple application of the law of cosine to the triangle with vertices $z_k$, $\varsigma_j$, and the origin, we get the following lemma:

**Lemma 4.4.** *For all $j, k \in \mathbb{N}$, we have*

$$(2\lambda_{j,k})^2 = 1 + 4(\delta_k + 1)(\delta_k + 1 - \cos t_{j,k})$$

*where $t_{j,k} = \theta_k - \theta_j$.*
Next, consider a very useful lemma for the subsequent results:

**Lemma 4.5.** *We have*

$$\lambda_{j,m} < \lambda_{j,2m-3} < \lambda_{j,m+1}$$

*for all $j \in \mathbb{N}$ and $m \in M_j$.*

**Proof.** The second inequality is easily observed from the fact that

$$\theta_j - \theta_{2m-3} < \theta_j - \theta_{m-2} = \theta_{m+1} - \theta_j$$

and for the first inequality it is sufficient to show that

$$(\delta_m + 1)(\delta_m + 1 - \cos t_{j,m}) < (\delta_{2m-3} + 1)(\delta_{2m-3} + 1 - \cos t_{j,2m-3}) \tag{7}$$

due to Lemma 4.4. Note that

$$\begin{aligned} q_{2m-3} &= q_m + 1 \\ r_{2m-3} &= 2r_m - 3 \end{aligned}$$

clearly. Now recall

$$\begin{aligned} t_{j,m} &= \theta_m - \theta_j = 2^{-q_m}\pi \\ \delta_m &= 2^{-2q_m}\pi^2 = t_{j,m}^2 \end{aligned}$$

and see that

$$\begin{aligned} \delta_{2m-3} &= \frac{1}{4}\delta_m = \frac{1}{4}t_{j,m}^2 \\ t_{j,2m-3} &= -\frac{5}{2}t_{j,m} \end{aligned}$$

which can be put in (7). Therefore it is sufficient to show

$$(t_{j,m}^2+1)(t_{j,m}^2+\frac{1}{2}t_{j,m}^2)<(\frac{1}{4}t_{j,m}^2+1)(\frac{1}{4}t_{j,m}^2+\frac{5^2}{2^3}t_{j,m}^2-\frac{5^4}{3\cdot 2^7}t_{j,m}^4)$$

or equivalently

$$\frac{625}{192}t_{j,m}^4+\frac{877}{48}t_{j,m}^2<15$$

which is ensured by the 1st and 2nd properties of $M_j$. □

As an immediate consequence of Lemma 4.5. we get the following lemma:

**Lemma 4.6.** *For all $j \in \mathbb{N}$ and $m \in M_j$ we have*

$$\frac{b_{m+1}}{\lambda_{j,2m-3}^{n_{j,m}}}<\frac{b_m}{8\lambda_{j,m}^{n_{j,m}}}$$

*for $n_{j,m}$ as defined in Lemma* 4.2.

Here's another useful lemma:

**Lemma 4.7.** *For all $j \in \mathbb{N}$ and $m \in M_j$ we have*

$$2^{n_{j,m}}b_{2m-2}<\frac{b_m}{8\lambda_{j,m}^{n_{j,m}}}$$

*for $n_{j,m}$ as defined in Lemma* 4.2.

**Proof.** By using Lemma 4.4. we get

$$\begin{aligned}\log(2\lambda_{j,m})^2 &< 4(\delta_m+1)(\delta_m+\frac{1}{2}t_{j,m}^2)\\ &< \frac{15}{2}2^{-2q_m}\pi^2\end{aligned}$$

which leads to

$$\frac{n_{j,m}}{2}\log(2\lambda_{j,m})^2<\frac{21}{22}m\log m$$

through Lemma 4.2. On the other hand we have

$$(2m-2)\log(2m-2)>m\log m+\frac{21}{22}m\log m+3\log 2$$

which completes the proof. □

Next, consider

**Lemma 4.8.** *For all $j \in \mathbb{N}$ and $m \in M_j$ we have*

$$\frac{3}{\lambda_{j,m-2}^{n_{j,m}}} < \frac{b_m}{\lambda_{j,m}^{n_{j,m}}}$$

*for $n_{j,m}$ as defined in Lemma 2.2.*

**Proof.** By Lemma 4.4.

$$\begin{aligned} \tfrac{(2\lambda_{j,m-2})^2}{(2\lambda_{j,m})^2} - 1 &= \tfrac{4(\delta_m+1)(\delta_m+1-\cos 3t_{j,m})-4(\delta_m+1)(\delta_m+1-\cos t_{j,m})}{1+4(\delta_m+1)(\delta_m+1-\cos t_{j,m})} \\ &= \tfrac{16(\delta_m+1)\cos t_{j,m}\sin^2 t_{j,m}}{1+4(\delta_m+1)(\delta_m+1-\cos t_{j,m})} \\ &< (168)2^{-2q_m} \end{aligned}$$

so we can write

$$\begin{aligned} \log \frac{(2\lambda_{j,m-2})^2}{(2\lambda_{j,m})^2} &> \left(\frac{99}{100}\right) \frac{16(\delta_m+1)\cos t_{j,m}\sin^2 t_{j,m}}{1+4(\delta_m+1)(\delta_m+1-\cos t_{j,m})} \\ &> \left(\frac{99}{100}\right) \frac{(16)(0.995)\left(\frac{99}{100}\right)^2 \pi^2 2^{-2q_m}}{1+4(\frac{101}{100})(\frac{1}{100}+\frac{1}{200})} \\ &> (140)2^{-2q_m} \end{aligned}$$

by estimation. Therefore by Lemma 2.2. we get

$$\begin{aligned} \frac{1}{2} n_{j,m} \log \frac{(2\lambda_{j,m-2})^2}{(2\lambda_{j,m})^2} &> \left(\frac{70}{44}\right) m \log m \\ &> \log \frac{3}{b_m} \end{aligned}$$

which completes the proof. $\square$

Another lemma:

**Lemma 4.9.** *Let $j \in \mathbb{N}$ and $m \in M_j$. Then*

$$\lambda_{j,m-2} < \lambda_{j,k}$$

*for all $k < m-2$.*

**Proof.** Recall that $q_{m-2} = q_m$. It is clear geometrically that the lemma is true when $q_k = q_m$. Now suppose that $q_k < q_m$. It is sufficient to prove the lemma for the case $q_k = q_m - 1$. By Lemma 4.4. we have

$$\begin{aligned}(2\lambda_{j,m-2})^2 &< 1 + 4(\delta_m + 1)(\delta_m + \frac{1}{2}(3t_{j,m})^2) \\ &= 1 + 4(\delta_m + 1)\frac{55}{10}\delta_m\end{aligned}$$

and

$$\begin{aligned}(2\lambda_{j,k})^2 &> 1 + 4(4\delta_m + 1)(4\delta_m + \frac{9}{10}\cdot\frac{1}{2}(2t_{j,m})^2) \\ &= 1 + 4(4\delta_m + 1)\frac{58}{10}\delta_m\end{aligned}$$

which completes the proof. $\square$
Now we can prove the main theorem of this section:

**Theorem 4.10.** *The natural boundary of $f$ is the unit circle.*

**Proof.** Let $j$ be a positive integer. Then it is sufficient to prove (6). Note that for each $m \in M_j$ we can write

$$\begin{aligned}\left|\sum_{k=1}^{\infty}\frac{b_k\cos n_{j,m}\tau_{j,k}}{\lambda_{j,k}^{n_{j,m}}}\right| &> \frac{b_m + b_{m-1}}{2\lambda_{j,m}^{n_{j,m}}} - \frac{\sum_{k=1}^{m-2} b_k}{\lambda_{j,m-2}^{n_{j,m}}} - \frac{\sum_{k=m+1}^{2m-3} b_k}{\lambda_{j,2m-3}^{n_{j,m}}} \\ &\quad - 2^{n_{j,m}}\sum_{k=2m-2}^{\infty} b_k \\ &> \frac{b_m}{\lambda_{j,m}^{n_{j,m}}} - \frac{3}{2\lambda_{j,m-2}^{n_{j,m}}} - \frac{3b_{m+1}}{2\lambda_{j,2m-3}^{n_{j,m}}} - \frac{3}{2}2^{n_{j,m}}b_{2m-2} \\ &> \frac{b_m}{\lambda_{j,m}^{n_{j,m}}} - \frac{b_m}{2\lambda_{j,m}^{n_{j,m}}} - \frac{3b_m}{16\lambda_{j,m}^{n_{j,m}}} - \frac{3b_m}{16\lambda_{j,m}^{n_{j,m}}} \\ &= \frac{b_m}{8\lambda_{j,m}^{n_{j,m}}}\end{aligned}$$

because of the 3rd property of $M_j$, and lemmas 4.2, 4.5, 4.6, 4.7, 4.8 and 4.9. To complete the proof, observe that $M_j$ is an infinite set of positive integers,

$$\lim_{m\to\infty}\lambda_{j,m} = \frac{1}{2},$$

and

$$\lim_{m\to\infty} b_m^{\frac{1}{n_{j,m}}} = 1,$$

due to (5). $\square$

**Acknowledgment**

The author thanks the Editorial Board of the Journal of Mathematical Extension for the honor to submit an invited paper. The research was funded in part by a grant from the Office of Research, University of Michigan-Flint.

## References

[1] T. Bagby, On interpolation by rational functions, *Duke Math. J.* 36 (1969), 95-104.

[2] S. N. Bernstein, Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné. *Mém. Acad. Roy. Belgique* (2) 4 (1912), 1-104.

[3] P. G. Bojadžiev, On rational approximation of analytic functions, *Math. USSR Sbornik*, 19 (1973), 157-163.

[4] A. A. Gončar, On a theorem of Saff, *Math. USSR Sbornik*, 23 (1974), 149-154.

[5] A. A. Gončar, Poles of rows of the Padé table and meromorphic continuation of functions, *Math. USSR Sbornik*, 43 (1982), 527-546.

[6] M. Götz, Note on the region of convergence of a polynomial series, *J. Approx. Theory*, 135 (2005), 140-144.

[7] R. K. Kovačeva, Generalized Padé approximants and meromorphic continuation of functions, *Math. USSR Sbornik*, 37 (1980), 337-348.

[8] R. K. Kovačeva, Diagonal rational Chebyshev approximants and holomorphic continuation of functions, *Analysis*, 10 (1990), 147-161.

[9] N. S. Landkof, *Foundations of Modern Potential Theory*, Springer-Verlag, New York, 1972.

[10] V. A. Prokhorov, Poles of Tchebycheff rational approximants and meromorphic continuation of functions, *Math. USSR Sbornik*, 69 (1991), 379-391.

[11] K. Reczek and M. Simkani, On rational approximation of meromorphic functions, *Constructive Theory of Functions*, Proceedings of the international conference, Varna 1991, Publishing House of the Bulgarian Academy of Sciences, Sofia, 1992, 243-248.

[12] E. B. Saff, Regions of meromorphy determined by the degree of best rational approximation, *Proc. Amer. Math. Soc.*, 29 (1971), 30-38.

[13] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Springer, Berlin, 1997.

[14] Y. C. Shen, On interpolation and approximation by rational functions with preassigned poles, *J. Chinese Math. Soc.*, 1 (1936), 154-173.

[15] M. Simkani, A note on a theorem of Bojadžiev, *Panamer. Math. J.*, 18 (2008), 83-94.

[16] M. Simkani, A unified theory for rational approximation on the extended complex plane, *East J. Approx.*, 15 (2009), 51-69.

[17] M. Tsuji, *Potential Theory in Modern Function Theory*, 2nd edition Chelsea Publishing Company, New York, 1975.

[18] J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, 4th edition, AMS Colloquium Publications, 20 American Mathematical Society, Providence, 1965.

**Mehrdad Simkani**
Department of Mathematics
University of Michigan-Flint
Flint, MI 48502-1950, USA
E-mail: simkani@umich.edu