Journal of Mathematical Extension
Vol. 4, No. 1 (2009), 115-126

A New Method for Solving Fuzzy DEA Models by Trapeziodal Approximation

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Abstract. Data envelopment analysis is a technique for measuring the relative efficiency of a set of decision making units with crisp data, but in this paper we explain a new method for evaluating of decision making units with fuzzy data.
At first, we introduce fuzzy data envelopment analysis models with parameters as trapezoidal membership function. Then we extend this method for solving models with general parameters.

AMS Subject Classification: 90C70.
Keywords and Phrases: Data envelopment analysis, fuzzy DEA, CCR, BCC, trapezoidal aproximation, ranking.

1. Introduction

These days according to continuous change in economical situations, evaluating the performance of industrial and economical units have become one of the most important factor in their improvement. In order to improve the performance and have good position in comparison to other units, industrial units have to be evaluated with scientific methods. One of the most important methods for evaluating the decision

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making units(DMUs) is data envelopment analysis (DEA) which was introduced by Charnes et al. ([4]) and developed by others ([3,5]). Although DEA is a powerful method in evaluating DMUs, it also has some limitations. One of the limitations of this method is all models presented in DEA deal with exact and known data so these models are not suitable for real situation. In most situation, data presented by natural languages including good, bad, to name but a few which reflected general situation of the DMU.
After introducing the fuzzy theory and it's progressive application in other sciences, researchers applied the fuzzy theory in evaluating performance of DMUs with fuzzy data, Kao and Liu ([10]) illustrate the point in usage of α-cut in solving CCR model and evaluating DMUs with fuzzy data. During the last years, some researchers have applied the conception of comparison fuzzy numbers and presented the methods for solving DEA models with fuzzy data ([12,13,14]).
Triangular and trapezoidal numbers are suitable way among fuzzy numbers for reflecting natural situations in real positions. Furthermore, doing mathematical operations is simpler than numbers. Hence during the last years researchers have concentrated on trapezoidal numbers and presented different methods for approximating non- trapezoidal fuzzy numbers with closest trapezoidal fuzzy number ([1,7]).
Based on mentioned notes, some researchers have applied triangular and trapezoidal numbers in formulating fuzzy DEA models (8) and (9). But we deal with general data in real situations which are not necessary triangular or trapezoidal. In this way, these models do not have enough efficiency to evaluate performance of DMUs which are presented by non-trapezoidal data.
In Section 2 of this paper, we will introduce fuzzy numbers, in Section 3 we first solve fuzzy CCR and BCC models trapezoidal data then we extend this method for general data according to approximations which are presented in [7] and finally we explain a numerical example.

2. Fuzzy Numbers

A fuzzy set A in X is a set of ordered pairs:

$$A = \{(x, \mu_A(x)) | x \in X\},$$

$\mu_A(x)$ is called the membership function of x in A.
LR-fuzzy number $\tilde{A}$ can be described with the following membership function:

$$\mu_{\widetilde{A}}(x) = \begin{cases} L(\frac{\underline{m}-x}{\beta}) & x \leqslant \underline{m} \\ 1 & \underline{m} \leqslant x \leqslant \overline{m} \\ R(\frac{x-\overline{m}}{\gamma}) & x \geqslant \overline{m} \end{cases},$$

where L,R: $[0,1] \rightarrow [0,1]$, with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$, are non-increasing, continuous shape functions. The LR-fuzzy number is denoted by $\tilde{A} = (\underline{m}, \overline{m}, \beta, \gamma)$.
The α-cut set of $\widetilde{A}$, denoted by $\widetilde{A}_\alpha$, is

$$\widetilde{A}_\alpha = \{x : \mu_{\tilde{A}}(x) \geqslant \alpha$$

and

$$\widetilde{A}_l(\alpha) = inf\{x : \mu_{\tilde{A}}(x) \geqslant \alpha\}.$$

$$\widetilde{A}_u(\alpha) = sup\{x : \mu_{\tilde{A}}(x) \geqslant \alpha\}.$$

Hence we have: $\tilde{A}_\alpha = [\tilde{A}_l(\alpha)\ \tilde{A}_u(\alpha]$.
The α-cut sets of a LR-fuzzy number can easily be computed as: $\tilde{A}_\alpha = [\underline{m} - L^{-1}(\alpha)\beta, \overline{m} + R^{-1}(\alpha)\gamma]$; $\alpha \in (0\ 1)$.

Definition 2.1. *For any arbitrary fuzzy number* $\tilde{A}, R(\tilde{A}) = \frac{1}{2}\int_0^1 (\tilde{A}_l(\alpha) + \tilde{A}_u(\alpha))d\alpha$.

Theorem 2.2. *Suppose* $\tilde{A}$ *and* $\tilde{B}$ *are fuzzy numbers so we have:*
$\tilde{A} \succeq \tilde{B}\ iff\ R(\tilde{A}) \geqslant R(\tilde{B})$.

Proof. See [15] (see also [6,11]).

Trapezoidal fuzzy number is a fuzzy number with the following membership function:

$$\mu_{\tilde{A}}(x)=\begin{cases} 0 & x\leqslant t_1\\ \frac{x-t_1}{t_2-t_1} & t_1\leqslant x\leqslant t_2\\ 1 & t_2\leqslant x\leqslant t_3\\ \frac{t_4-x}{t_4-t_3} & t_3\leqslant x\leqslant t_4\\ 0 & x\geqslant t_4\end{cases}$$

Since the trapezoidal fuzzy number characterized by four real numbers $t_1\leqslant t_2\leqslant t_3\leqslant t_4$, it is often denoted by: $\tilde{A}=(t_1,t_2,t_3,t_4)$.
Consequently:

$$\tilde{A}=[t_1+\alpha(t_2-t_1),t_4-\alpha(t_4-t_3)]$$
$$R(\tilde{A})=\tfrac{1}{4}(t_1+t_2+t_3+t_4).$$

3. Fuzzy DEA Model

Consider the set of DMUs including DMU$_j$ j=1,...,n with fuzzy inputs and outputs.

$\tilde{x}_{ij}=(\underline{m}_{ij},\overline{m}_{ij},\alpha_{ij},\beta_{ij})$ and $\tilde{y}_{rj}=(\underline{n}_{rj},\overline{n}_{rj},\varphi_{rj},\gamma_{rj})$ are inputs and outputs respectively, and both of them have trapezoidal property .
Now, concider CCR model with fuzzy data as follows:

$$\begin{array}{lll} \min & \theta \quad \text{satisfying:} & \\ & \sum_{j=1}^{n}\lambda_j\tilde{x}_{ij}\preceq\theta\tilde{x}_{io}, & i=1,...,m\in\mathbb{N},\\ & \sum_{j=1}^{n}\lambda_j\tilde{y}_{rj}\succeq\tilde{y}_{ro}, & r=1,...,s\in\mathbb{N},\\ \text{where} & \lambda_j\geqslant 0, & j=1,...n. \end{array} \tag{1}$$

We have

$$\sum_{j=1}^{n}\lambda_j\tilde{x}_{ij}=(\sum_{j=1}^{n}\lambda_j\underline{m}_{ij},\sum_{j=1}^{n}\lambda_j\overline{m}_{ij},\sum_{j=1}^{n}\lambda_j\alpha_{ij},\sum_{j=1}^{n}\lambda_j\beta_{ij}).$$

By using Theorem 2.2. we rewrite model 1 as below:

$$\begin{array}{lll} \min & \theta \quad \text{satisfying:} & \\ & \sum_{j=1}^{n} \lambda_j(\underline{m}_{ij} + \overline{m}_{ij} + \frac{1}{2}(\beta_{ij} - \alpha_{ij})) & \\ & \leqslant \theta(\underline{m}_{io} + \overline{m}_{io} + \frac{1}{2}(\beta_{io} - \alpha_{io})), & i = 1, ..., m \in \mathbb{N} \\ & \sum_{j=1}^{n} \lambda_j(\underline{n}_{rj} + \overline{n}_{rj} + \frac{1}{2}(\theta_{rj} - \gamma_{rj})) & \\ & \geqslant (\underline{n}_{ro} + \overline{n}_{ro} + \frac{1}{2}(\theta_{ro} - \gamma_{ro})), & r = 1, ..., s \in \mathbb{N} \\ \text{where} & \lambda_j \geqslant 0, & j = 1, ...n. \end{array} \tag{2}$$

Consider the following changes of variable

$$\hat{x}_{ij} = \underline{m}_{ij} + \overline{m}_{ij} + \frac{1}{2}(\beta_{ij} - \alpha_{ij})$$

$$\hat{y}_{rj} = \underline{n}_{rj} + \overline{n}_{rj} + \frac{1}{2}(\theta_{rj} - \gamma_{rj}).$$

According to mentioned variable changes model (2) can be converted into a linear form as follows:

$$\begin{array}{lll} \min & \theta \quad \text{satisfying:} & \\ & \sum\limits_{j=1}^{n} \lambda_j \hat{x}_{ij} \leqslant \theta \hat{x}_{io}, & i = 1, ..., m \in \mathbb{N}, \\ & \sum\limits_{j=1}^{n} \lambda_j \hat{y}_{rj} \geqslant \hat{y}_{ro}, & r = 1, ..., s \in \mathbb{N}, \\ \text{where} & \lambda_j \geqslant 0, & j = 1, ...n. \end{array} \tag{3}$$

At the first of this section, DMUs with only trapezoidal fuzzy inputs and outputs was regarded. In the following method we extend it for general data.

Suppose $\tilde{A}$ be a non-trapezoidal fuzzy number. By applying presented approximation in [7], we have:

if $\int_0^1 (\tilde{A}_u(\alpha) - \tilde{A}_l(\alpha))(\frac{1}{3} - \alpha)d\alpha > 0$,

nearest trapezoidal fuzzy number to fuzzy number $\tilde{A}$ is a trapezoidal fuzzy number $T(\tilde{A}) = T(t_1, t_2, t_3, t_4)$, with t_1, t_2, t_3, t_4 given by:

$t_1 = 3\int_0^1 \tilde{A}_l(\alpha)d\alpha - 3\int_0^1 \alpha\tilde{A}_l(\alpha)d\alpha - 3\int_0^1 \alpha\tilde{A}_u(\alpha)d\alpha + \int_0^1 (\tilde{A}_u\alpha)d\alpha$

$t_2 = t_3 = 3\int_0^1 \alpha\tilde{A}_l(\alpha)d\alpha + 3\int_0^1 \alpha\tilde{A}_u(\alpha)d\alpha - \int_0^1 \tilde{A}_l(\alpha)d\alpha - \int_0^1 \tilde{A}_u(\alpha)d\alpha$

$t_4 = 3\int_0^1 \tilde{A}_u(\alpha)d\alpha - 3\int_0^1 \alpha\tilde{A}_l(\alpha)d\alpha - 3\int_0^1 \alpha\tilde{A}_u(\alpha)d\alpha + \int_0^1 \tilde{A}_l(\alpha)d\alpha$

otherwise t_1, t_2, t_3, t_4 given by:

$t_1 = -6\int_0^1 \alpha \tilde{A}_l(\alpha)d\alpha + 4\int_0^1 \tilde{A}_l(\alpha)d\alpha$
$t_2 = 6\int_0^1 \alpha \tilde{A}_l(\alpha)d\alpha - 2\int_0^1 \tilde{A}_l(\alpha)d\alpha$
$t_3 = 6\int_0^1 \alpha \tilde{A}_u(\alpha)d\alpha - 2\int_0^1 \tilde{A}_u(\alpha)d\alpha$
$t_4 = -6\int_0^1 \alpha \tilde{A}_u(\alpha)d\alpha + 4\int_0^1 \tilde{A}_u(\alpha)d\alpha.$

Let $\tilde{A}$ be a fuzzy number and $T(\tilde{A})$ be its trapezoidal approximation; so we have:

$$R(T(\tilde{A})) = \frac{1}{4}(t_1 + t_2 + t_3 + t_4).$$

In both of them we observe that

$$R(T(\tilde{A})) = \frac{1}{2}(\int_0^1 \tilde{A}_l(\alpha)d\alpha + \int_0^1 \tilde{A}_u(\alpha)d\alpha).$$

Now, consider DMU$_j$ with input $\tilde{x}_{ij}$ and output $\tilde{y}_{rj}$ which have not necessarily trapezoidal property. So we approximate them with closest numbers related to $\tilde{x}_{ij}$, $\tilde{y}_{rj}$ respectively.
Now, it is possible to apply properties of trapezoidal numbers.

$T(\tilde{x}_{ij}) = (t_{ij}^1, t_{ij}^2, t_{ij}^3, t_{ij}^4)$
$T(\tilde{y}_{rj}) = (\gamma_{rj}^1, \gamma_{rj}^2, \gamma_{rj}^3, \gamma_{rj}^4).$

So we have:

$$\begin{array}{lll} \min & \theta \quad \text{satisfying:} & \\ & \sum_{j=1}^{n} \lambda_j T(\tilde{x}_{ij}) \preceq \theta T(\tilde{x}_{io}), & i = 1, ..., m \in \mathbb{N} \\ & \sum_{j=1}^{n} \lambda_j T(\tilde{y}_{rj}) \succeq T(\tilde{y}_{ro}), & r = 1, ..., s \in \mathbb{N} \\ \text{where} & \lambda_j \geqslant 0, & j = 1, ...n. \end{array} \tag{4}$$

By using Theorem 2.2. model (4) changes as below:

$$\begin{aligned}
&\min \quad \theta \quad \text{satisfying:}\\
&\sum_{j=1}^{n} \lambda_j(t_{ij}^1 + t_{ij}^2 + t_{ij}^3 + t_{ij}^4) \leqslant \theta(t_{io}^1 + t_{io}^2 + t_{io}^3 + t_{io}^4), \quad i = 1, ..., m \in \mathbb{N}\\
&\sum_{j=1}^{n} \lambda_j(\gamma_{rj}^1 + \gamma_{rj}^2 + \gamma_{rj}^3 + \gamma_{rj}^4) \geqslant (\gamma_{ro}^1 + \gamma_{ro}^2 + \gamma_{ro}^3 + \gamma_{ro}^4), \quad r = 1, ..., s \in \mathbb{N} \qquad (5)\\
&where \quad \lambda_j \geqslant 0, \quad j = 1, ...n.
\end{aligned}$$

So we have:

$$\begin{aligned}
&\min \quad \theta \quad \text{satisfying:}\\
&\quad \textstyle\sum_{j=1}^{n} \lambda_j(\int_0^1 \underline{x}_{ij}(\alpha)d\alpha + \int_0^1 \overline{x}_{ij}(\alpha)d\alpha) \leqslant \theta(\int_0^1 \underline{x}_{io}(\alpha)d\alpha\\
&\quad + \textstyle\int_0^1 \overline{x}_{io}(\alpha)d\alpha), \qquad i = 1, ..., m \in \mathbb{N}\\
&\quad \textstyle\sum_{j=1}^{n} \lambda_j(\int_0^1 \underline{y}_{rj}(\alpha)d\alpha + \int_0^1 \overline{y}_{rj}(\alpha)d\alpha) \geqslant (\int_0^1 \underline{y}_{ro}(\alpha)d\alpha\\
&\quad + \textstyle\int_0^1 \overline{y}_{ro}(\alpha)d\alpha), \qquad r = 1, ...s \in \mathbb{N} \qquad (6)\\
&where \quad \lambda_j \geqslant 0, \qquad j = 1, ...n.
\end{aligned}$$

Theorem 3.1. *Model* (6) *can be transformed into a linear model.*

Proof. We define

$$\hat{x}_{ij} = \int_0^1 \underline{x}_{ij}(\alpha)d\alpha + \int_0^1 \overline{x}_{ij}(\alpha)d\alpha$$

$$\hat{y}_{ij} = \int_0^1 \underline{y}_{rj}(\alpha)d\alpha + \int_0^1 \overline{y}_{rj}(\alpha)d\alpha.$$

By substitution, model 6 will be changed as follow:

$$\begin{aligned}
\min \quad & \theta \quad \text{satisfying:}\\
& \sum_{j=1}^{n} \lambda_j \hat{x}_{ij} \leqslant \theta \hat{x}_{io}, \quad i = 1, ..., m \in \mathbb{N}\\
& \sum_{j=1}^{n} \lambda_j \hat{y}_{rj} \geqslant \hat{y}_{ro}, \quad r = 1, ..., s \in \mathbb{N} \qquad (7)\\
\text{where} \quad & \lambda_j \geqslant 0, \quad j = 1, ...n.
\end{aligned}$$

It is obvious that the above model is linear. □

The method presented in this section can be applied for fuzzy DEA models with triangular and trapezoidal data and expanded to general

situations.
Another basic DEA models is BCC model (2). By adding $\sum_{j=1}^{n} \lambda_j = 1$ to model (1) and applying the similar method mentioned in CCR with fuzzy data, the model BCC is made with fuzzy data.

4. Ranking of Efficient DMUs

There are several methods for ranking of DMUs. One of the methods for ranking DMUs is AP model (1). The AP model is as follows:

$$\begin{array}{lll}
\min & \theta \quad \text{satisfying:} & \\
 & \sum\limits_{\substack{j=1,\\ j\neq o}}^{n} \lambda_j x_{ij} \leqslant \theta x_{io}, & i = 1, ..., m \in \mathbb{N} \\
 & \sum\limits_{\substack{j=1,\\ j\neq o}}^{n} \lambda_j y_{rj} \geqslant y_{ro}, & r = 1, ..., s \in \mathbb{N} \\
\text{where} & \lambda_j \geqslant 0, & j = 1, ...n.
\end{array} \tag{8}$$

The AP model is not convenient for DMUs with common inputs and outputs. But we extended this model for DMUs with fuzzy parameters.

$$\begin{array}{lll}
\min & \theta \quad \text{satisfying:} & \\
 & \sum\limits_{\substack{j=1,\\ j\neq o}}^{n} \lambda_j \tilde{x}_{ij} \preceq \theta \tilde{x}_{io} & i = 1, ..., m \in \mathbb{N} \\
 & \sum\limits_{\substack{j=1,\\ j\neq o}}^{n} \lambda_j \tilde{y}_{rj} \succeq \tilde{y}_{ro} & r = 1, ..., s \in \mathbb{N} \\
\text{where} & \lambda_j \geqslant 0 & j = 1, ...n
\end{array} \tag{9}$$

We can easily solve this model by the proposed method in this paper.

5. Numerical Example

In this section, to illustrate the usage of the methodology developed here, a numerical example is considered. We are evaluating the efficiency of the method with the data listed in the following table. In this example we have $L(x) = 1 - x^2$ *and* $R(x) = 1 - x^{\frac{1}{2}}$ for input1, $L(x) = R(x) = (1 - x)^2$ for Input2, $L(x) = 1 - x^3$ *and* $R(x) = 1 - x^{\frac{1}{3}}$ for output.
The results of computation are explained in the following table.

DMUs	Input1	Input2	Output	eff-FCCR	Ranking-CCR	eff-FBCC	Ranking-BCC
A	5.66	5	14.25	1	1.5094	1	1.666
B	10	9.33	11.75	0.4622	0.4622	0.5549	0.5549
C	8.66	8.33	13.75	0.6210	0.6210	0.6336	0.6336
D	2.33	10.33	10	1	1.7047	1	2.4292
E	11	3.33	7.25	0.7639	0.7639	1	1.3994
F	6.66	7.33	4.25	0.2430	0.2430	0.7815	0.7815
G	11	7.33	7	0.3351	0.3351	0.6286	0.6286
H	9.66	4.66	13.25	0.9977	0.9977	1	1.0218

As it is shown in the above table efficiency score for DMU's A, D is equal to one, so these DMUs are FCCR efficient and also evaluating A, D, E, H shows that they are FBCC efficient.

6. Conclusion

In practice fuzzy inputs and outputs based on their nature are represented in different forms (for example trapezoidal form, triangular form, ...). But working with these forms simultaneously is not easy; then we approximate any fuzzy number with trapezoidal number and we compare DMUs in fuzzy environment by linear programming.
The implementation of the method is illustrated by a numerical example. Some fuzzy DEA models like SBM, to name but few, have fuzzy objective function. In oder to solve them, applying various methods for minimizing or maximizing is inevitable; in the next paper, some kinds of these methods will be explained.

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