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Auto-Average Length of Finite Groups

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Abstract. Let g and h be arbitrary elements of a given finite group G. Then g and h are said to be autoconjugate if there exists some automorphism α of G such that $h = g^{\alpha}$. In this article, we introduce and study auto-average length of autoconjugacy classes of finite groups. Also, we construct some sharp bounds for the auto-average length of finite groups.

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1. Introduction

Let G be any group then the *autocommutator* of the element $g \in G$ and the automorphism α in $\mathrm{Aut}(G)$ is defined to be

$$[g, \alpha] = g^{-1}g^{\alpha} = g^{-1}\alpha(g).$$

Using this definition, the subgroup

$$K(G) = \langle [x, \alpha] : \ x \in G, \ \alpha \in \mathrm{Aut}(G) \rangle,$$

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is called the *autocommutator subgroup* of G. The concept of autocommutator subgroups has been already studied in [4, 5]. Also

$$L(G) = \{g \in G \ : \ [g, \alpha] = 1, \ \forall \alpha \in \mathrm{Aut}(G)\},$$

is called the *autocentre* of G. Clearly if α runs over the inner automorphisms of G, then $K(G)$ and $L(G)$ will be the commutator subgroup, G', and the centre, $Z(G)$, of G, respectively. One notes that, $K(G)$ and $L(G)$ are characteristic subgroups of G.

A group G acts on a non-empty set Ω, if for every pair $(\omega, g) \in \Omega \times G$, the element $\omega^g \in \Omega$ such that

$$O_1. \quad \omega^{1_G} = \omega;$$
$$O_2. \quad (\omega^{g_1})^{g_2} = \omega^{g_1 g_2},$$

for all $g_1, g_2 \in G$ and $\omega \in \Omega$. Clearly, $\omega^G = \{\omega^g | \ g \in G\}$ is the *orbit* of $\omega \in \Omega$ and $G_\omega = \{g \in G| \ \omega^g = \omega\}$ is the *stabilizer* of ω in G. From now on we assume that G is a finite group, then it is easily seen that $|\omega^G| = [G : G_\omega]$ and $|G| = |\omega^G||G_\omega|$, for all ω in Ω.

As $\mathrm{Aut}(G)$ acts on the group G, the set of all elements of G which are autoconjugate to the fixed element g in G is called the *autoconjugacy class* of g and

$$|g^{\mathrm{Aut}(G)}| = |\mathrm{Aut}(G) : \mathrm{Aut}(G)_g|,$$

in which $\mathrm{Aut}(G)_g = C_{\mathrm{Aut}(G)}(g)$, and it is the stabilizer of g in $\mathrm{Aut}(G)$.

We denote $\kappa(G)$ to be the *conjugacy number* of G. Then $\mu(G) = |G|/\kappa(G)$ is the *average length* of conjugacy classes of the finite group G. One notes that conjugacy classes length gives some characterization of the group. Furthermore, the average length of a group has strong restriction to the group. Shi and Xiao [7] proved that if $Z(G)$ is trivial, then $\mu(G) = 2$ if and only if $G/Z(G) \cong S_3$. Du [3] generalized this result, so that if $|Z(G)|$ is odd, then $\mu(G) = 2$ if and only if $G/Z(G) \cong S_3$.

We define $\mu_a(G) = |G|/\kappa_a(G)$, where $\kappa_a(G)$ is the autoconjugacy number of G. By class equation: $|G| = C_{g_1} + C_{g_2} + ... + C_{g_k}$, where C_{g_i}'s are the length of autoconjugacy classes of elements $g_1, ..., g_k$ of G. We call $\mu_a(G)$ to be the *auto-average length* of autoconjugacy classes of the finite group G. It is easy to see that

$$\mu_a(G) = \frac{1}{\kappa_a(G)} \sum_{i=1}^{\kappa_a(G)} |g_i^{\mathrm{Aut}(G)}|. \tag{1}$$

2. Main Results

In this section, we study the auto-average length of autoconjugacy classes of finite groups. Also, we construct some sharp bounds for the auto-average length of finite groups.

In the following we construct upper and lower bounds for $\mu_a(G)$.

Theorem 2.1. *Let G be a finite non-trivial group. Then*

$$1 \leqslant \mu_a(G) \leqslant |K(G)|.$$

Proof. Consider

$$[g, \mathrm{Aut}(G)] = \langle [g, \alpha] : \alpha \in \mathrm{Aut}(G) \rangle,$$

which is the autocommutator subgroup of g and $\mathrm{Aut}(G)$. On the other hand, we have

$$|g^{\mathrm{Aut}(G)}| = |g^{-1} g^{\mathrm{Aut}(G)}| = |[g, \mathrm{Aut}(G)]| \leqslant |K(G)|.$$

Using equation (1),

$$\begin{aligned} \mu_a(G) &= \frac{1}{\kappa_a(G)} \sum_{i=1}^{\kappa_a(G)} |g_i^{\mathrm{Aut}(G)}| \\ &\leqslant \frac{1}{\kappa_a(G)} \sum_{i=1}^{\kappa_a(G)} |K(G)| \leqslant |K(G)|. \end{aligned}$$

It is clear that $\mu_a(G) \geqslant 1$ and the equality holds exactly, when G is trivial or isomorphic with $\mathbb{Z}_2$. Thus we obtain our claim. $\square$

Theorem 2.2. *Let $g_1, g_2, ..., g_k$ be a complete set of representatives for autoconjugacy classes of G. Then*

$$\mu_a(G) = \frac{|\mathrm{Aut}(G)|}{\kappa_a(G)} \sum_{i=1}^{\kappa_a(G)} \frac{1}{|C_{\mathrm{Aut}(G)}(g_i)|}.$$

Proof. Using the equation (1), we have

$$\mu_a(G) = \frac{1}{\kappa_a(G)} \sum_{i=1}^{\kappa_a(G)} |g_i^{\mathrm{Aut}(G)}|$$

$$\begin{aligned} &= \frac{1}{\kappa_a(G)} \sum_{i=1}^{\kappa_a(G)} |\mathrm{Aut}(G) : C_{\mathrm{Aut}(G)}(g_i)| \\ &= \frac{1}{\kappa_a(G)} \sum_{i=1}^{\kappa_a(G)} \frac{|\mathrm{Aut}(G)|}{|C_{\mathrm{Aut}(G)}(g_i)|} \\ &= \frac{|\mathrm{Aut}(G)|}{\kappa_a(G)} \sum_{i=1}^{\kappa_a(G)} \frac{1}{C_{\mathrm{Aut}(G)}(g_i)}. \quad \square \end{aligned}$$

The following remark is helpful for calculating the auto-average length of elements of a direct product of groups, when the orders of direct factors are coprime.

Remark 2.3. *Let H and K be two finite groups with coprime orders. Using Theorem 2.1 of [6], we have $\mathrm{Aut}(H \times K) = \mathrm{Aut}(H) \times \mathrm{Aut}(K)$. Hence*

$$\mu_a(H \times K) = \frac{|H \times K|}{\kappa_a(H \times K)} = \frac{|H| \times |K|}{\kappa_a(H) \times \kappa_a(K)} = \mu_a(H) \times \mu_a(K).$$

As an example, one can calculate that $\mu_a(\mathbb{Z}_6) = \frac{3}{2}$ and it is easy to see that $\mu_a(\mathbb{Z}_2) \times \mu_a(\mathbb{Z}_3) = \frac{3}{2}$.

In the following results we construct some upper and lower bounds for $\mu_a(G)$, which are more precise than the one given in Theorem 2.1.

Proposition 2.4. *Let G be a finite group. Then*

$$\mu_a(G) \leqslant \frac{1}{\kappa_a(G)} \left(|L(G)| + |Z(G) \setminus L(G)| \frac{|\mathrm{Aut}(G)|}{|\mathrm{Inn}(G)|} + |G \setminus Z(G)| \frac{|\mathrm{Aut}(G)|}{2} \right).$$

Proof. By equation (1), one has

$$\mu_a(G) = \frac{1}{\kappa_a(G)} \left(\sum_{g_i \in L(G)} |g_i^{\mathrm{Aut}(G)}| + \sum_{g_i \in Z(G) \setminus L(G)} |g_i^{\mathrm{Aut}(G)}| + \sum_{g_i \in G \setminus Z(G)} |g_i^{\mathrm{Aut}(G)}| \right).$$

Clearly, for every $g \in Z(G)$ and $\phi_x \in \mathrm{Inn}(G)$ we have $g^{\phi_x} = g^x = g$. Thus $\mathrm{Inn}(G) \subseteq C_{\mathrm{Aut}(G)}(g)$ and for all $g \in Z(G) \setminus L(G)$,

$$|g^{\mathrm{Aut}(G)}| = \frac{|\mathrm{Aut}(G)|}{|C_{\mathrm{Aut}(G)}(g)|} \leqslant \frac{|\mathrm{Aut}(G)|}{|\mathrm{Inn}(G)|}.$$

Also for every $g \in G \setminus Z(G)$, one can easily check that $|C_{\mathrm{Aut}(G)}(g)| > 2$ and

$$|g^{\mathrm{Aut}(G)}| = \frac{|\mathrm{Aut}(G)|}{|C_{\mathrm{Aut}(G)}(g)|} \leqslant \frac{|\mathrm{Aut}(G)|}{2}.$$

Therefore

$$\mu_a(G) \leqslant \frac{1}{\kappa_a(G)}\left(|L(G)| + |Z(G) \setminus L(G)|\frac{|\mathrm{Aut}(G)|}{|\mathrm{Inn}(G)|} + |G \setminus Z(G)|\frac{|\mathrm{Aut}(G)|}{2}\right). \quad \square$$

Proposition 2.5. *Let G be a finite group, then*

$$\mu_a(G) \geqslant 2 - \frac{|L(G)|}{\kappa_a(G)} \geqslant \frac{3}{\kappa_a(G)}.$$

Proof. Using equation (1)

$$\mu_a(G) = \frac{1}{\kappa_a(G)} \sum_{i=1}^{\kappa_a(G)} |g_i^{\mathrm{Aut}(G)}| \geqslant \frac{1}{\kappa_a(G)}$$

$$\left(\sum_{g_i \in L(G)} |g_i^{\mathrm{Aut}(G)}| + \sum_{g_i \in G \setminus L(G)} |g_i^{\mathrm{Aut}(G)}|\right).$$

It is clear that for every $g \in G \setminus L(G)$, one has $|g^{\mathrm{Aut}(G)}| \geqslant 2$ and hence

$$\mu_a(G) \geqslant \frac{1}{\kappa_a(G)}\left(|L(G)| + (\kappa_a(G) - |L(G)|)^2\right) \geqslant 2 - \frac{|L(G)|}{\kappa_a(G)} \geqslant \frac{3}{\kappa_a(G)}.$$

The equality holds exactly when $|g^{\mathrm{Aut}(G)}| = 2$, for every $g \in G \setminus L(G)$. $\square$

For example, the equality in the above theorem holds for the groups $\langle 1 \rangle, \mathbb{Z}_2, \mathbb{Z}_3$ or $\mathbb{Z}_4$.

If $|C_G(x)| = 2$, then clearly $x \in C_G(x)$ and hence $|x| = 2$. Therefore for the involution x such that $|C_G(x)| = 2$, we say that x is *self centralizing involution.*

Theorem 2.6. *Let G be a finite centreless group with no self centralizing involutions, then*

$$\mu_a(G) < \frac{1}{\kappa_a(G)}\left(1 + (\kappa_a(G) - 1)\frac{|\mathrm{Aut}(G)|}{3}\right).$$

Proof. It is clear that for every non-trivial element g in G, we have $|C_{\mathrm{Aut}(G)}(g)| \geqslant 3$. Now, the equation (1) implies that

$$\begin{aligned}
\mu_a(G) &= \frac{1}{\kappa_a(G)} \sum_{i=1}^{\kappa_a(G)} |g_i^{\mathrm{Aut}(G)}| \\
&= \frac{1}{\kappa_a(G)}\left(1 + \sum_{i=1}^{\kappa_a(G)-1} |g_i^{\mathrm{Aut}(G)}|\right) \\
&= \frac{1}{\kappa_a(G)}\left(1 + \sum_{i=1}^{\kappa_a(G)-1} \frac{|\mathrm{Aut}(G)|}{|C_{\mathrm{Aut}(G)}(g_i)|}\right) \\
&< \frac{1}{\kappa_a(G)}\left(1 + (\kappa_a(G)-1)\frac{|\mathrm{Aut}(G)|}{3}\right).
\end{aligned}$$

Note that the inequality is strict, since for every non trivial element g in G,

$$|g^{\mathrm{Aut}(G)}| = \frac{|\mathrm{Aut}(G)|}{3} \Rightarrow |C_{\mathrm{Aut}(G)}(g)| = 3.$$

This implies that G is a 3-group, which is a contradiction. □

Chaboksavar et. al [1] in 2014, classified all finite groups G whose absolute central factors are isomorphic to a cyclic group, $\mathbb{Z}_p \times \mathbb{Z}_p, D_8, Q_8$, or a non-abelian group of order pq, for some distinct primes p and q.

Now, using Theorem 3.1 of [1], we classify all finite groups G with $\mu_a(G) < \frac{16}{9}$.

Theorem 2.7. *Let G be a finite group with $\mu_a(G) < \frac{16}{9}$. Then G is one of the following groups: $\mathbb{Z}_4, \mathbb{Z}_5, \mathbb{Z}_6, \mathbb{Z}_7, \mathbb{Z}_8, \mathbb{Z}_{10}, \mathbb{Z}_{12}, \mathbb{Z}_{14}, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_2, S_3, D_8, Q_8$.*

Proof. Let $2 \leqslant |\frac{G}{L(G)}| \leqslant 7$, Then Theorem 3.1 [1] implies that G is one of the following groups:

$$\mathbb{Z}_4, \mathbb{Z}_5, \mathbb{Z}_6, \mathbb{Z}_7, \mathbb{Z}_8, \mathbb{Z}_{10}, \mathbb{Z}_{12}, \mathbb{Z}_{14}, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_2, S_3, D_8, Q_8.$$

Now, assume that $|G/L(G)| = 8$. By Proposition 2.5, one can calculate that

$$\begin{aligned}
\mu_a(G) \geqslant 2 - \frac{|L(G)|}{\kappa_a(G)} &= 2 - (\frac{|L(G)|}{|G|} \times \frac{|G|}{\kappa_a(G)}) \\
&= 2 - \frac{1}{8}\mu_a(G),
\end{aligned}$$

and hence $\mu_a(G) \geqslant \frac{16}{9}$ gives the result. □

Theorem 2.8. *Let $G \cong \mathbb{Z}_{p^{n_1}} \times \mathbb{Z}_{p^{n_2}} \times ... \times \mathbb{Z}_{p^{n_k}}$ be a finite abelian p-group with $n_1 \geqslant n_2 \geqslant ... \geqslant n_k$. Then*

$$\mu_a(G) \geqslant \frac{2(p^{n_1} \times p^{n_2} \times ... \times p^{n_k})}{1 + (p^{n_1} \times p^{n_2} \times ... \times p^{n_k})}.$$

Proof. Assume that $G \cong \mathbb{Z}_{p^{n_1}} \times \mathbb{Z}_{p^{n_2}} \times ... \times \mathbb{Z}_{p^{n_k}}$ is a finite abelian p-group with $n_1 \geqslant n_2 \geqslant ... \geqslant n_k$. Then Corollary 3.4 [2] implies that $L(G) = 1$, when p is odd or $p = 2$ and $n_1 = n_2$. Hence Proposition 2.5 implies that

$$\mu_a(G) \geqslant \frac{2(p^{n_1} \times p^{n_2} \times ... \times p^{n_k})}{1 + (p^{n_1} \times p^{n_2} \times ... \times p^{n_k})}.$$

Also $L(G) = \mathbb{Z}_2$, when $p = 2$ and $k = 1$ or $n_1 > n_2$. Now Proposition 2.5 gives

$$\mu_a(G) \geqslant \frac{2(p^{n_1} \times p^{n_2} \times ... \times p^{n_k})}{2 + (p^{n_1} \times p^{n_2} \times ... \times p^{n_k})}. \quad \square$$

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