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Third Term of the Lower Autocentral Series of Abelian Groups

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Abstract. Let G be a group and $Aut(G)$ be the group of automorphisms of G. Then $[g, \alpha, \beta] = (g^{-1}g^{\alpha})^{-1}(g^{-1}g^{\alpha})^{\beta}$ is the autocommutator of the element $g \in G$ and $\alpha, \beta \in Aut(G)$ of weight 3. Also, we define $K_2(G) =< [g, \alpha, \beta] : g \in G, \alpha, \beta \in Aut(G) >$ to be the third term of the lower autocentral series of subgroups of G. In this paper, it is shown that every finite abelian group is isomorphic to the third term of the autocentral series of some finite abelian group.

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1. Introduction

For every element g of a group G and any automorphism $\alpha \in Aut(G)$,

$$[g, \alpha] = g^{-1}g^{\alpha}$$

is the *autocommutator* of the element g and the automorphism α. Clearly, if α is taken to be an inner automorphism then we obtain the commutator of two elements of G. Now the *autocommutator subgroup* of G is defined as follows:

$$K_1(G) =< [g, \alpha] : g \in G, \alpha \in Aut(G) >,$$

which is a characteristic subgroup of G.
There are some results on the autocommutator subgroup of a finite group G (see [2, 3, 4]). Recently, C. Chis, M. Chis and G. Silberberg ([1]) in 2008 showed that, every finite abelian group is the autocommutator subgroup of some finite abelian group. In the present paper, it is shown that a similar result holds for the autocommutator of weight three.
The following definition is vital in our investigation.

Definition 1.1. *Let g be an element of a given group G and $\alpha, \beta \in Aut(G)$. Then we define*

$$[g, \alpha, \beta] = [g, \alpha]^{-1}[g, \alpha]^{\beta}$$

to be the autocommutator of the element g and the automorphisms α and β of weight 3. Clearly, when α and β are taken to be inner automorphisms of G, one obtains the usual commutator of weight 3.
Now we define the third term of the lower autocentral series of subgroups as follows:

$$K_2(G) =< [g, \alpha, \beta] : g \in G, \alpha, \beta \in Aut(G) > .$$

Note that one may define the lower autocentral series of subgroups of a given group, which are all characteristic subgroups. The main objective of this paper is to prove the following result.

Main Theorem. *Every finite abelian group is the third term of the lower autocentral series of some finite abelian group.*

2. Preparatory Results

To prove our main theorem we first establish some preparatory results. All groups, which are considered in this paper are finite.

Lemma 2.1. *Let A and B be characteristic subgroups of a given group G such that $G = A \times B$. Then $K_2(G) = K_2(A) \times K_2(B)$.*

Proof. Let $\varphi : Aut(G) \longrightarrow Aut(A) \times Aut(B)$, given by $\alpha \mapsto (\alpha_{|A}, \alpha_{|B})$ and $\theta : Aut(A) \times Aut(B) \longrightarrow Aut(G)$, given by $(\mu, \eta) \mapsto \bar{\mu} o \bar{\eta}$,
where $\bar{\mu} o \bar{\eta}$ is an automorphism of G, defined as follows:

$$\bar{\mu}(ab) = a^{\mu} b,$$

$$\bar{\eta}(ab) = ab^{\eta},$$

for all $a \in A$ and $b \in B$. Since A and B are characteristic subgroups, it follows that φ and θ are inverse isomorphisms and hence we may identify $Aut(G)$ with $Aut(A) \times Aut(B)$.
Now, for any $g = ab = ba \in G = A \times B$ and $\alpha, \beta \in Aut(G)$ we have

$$\begin{aligned} [g, \alpha, \beta] &= [ab, \alpha, \beta] = [ab, \alpha]^{-1} [ab, \alpha]^{\beta} \\ &= (ab)^{-\alpha} (ab).(ab)^{-\beta} (ab)^{\alpha\beta} \\ &= b^{-\alpha} a^{-\alpha} ab.a^{-\beta} b^{-\beta}.a^{\alpha\beta} b^{\alpha\beta} \\ &= a^{-\alpha} a a^{-\beta} a^{\alpha\beta}.b^{-\alpha} b b^{-\beta} b^{\alpha\beta} \\ &= [a, \alpha_{|A}, \beta_{|B}][b, \alpha_{|A}, \beta_{|B}]. \end{aligned}$$

Thus $K_2(G) \subseteq K_2(A) \times K_2(B)$.
Now, for any $a \in A$ and $\mu, \mu' \in Aut(A)$,

$$[a, \mu, \mu'] = [a, \bar{\mu}, \bar{\mu'}] \in K_2(G).$$

Hence $K_2(A)$ is contained in $K_2(G)$. Similarly $K_2(B) \subseteq K_2(G)$ and so $K_2(G) = K_2(A) \times K_2(B)$. □

Lemma 2.2. *Let G be a finite cyclic group. Then $K_2(G) = G^4$.*

Proof. Let $G = < x : x^n = 1 >$ be the cyclic group of order n. Clearly φ is an automorphism of G if and only if

$$\varphi : x \mapsto x^i, \quad 1 \leqslant i \leqslant n,$$

where $(i, n) = 1$, (see [5]). Assume n is an odd number, then since the group G is abelian, the map α given by $x \mapsto x^{-1}$ is an automorphism of G. Hence, for all $g \in G$,

$$g^4 = [g, \alpha, \alpha] \in K_2(G).$$

Thus $G^4 \subseteq K_2(G)$. By the assumption $(4,n)=1$, then there exist some integers $s,r \in Z$ such that $4s+nr=1$. Thus $g = g^{4s+nr} = (g^s)^4 \in G^4$. This shows that G and hence $K_2(G)$ is contained in G^4. Therefore in this case, $K_2(G) = G^4$.
Now, we assume n is even. Hence for a non-trivial automorphism of G given by $x \mapsto x^i$, the integer i must be odd and greater than 2. Therefore for any $\alpha,\ \beta \in Aut(G)$, with $\alpha(x) = x^i$ and $\beta(x) = x^j$,

$$[x,\alpha,\beta] = [x^{i-1},\beta] = x^{1-i}x^{(i-1)j} = x^{(i-1)(j-1)} \in G^4.$$

Thus $K_2(G) = G^4$. □

Lemma 2.3. *Let G be an abelian group of odd order and Z_2 the cyclic group of order 2. Then $K_2(G)$ and $K_2(G\times Z_2)$ are both isomorphic with G.*

Proof. Since G is abelian of odd order n, say, we conclude that $(4,n) = 1$. Lemma 2.2. implies that $K_2(G) = G$. On the other hand, Z_2 and G are characteristic subgroups in $G \times Z_2$. Thus Lemma 2.1. gives the assertion. □

Lemma 2.4. *Let G be a cyclic group of order 2^m and H an abelian 2-group of exponent 2^n, with $n < m$. Then $K_2(G \times H) = G^4 \times H^2$.*

Proof. Let $G =< x : x^{2^m} = 1 >$, then for every element $h \in H$, we may define a unique automorphism $\alpha_h \in Aut(G \times H)$ in the following way:

$$\alpha_h : (x, h_1) \longrightarrow (x, h^{-1}h_1^{-1}).$$

Thus $h^2 = [x, \alpha_h, \alpha_{e_H}] \in K_2(G \times H)$, and so

$$H^2 \subseteq K_2(G \times H).$$

Using Lemma 2.2. we have $K_2(G) = G^4$. Hence

$$G^4 = K_2(G) \subseteq K_2(G \times H).$$

So $G^4 \times H^2 \subseteq K_2(G\times H)$. On the other hand, for all $\alpha, \beta \in Aut(G\times H)$ and noting the structures of the groups G and H we have

$$[g, \alpha, \beta] \in G^4 \times H^2, \quad \text{for all g} \in \text{G} \times \text{H}.$$

Hence the result holds. □
Using the above lemma, we obtain the following.

Proposition 2.5. *For natural numbers $m > n_1 \geqslant n_2 \geqslant ... \geqslant n_r$, such that $m \geqslant 2$, we have*

$$K_2(Z_{2^m} \times Z_{2^{n_1}} \times ... \times Z_{2^{n_r}}) = Z_{2^{m-2}} \times Z_{2^{n_1-1}} \times ... \times Z_{2^{n_r-1}}.$$

3. Proof of the Main Theorem

Let G be a finite abelian group. Then G can be written as a direct product of its Sylow p-subgroups.
If $(4, |G|) = 1$, then G is of odd order and has no Sylow 2-subgroups. Since G is abelian, all of its Sylow p-subgroups are characteristic and of odd order, which by Lemma 2.3. implies that $K_2(G) = G$. Otherwise, G has a Sylow 2-subgroup, say P, and it can be written as follows:

$$P = Z_{2^m} \times Z_{2^{n_1}} \times ... \times Z_{2^{n_r}}.$$

Now, we construct the abelian group

$$H = Z_{2^{m+2}} \times Z_{2^{n_1+1}} \times ... \times Z_{2^{n_r+1}} \times P_1 \times ... \times P_s,$$

where P_i's are all Sylow p_i-subgroups of G, except P. Using Proposition 2.5, we obtain

$$K_2(H) \cong G,$$

which proves the theorem. □

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