FUNDAMENTAL THEOREM OF CALCULUS IN TOPOLOGICAL VECTOR SPACES

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ABSTRACT. We extend the notions of integration and differentiation to cover the class of functions taking values in topological vector spaces. We give versions of the Lebesgue-Nikodym Theorem and the Fundamental Theorem of Calculus in such a more general setting.

1. INTRODUCTION

It is hardly possible to overemphasize the importance of the notions of integral and derivative in mathematical analysis. These two notions constitute the twin pillars on which analysis is built. The Fundamental Theorem of Calculus (FTC) shows that integration and differentiation are essentially inverses of one another. Since its discovery in the 17th century, several authors have attempted to give a more general setting to the FTC. A systematic study of the vector valued case of these two notions have started since the first half of the 20th centuries. Details studies of integration/differentiation for functions valued in normed spaces are presented in several books (e.g. [1, 2, 3]). For more recent results on vector valued integration theory, the reader is referred to [7, 8]. As in many areas of mathematics, it is always desirable and useful to have at our disposal a theory at a level of generality that will allow a wide of a spectrum of applications as possible. Our main purpose in this paper is to further enlarge the class of integrable and differentiable functions to include functions taking values in topological vector spaces, and give more general setting to the statement of the Lebesgue-Nikodym Theorem, as well as sufficient and necessary conditions under which the Fundemental Theorem of Calculus holds in such a setting.

The exposition will be organized as follows. In Section 2, we review few elementary concepts related to limit of nets of elements of a linear topological space. Some the results in this section can be considered as of independent interest. In Section 3, we show that a suitable concept of integration can be defined for functions taking values in a topological vector space. In Section 4, we state and prove a newer version of the Lebesgue-Nikodým theorem. The last Section 5 is devoted to some extension of the Fundamental Theorem of Calculus.

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2. Preliminaries

Since integration and differentiation are both limit operations, it is very sound to want to develop a good understanding of the concept of limit. There are many important topological vector spaces where the notion of convergence are not generated by a norm nor even by a seminorm. To treat those cases, it turns out to be very convenient to use the definition of limit in its most general form as devised by E.H. Moore and H.L. Smith [5]. Recall that a nonempty set Ω is said to be *directed* by a binary relation $\succ$, if $\succ$ has the following properties:

(1) for $x, y, z \in \Omega$ if $x \succ y$ and $y \succ z$, then $x \succ z$ *(transitivity);*
(2) for $x, y \in \Omega$, there exists $z \in \Omega$ such that $z \succ x$ and $z \succ y$ *(upper bound property).*

Given a set X, a *net* of elements of X is an X-valued function defined on a directed set $(\Omega, \succ)$. For our purposes, we will define the notion of limit for the setting of a topological vector space. We denote by $\mathcal{N}_0(X)$ the set of all balanced neighborhoods of 0 in X.

Definition 1. Let X be a topological vector space. A net $f : (\Omega, \succ) \to X$ is said to be convergent if there exists a vector $\lim\limits_{(\Omega,\succ)} f \in X$ such that for every $N_0 \in \mathcal{N}_0(X)$, there exists $\omega_0 \in \Omega$ such that $f(\omega) \in \lim\limits_{(\Omega,\succ)} f + N_0$ whenever $\omega \succ \omega_0$.

For simplicity of notation, we shall omit such parts of the symbolism under lim as can be without any danger of confusion. We also introduce the following notions of boundedness.

Definition 2. Let X be a topological vector space. A net $f :(\Omega, \succ) \to X$ is said to be

(1) ***bounded*** if its range $f(\Omega)$ is bounded in X, that is, if for every $N_0 \in \mathcal{N}_0(X)$, there exists $s \in [0, \infty)$ such that for every $t > s$, $f(\Omega) \subset tN_0$.
(2) ***eventually bounded*** if for every $N_0 \in \mathcal{N}_0(X)$, there exists $\omega_0 \in \Omega$, and there exists $s \in [0, \infty)$ such that for every $t > s$, $f(\{\omega \in \Omega : \omega \succ \omega_0\}) \subset tN_0$.

Note that as opposed to the notion of eventual boundedness, the notion of boundedness is independent of the direction. Clearly, a bounded net is eventually bounded. The converse is not true: the net $x \in (\mathbb{R}, >) \mapsto e^{-x}$ is eventually bounded towards ∞, but obviously not bounded. However, if the net is a sequence, that is, if $f : (\mathbb{N}, >) \to X$ is directed towards∞, then it is an easy exercise to show that

Proposition 3. *A sequence $f :(\mathbb{N}, >) \to X$ is bounded if and only if it is eventually bounded.*

Proof. We need only to show the sufficiency. Assume that $n \mapsto f(n)$ is eventually bounded, i.e. for every $N_0 \in \mathcal{N}_0(X)$, there exists $n_0 \in \mathbb{N}$, and there exists $s \in [0, \infty)$ such that for every $t > s$, $f(\{n \in \mathbb{N} : n \succ n_0\}) \subset tN_0$. Let $\tau \in [0, \infty)$ such that $f(1), f(2), \ldots, f(n_0) \in \tau N_0$. Then clearly, for all $n \in \mathbb{N}$, $f(n) \in (\tau \vee t)N_0$. The proof is complete. $\square$

Proposition 4. *Let X be a topological vector space. Assume that a net $f : (\Omega, \succ) \to X$ is convergent. Then for every $N_0 \in \mathcal{N}_0(X)$, there exists $\omega_0 \in \Omega$ such that $f(\omega) \in f(\varpi) + N_0$ whenever $\omega, \varpi \succ \omega_0$.*

Proof. Suppose $\lim\limits_{(\Omega,\succ)} f = a$. Fix a $N_0 \in \mathcal{N}_0(X)$. We can choose a neighborhood U_0 of 0 such that $U_0 + U_0 \subset N_0$. Then there is $\omega_0 \in \Omega$ such that for $\omega \succ \omega_0$, we have $f(\omega) \in a + U_0$. Thus for $\omega, \varpi \succ x_0$, we have

$$f(\omega) - f(\varpi) = f(\omega) - a - (f(\varpi) - a) \in U_0 + U_0 \subset N_0.$$

as desired. □

Definition 5. A net $f : (\Omega, \succ) \to X$ that satisfies the conclusion of the above Proposition 4 is said to be *topologically Cauchy*.

Recall that a net $g : (\Gamma, \succcurlyeq) \to X$ is said to be a *subnet* of a net $f : (\Omega, \succ) \to X$ if there exists a function $\varphi : \Gamma \to \Omega$ such that $g = f \circ \varphi$ and such that for each $\omega_0 \in \Omega$, there exists $\gamma_0 \in \Gamma$ such that whenever $\gamma \succcurlyeq \gamma_0$ then $\varphi(\gamma) \succ \omega_0$.

Proposition 6. *Let X be a topological vector space and let $f : (\Omega, \succ) \to X$ be a topological Cauchy net. Assume that f admits a subnet $g : (\Gamma, \succcurlyeq) \to X$ which converges to say a. Then f converges to a.*

Proof. Let $f : (\Omega, \succ) \to X$ be a topological Cauchy net and assume that a subnet $g : (\Gamma, \succcurlyeq) \to X$ of f converges to $a \in X$. Let $\varphi : \Gamma \to \Omega$ be the function defining the subnet g. Fix a $N_0 \in \mathcal{N}_0(X)$ and choose $N \in \mathcal{N}_0(X)$ such that $N + N \subset N_0$. Then there exists $\omega_0 \in \Omega$ such that $f(\omega) - f(\varpi) \in N$ whenever $\omega, \varpi \succ \omega_0$. There exists $\gamma_0 \in \Gamma$ such that whenever $\gamma \succcurlyeq \gamma_0$ then $\varphi(\gamma) \succ \omega_0$. On the other hand, since the subnet $g = f \circ \varphi$ converges to a, there exists $\gamma_N \in \Gamma$, such that whenever $\gamma \succ \gamma_N$, $f \circ \varphi(\gamma) \in a + N$. It follows that if $\omega \succ \omega_0$, then $\varphi(\gamma) \succ \omega_0$, and we have

$$f(\omega) - a = f(\omega) - f \circ \varphi(\gamma) + f \circ \varphi(\gamma) - a \in N + N \subset N_0.$$

This completes the proof. □

Proposition 4 states that every convergent net is topologically Cauchy. The converse of such a statement does not hold in general. A topological vector space X is said to be *topologically complete* (resp. *sequentially complete*) if every topological Cauchy net (resp. Cauchy sequence) of elements of X is convergent. Clearly, every topologically complete vector space is sequentially complete. The following result shows that in fact the two properties are exactly the same.

Theorem 7. *Every sequentially complete vector space is topologically complete.*

Proof. Let $f : (\Omega, \succ) \to X$ be a topological Cauchy net. Fixed a neighborhood N_0 neighborhoods of 0, and for each $n \in \mathbb{N}$, let $N_n = \frac{1}{n} N_0$ and let N'_n be a neighborhood of 0 such that $N'_n + N'_n \subset N_n$. We then choose successively $\omega_1, \omega_2, \omega_3, \ldots \in \Omega$ such that $\omega_n \succ \omega_{n-1}$ and $f(\omega) - f(\varpi) \in N_n'$ whenever $\omega, \varpi \succ \omega_n$. Then the sequence $n \mapsto f(\omega_n)$ is subnet of f which is Cauchy. By the sequential completeness of X, $n \mapsto f(\omega_n)$ converges to a limit. Proposition 6 now completes the proof. □

Proposition 8. *Every topological Cauchy net of elements of a topological vector space is eventually bounded.*

Proof. Let $f : (\Omega, \succ) \to X$ be a net. Let N_0 and U_0 be neighborhoods of 0 such that $U_0 + U_0 \subset N_0$. Then there exists $\omega_0 \in \Omega$ such that for all $\omega, \varpi \succ \omega_0$, $f(\omega) - f(\varpi) \in U_0$, and in particular for all $\omega \succ \omega_0$

$$f(\omega) \in f(\omega_0) + U_0.$$

Choose $s > 1$ such that $\omega_0 \in sU_0$. Then for $\omega \succ \omega_0$, we have

$$f(\omega) \in sU_0 + U_0 \subset sU_0 + sU_0 \subset sN_0.$$

The proof is complete. □

In view of Proposition 3, we have

Corollary 9. *Every Cauchy sequence of elements of a topological vector space is bounded.*

Proposition 10. *Let E be a subset of a topological vector space X. Then E is bounded if for every sequence $n \mapsto x_n$ of elements of E, and every sequence of scalar $n \mapsto \alpha_n$ converging to 0, the sequence $n \mapsto \alpha_n x_n$ converges to 0.*

Proof. Suppose that E is bounded, and let $n \mapsto \alpha_n$ converge to 0. Let N_0 be a balanced neighborhood of 0. Then $E \subset tN_0$ for some $t > 0$. Choose n large enough so that $|\alpha_n t| < 1$. Then $\alpha_n E \subset \alpha_n t N_0 \subset N_0$. Thus for every sequence $n \mapsto x_n$ in E, $\alpha_n x_n \in N_0$, showing that $n \mapsto \alpha_n x_n$ converges to 0.

Conversely, suppose that for every sequence $n \mapsto x_n$ of elements of E, and every sequence of scalar $n \mapsto \alpha_n$ converging to 0, the sequence $n \mapsto \alpha_n x_n$ converges to 0. Suppose to the contrary that E is not bounded. Then there exist a $N_0 \in \mathcal{N}_0(X)$ and a sequence $n \rightharpoondown \alpha_n$ of positive real number diverging to ∞ such that no $\alpha_n N_0$ contains E. Then for each n, take $x_n \in E$ such that $x_n \notin \alpha_n N_0$, or equivalently $\alpha_n^{-1} x_n \notin N_0$. It follows that the sequence $n \mapsto \alpha_n^{-1} x_n$ does not converge to 0. A contradiction! □

3. Integrability

In what follows Ω is a nonempty set; the power set of Ω, that is, the set of all subsets of Ω will be denoted by 2^Ω. Let $\Sigma \subset 2^\Omega$. By a *size-function*, we mean a set-function $\lambda : \Sigma \to [0, +\infty]$ that satisfies the following conditions:$\lambda(\emptyset) = 0$; and $\lambda(A) \leq \lambda(A \cup B) \leq \lambda(A) + \lambda(B)$ whenever A, B in Σ. Obviously, any measure defined on a σ-ring is a size-function. The length function defined on the σ-ring generated by the bounded intervals in $\mathbb{R}$ is another example of size function. It is also easy to see that an outer-measure is a size-function defined on $\Sigma = 2^{\Omega}$.

Given two sets E and F, we write $E \sqcup F$ in place of $E \cup F$ when E and F are disjoint. A Σ, λ-*subpartition* P of a subset $A \subset \Omega$ is any finite collection $\{E \in \Sigma; E \subset A\}$ with the following properties:

(1) $\lambda(E) < \infty$ for all $E \in P$
(2) $E \cap F = \emptyset$ whenever $E \neq F$ in P.

We denote by $\bigsqcup P$ the subset of A obtained by taking the union of all elements of P. A Σ, λ-subpartition P is said to be *tagged* if a point $t \in E$ is chosen for each $E \in P$. We shall write $E = (E, t)$ if t is a tagging point for E. We denote by $\Pi(A, \Sigma, \lambda)$ the collection of all tagged Σ, λ-subpartitions of the set A. The *mesh* or the *norm* of $P \in \Pi(A, \Sigma, \lambda)$ is defined to be $\|P\| = \max\{\lambda(I) : I \in P\}$. If $P, Q \in \Pi(A, \Sigma, \lambda)$, we say that Q is a *refinement* of P and we write $Q \succ P$ if $\|Q\| \leq \|P\|$ and $\bigsqcup P \subset \bigsqcup Q$. It is readily seen that such a relation does not depend on the tagging points. It is also easy to see that the relation $\succ$ is transitive on $\Pi(A, \Sigma, \lambda)$. If $P, Q \in \Pi(A, \Sigma, \lambda)$. We quickly notice that If , then clearly, $P \vee Q := \{I \setminus J, I \cap J, J \setminus I : I \in P, J \in Q\} \in \Pi(A, \Sigma, \lambda)$, $P \vee Q \succ P$ and $P \vee Q \succ Q$. That is, the relation $\succ$ has the upper bound property on $\Pi(A, \Sigma, \lambda)$. We then infer that the set $\Pi(A, \Sigma, \lambda)$ is directed by the binary relation $\succ$. If there is no risk of confusion, we shall drop λ in all of the above notations.

Given a function $f : \Omega \to X$, and a tagged Σ-subpartition $P \in \Pi(A, \Sigma)$, the (Σ, λ)-*Riemann sum* of f on P is defined to be the vector $f_\lambda(P) = \sum_{(I,t) \in P} \lambda(I) f(t)$. The function $P \mapsto f_\lambda(P)$ is an X-valued net defined on the directed set $(\Pi(A, \Sigma, \lambda), \succ)$. For convenience, we are going to denote $\int_A f d\lambda_\Sigma := \lim\limits_{(\Pi(A,\Sigma,\lambda),\succ)} f_\lambda$ whether or not such a limit exists. The limit is of course in the sense of the topology of X. If such a limit does exist, we say that the function f is Σ, λ-*integrable over the set* A, and its limit $\int_A f d\lambda_\Sigma$ is called the Σ, λ-*integral of* f *over* A. Again we shall drop the symbols Σ, λ whenever no risks of confusion arise. More formally,

Definition 11. We say that a function $f : \Omega \to X$ is (Σ, λ)-*integrable* over a set $A \subset \Omega$, if $\lim\limits_{(\Pi(A,\Sigma,\lambda),\succ)} f_\lambda$ represents a vector in X. The vector $\lim\limits_{(\Pi(A,\Sigma,\lambda),\succ)} f_\lambda$ will be denoted by $\int_A f d\lambda_\Sigma$and called the (Σ, λ)-*integral of* f *over the set* A.

In other words, $f : \Omega \to X$ is (Σ, λ)-integrable over the set A with (Σ, λ)-integral $\int_A f d\lambda_\Sigma$ if for every $N_0 \in \mathcal{N}_0(X)$ there exists $P_0 \in \Pi(A, \Sigma, \lambda)$, such that for every $P \in \Pi(A, \Sigma, \lambda)$, $P \succ P_0$ we have

$$f_\mu(Q) \in \int_A f d\lambda_\Sigma + N_0. \tag{3.1}$$

We shall denote by $\mathcal{I}(A, X, \Sigma, \lambda)$ the set of all (Σ, λ)-integrable functions over the set A. Many classical properties of the integral follow immediately from the properties of net limits and therefore their proofs are obtained at no extra cost.

Proposition 12. *If $f \in \mathcal{I}(A, X, (\Sigma, \lambda))$ then for every $N_0 \in \mathcal{N}_0(X)$ there exists $P_0 \in \Pi(A, \Sigma)$ such that $f_\lambda(Q)_W \in N_0$ for every $Q \in \Pi(A, \Sigma)$ that does not intersect P_0 and such that $\|Q\| \leq \|P_0\|$.*

Proof. Fix neighborhoods N_0, N of 0, such that $N + N \subset N_0$. Let $P_1 \in \Pi(A, \Sigma)$ be such that for every $P \succ P_1$ in $\Pi(A, \Sigma)$ we have $f_\lambda(P) \in \int_A f d\lambda_\Sigma + N$. Fix $P_0 \succ P_1$. Then for every $Q \in \Pi(A, \Sigma)$ that does not intersect P_0, and such that $\|Q\| \leq \|P_0\|$, we have $P_0 \vee Q \succ P_1$, and therefore

$$f_{\lambda,\tau}(P_0 \vee Q) \in \int_A f d\lambda_\Sigma + N.$$

It follows that

$$\begin{aligned} f_\lambda(Q) &= f_\lambda(P_0 \vee Q) - f_\lambda(P_0) \\ &= f_\lambda(P_0 \vee Q) - \int_A f d\lambda + \int_A f d\lambda - f_\lambda(P_0) \\ &\in N + N \subset N_0. \end{aligned}$$

The proof is complete. □

The above proposition suggests the following definition.

Definition 13. Let X be a topological vector space. A function $f : \Omega \to X$ is said to satisfy the *Cauchy criterion for integrability* on $A \subset \Omega$ if for every $N_0 \in \mathcal{N}_0(X)$ there exists $P_0 \in \Pi(A, \Sigma)$ such that $f_\lambda(Q) \in N_0$ for every $Q \in \Pi(A, \Sigma)$ that does not intersect P_0 and such that $\|Q\| \leq \|P_0\|$.

Thus Proposition 12 states that every integrable function satisfies the Cauchy criterion for integrability. Conversely, we notice that for if $P, Q \in \Pi(A, \Sigma)$ is such that no set in P intersects a set in Q, then $f_\lambda(P \vee Q) - f_\lambda(P) = f_\lambda(Q)$. It is then quickly seen that the Cauchy criterion for integrability of a function f is equivalent to the Cauchy condition for the net $P \mapsto f_\lambda(P)$. It then follows that Cauchy nets taking values in a complete topological vector space is convergent. Hence we have the following characterization theorem.

Theorem 14. *Let X be complete topological vector space. Then $f \in \mathcal{I}(A, X, (\Sigma, \lambda))$ if and only if it satisfies the Cauchy criterion for integrability on A.*

It should be noticed that if the set A is such that $\lambda(A) = 0$, then for all subpartitions $P \in \Pi(A)$, $\lambda_f(P) = 0$, and thus $\int_A f d\lambda = 0$. It follows that the integral does not distinguish between functions which differ only on set of size zero. More precisely,

$$\int_A f d\lambda = \int_A g d\lambda \text{ whenever } \lambda\{x \in A : f(x) \neq g(x)\} = 0.$$

We say that a function f is λ-essentially equal on A to another function g, and we write $f \overset{\lambda}{\sim} g$, if $\lambda\{x \in A : f(x) \neq g(x)\} = 0$. It is readily seen that the relation $f \overset{\lambda}{\sim} g$ is an equivalence relation on $\mathcal{I}(A, X, \lambda)$. We shall denote by $I(A, \lambda, X)$ the quotient space $\mathcal{I}(A, X, \lambda)/ \overset{\lambda}{\sim}$.

Our next result shows that if X is a topological vector space, then the space $\mathcal{I}(A, X, (\Sigma, \lambda))$ can also be naturally given a structure of topological vector space. If $f \in \mathcal{I}(A, X, (\Sigma, \lambda))$, for each $N_0 \in \mathcal{N}_0(X)$, we define a *neigborhood of* f as follows

$$B(f, P, N_0) := \{g \in \mathcal{I}(A, X, (\Sigma, \lambda)) : (g - f)_\lambda(Q) \in N_0, \text{ for all } Q, Q \cap P = \emptyset\}.$$

Theorem 15. *The collection* $\mathcal{B} = \{B(f, P, N_0) : f \in \mathcal{I}(A, X, (\Sigma, \lambda)), P \in \Pi(A, \Sigma, \lambda), N_0 \in \mathcal{N}_0(X)\}$ *is a basis for a topology on* $\mathcal{I}(A, X, (\Sigma, \lambda))$.

Proof. Let $B(f, P_1, N_1)$ and $B(g, P_2, N_2)$ be in $\mathcal{B}$, and let $h \in B(f, P_1, N_1) \cap B(f, P_2, N_2)$. Then $(h - f)_\lambda(P) \in N_1$ for all P that does not intersect P_1 and $(h - f)_\lambda(P) \in N_2$ for all P that does not intersect P_2. It follows that for all P that does not intersect $P_1 \vee P_2$ we have $(h-f)_\lambda(P) \in N_1 \cap N_2$, that is, $h \in B(f, P_1 \vee P_2, N_1 \cap N_2)$. This proves the theorem. □

We denote by Θ be the topology on $\mathcal{I}(A, X, (\Sigma, \lambda))$ generated by $\mathcal{B}$.

Theorem 16. *The vector space operations are continuous on* $(\mathcal{I}(A, X, (\Sigma, \lambda)), \Theta)$.

Proof. Let $B(h, P, N_0) \in \mathcal{B}$. Suppose $f + g \in B(h, P, N_0)$. Then for some $0 < \lambda < 1$, we have $(f + g - h)_\lambda(Q) \in \lambda N_0$ for all $Q \in \Pi(A, \Sigma)$ that does not intersect P. Let $N_1 \in \mathcal{N}_0(X)$ such that $\lambda N_0 + N_1 + N_1 \subset N_0$. Consider $B := B(f, P, N_1) \times B(g, P, N_1)$. It $(f_1, g_1) \in B$, then for all $Q \in \Pi(A, \Sigma)$ that does not intersect P, we have

$$\begin{aligned} (f_1 + g_1 - h)_\lambda(Q) &= (f + g - h)_\lambda(Q) + (f_1 - f)_\lambda(Q) + (g_1 - g)_\lambda(Q) \\ &\in \lambda N_0 + N_1 + N_1 \subset N_0. \end{aligned}$$

This proves that the addition is continuous.

Now let α be a scalar and suppose that $\alpha f \in B(h, P, N_0)$. Then for some $0 < \lambda < 1$, we have $(\alpha f - h)_\lambda(Q) \in \lambda N_0$ for all $Q \in \Pi(A, \Sigma, \lambda)$ that does not intersect P. Let $N_1 \in \mathcal{N}_0(X)$ such that $N_1 \subset N_0$. Choose $0 < \delta$ small enough so that $\beta N_1 + \delta \lambda N_0 \subset N_0$. Consider $N := B(\alpha, \delta) \times B(f, P, N_1)$. It $(\beta, g) \in N$, then for all $Q \in \Pi(A, \Sigma, \lambda)$ that does not intersect P, we have

$$\begin{aligned} (\beta g - h)_\lambda(Q) &= (\alpha f - h)_\lambda(Q) + (\beta g - \alpha f)_\lambda(Q) \\ &= (\alpha f - h)_\lambda(Q) + \beta(g - f)_\lambda(Q) + (\beta - \alpha) f_\lambda(Q) \\ &\in \lambda N_0 + \beta N_1 + \delta \lambda N_0 \subset N_0. \end{aligned}$$

This proves that the the scaling function is continuous. The proof is complete. □

Theorem 17. *Let* $\lambda : \Sigma \subset 2^\Omega \to [0, \infty]$ *be a size-function and* X *a topologically complete topological vector space. Let* $A \in \Sigma$ *be such that* $\lambda(A) < \infty$. *Then the topological function space* $(\mathcal{I}(A, X, (\Sigma, \lambda)), \Theta)$ *is topologically complete.*

Proof. Let $n \mapsto f_n$ be a Cauchy sequence in $(\mathcal{I}(A, X, (\Sigma, \lambda)), \Theta)$. Fix $N_0 \in \mathcal{N}_0(X)$. Choose $N_{00} \in \mathcal{N}_0(X)$ such that $N_{00} + N_{00} + N_{00} + N_{00} \subset N_0$. Let $P \in \Pi(A, \Sigma, \lambda)$, and let $N > 0$ such that for $m, n > N$ in $\mathbb{N}$,

$$(f_n - f_m)_\lambda(P) \in N_{00}. \tag{3.2}$$

In particular, taking the subpartition $P = \{(A, \omega)\}$, we see that the sequence $n \mapsto f_n(\omega)$ is Cauchy in X. Since X is a topologically complete, we can define a function $\omega \mapsto f(\omega) = \lim_{n\to\infty} f_n(\omega)$.

On the other hand, since $f_n, f_m \in \mathcal{I}(A, X, \Sigma, \lambda)$, there exist $P_n, P_m \in \Pi(A, \Sigma, \lambda)$ such that $(f_n)_\lambda(P) - \int_A f_n d\lambda \in N_{00}$, and $(f_m)_\lambda(P) - \int_A f_m d\lambda \in N_{00}$ whenever $P \succ P_n \vee P_m$. Combining with (3.2), it follows that for $m, n > N_\epsilon$ in $\mathbb{N}$ and for every $P \succ P_n \vee P_m$, we have

$$\int_A f_n d\lambda - \int_A f_m d\lambda = (f_n)_\lambda(P) - \int_A f_n d\lambda + (f_n - f_m)_\lambda(P) + \int_A f_m d\lambda - (f_m)_\lambda(P) \in N_{00}.$$

This proves that the sequence $n \mapsto \int_A f_n d\lambda$ is Cauchy in X, and thus converges to say $a \in X$.

Now since for each $\omega \in A$, $f(\omega) = \lim_{n\to\infty} f_n(\omega)$, there exists $N_\omega > N_\epsilon$ such that for $m, n > N_\omega$ in $\mathbb{N}$,

$$f_n(\omega) - f_m(\omega) \in \lambda(A)^{-1} N_{00}.$$

It follows that for $P \in \Pi(A, \Sigma)$, and for every $m, n > \max\{N_t : (I, t) \in P\} =: N_P$, we have

$$(f_n - f_m)_\lambda(P) = \sum_{(I,t)\in P} \lambda(I)(f_n(t) - f_m(t)) \in N_{00}.$$

If we let $m \to \infty$, we obtain $(f_n - f)_\lambda(P) \in N_{00}$. Since $a = \lim_{m\to\infty} \int_A f_m d\lambda$, there exists $N > N_P$ such that $\int_A f_m d\lambda - a \in N_{00}$ whenever $m > N$. Thus for $n, m > N$,

$$\begin{aligned} f_\lambda(P) - a &= (f - f_n)_\lambda(P) + (f_n - f_m)_\lambda(P) + (f_m)_\lambda(P) - \int_A f_m d\lambda + \int_A f_m d\lambda - a. \\ &\in N_{00} + N_{00} + N_{00} + N_{00} \subset N_0 \end{aligned}$$

Since N_0 is arbitrary, this shows that $f \in \mathcal{I}(A, X, \Sigma, \lambda)$ and that $\int_A f d\lambda = a$. □

4. Lebesgue-Nikodým theorem

We say that a set function $F : \Sigma \to X$ is *λ-absolutely continuous on Σ* if for every $N_0 \in \mathcal{N}_0(X)$, there exists $\delta > 0$ such that $F(A) \in N_0$ whenever $A \in \Sigma$ with $\lambda(A) < \delta$. The classical Lebesgue-Nikodým theorem states that for an additive real valued set function (not necessarily countably additive) $F : \Sigma \to \mathbb{R}$, λ-absolutely continuity implies λ-Lebesgue differentiability (see for example [4]). Such a result was extended to the Banach space valued case in [6]. Our next result further extends such a result to the case of functions taking values in a topological vector space. Completeness is crucial for the following extended result.

Theorem 18. *Let X be a complete topological vector space. Assume that $F : \Sigma \to X$ is an additive set function that is λ-absolutely continuous on Σ. Then there exists $f \in (\mathcal{I}(A, X, (\Sigma, \lambda)), \Theta)$ such that $F(A) = \int_A f d\lambda$ for every $A \in \Sigma$, with $\lambda(A) < \infty$.*

Proof. We may assume without loss of generality that $\Sigma = 2^{\Omega}$. Fix $A \in 2^{\Omega}$, with $\lambda(E) < \infty$. For every subpartition $P \in \Pi(\Omega, 2^{\Omega}, \lambda)$, consider the function defined on Ω by

$$F_P(\omega) = \sum_{I \in P} \frac{1_I(\omega)}{\lambda(I \cap A)} F(I \cap A).$$

Here 1_I denotes the indicator function of the set I. Then it is easily seen that $F_P \in (\mathcal{I}(A, X, (\Sigma, \lambda)), \Theta)$. The λ-absolute continuity of F ensures that

$$\int_A F_P d\lambda = \sum_{I \in P} F(I \cap A) = F\left(\bigsqcup_{I \in P} I \cap A\right) \to F(A) \tag{4.1}$$

as $\lambda(\bigsqcup_{I \in P} I \cap A) \to \lambda(A)$. We claim that the net $P \mapsto F_P$ is Cauchy in $(\mathcal{I}(A, X, (\Sigma, \lambda)), \Theta)$. Fix $N_0 \in \mathcal{N}_0(X)$. Choose $N_{00} \in \mathcal{N}_0(X)$ such that $N_{00} + N_{00} + N_{00} + N_{00} \subset N_0$. By the λ-absolutely continuity of F, we can find P and Q so refined that

$$F\left(\bigsqcup_{I \in P} I \cap A\right) - F\left(\bigsqcup_{J \in Q} J \cap A\right) = F\left(\bigsqcup_{I \in P} I \cap A \backslash \bigsqcup_{J \in Q} J \cap A\right) \in N_{00}. \tag{4.2}$$

For such P and Q, there exists $R_0 \in \Pi(A)$ such that for $R \succ R_0$,

$$(F_P)_\lambda(R) - \int_A F_P d\lambda,\ (F_Q)_\lambda(R) - \int_A F_Q d\lambda \in N_{00}. \tag{4.3}$$

It follows from (4.2), (4.3) that for $R \succ R_0$,

$$(F_P - F_Q)_\lambda(R) = (F_P)_\lambda(R) - \int_A F_P d\lambda + \int_A F_P d\lambda - \int_A F_Q d\lambda + \int_A F_Q d\lambda - (F_Q)_\lambda(R) \in N_0.$$

This proves our claim.

By Theorem 17, there exists $f \in (\mathcal{I}(A, X, (\Sigma, \lambda)), \Theta)$ such that the net $P \mapsto F_P$ converges to f . For a given $N_0 \in \mathcal{N}_0(X)$, choose $N_{00} \in \mathcal{N}_0(X)$ such that $N_{00} + N_{00} + N_{00} \subset N_0$. Then there exists $P_1 \in \Pi(A, \Sigma, \lambda)$ such that for $R \succ P_1$

$$(F_P)_\lambda(R) - f_\lambda(R) \in N_{00}. \tag{4.4}$$

On the other hand, it follows from (4.1) that there exists $P_2 \in \Pi(A, \Sigma, \lambda)$ such that for $R \succ P_2$

$$(F_P)_\lambda(R) - F(A) \in N_{00}. \tag{4.5}$$

Finally, by definition of the integral, there exists $P_3 \in \Pi(A)$ such that for $R \succ P_3$

$$(F_P)_\lambda(R) - \int_A f d\lambda \in N_{00}. \tag{4.6}$$

Combining (4.4), (4.5), and (4.6), we have for $R \succ P_1 \vee P_2 \vee P_3$

$$\int_A f d\eta - F(A) = \int_A f d\lambda - f_\lambda(R) + f_\lambda(R) - (F_P)_\lambda(R) + (F_P)_\lambda(R) - F(A) \in N_0.$$

The desired result follows since N_0 is arbitrary chosen in $\mathcal{N}_0(X)$. The proof is complete. $\square$

Remark. Note that, unlike the Radon-Nikodým derivative of a vector measure [1], the above density function f need not be measurable.

A function $f : A \to X$ that satisfies the conclusion of Theorem 18 will be called a λ-*density* of the set function F and the function F is then said to be the λ-*indefinite integral* of f.

It goes without saying that if $f : A \to X$ is a λ-density of F, then any function $g : A \to X$ such that $f = g$ λ-a.e. is also a λ-density of F. The symbol $d_\lambda F$ will be used to denote the class of all λ-density functions of the set function F.

5. Fundamental Theorem of Calculus

Let now assume that the domain space Ω itself is a topological space, and $\Sigma \subset 2^\Omega$contains the open sets of Ω. We say that a size-function $\lambda : \Sigma \subset 2^\Omega \to [0, \infty]$ is

- *regular* if it is non-zero on open sets of Ω.
- *translation invariant* if $\lambda(\omega + U) = \lambda(U)$, for every U and for every $\omega \in \Omega$.

Henceforth, we shall only consider a finite regular translation invariant size-function. We introduce the following definition.

Definition 19. Let Ω, X be topological vector spaces and $\lambda : 2^\Omega \to [0, \infty)$ a regular translation invariant size-function. Let $\Sigma \subset 2^\Omega$ contain the topology of Ω. A set function $F : \Sigma \to X$ is said to be λ-*differentiable* at a point $\omega \in \Omega$, if there exists a vector $\frac{dF}{d\lambda}(\omega) \in X$ such that for every $N_0 \in \mathcal{N}_0(X)$, there exists $U_0 \in \mathcal{N}_0(\Omega)$ such that for every $U \in \mathcal{N}_0(\Omega)$ contained in U_0

$$\frac{F(\omega + U)}{\lambda(U)} \in \frac{dF}{d\lambda}(\omega) + N_0.$$

We notice that when the set $\mathcal{N}_0(\Omega)$ is directed by inclusion, the statement in the above definition corresponds exactly to the definition of the net limit

$$\frac{dF}{d\lambda}(\omega) = \lim_{(\mathcal{N}_0(\Omega), \subset)} \frac{F(\omega + U)}{\lambda(U)}.$$

We denote by Δ_F the set of all $\omega \in \Omega$ for which $\frac{dF}{d\lambda}(\omega)$ exists. We shall call such a set Δ_F *the domain of differentiability of* F. By the uniqueness of limit, the correspondence $\omega \mapsto \frac{dF}{d\lambda}(\omega)$ defines a function $\frac{dF}{d\lambda}$ on Δ_F which we shall call the *derivative* of F with respect to λ, or simply the λ-*derivative* of F. It is easily checked that $\frac{dF}{d\lambda}$ is

- *homogeneous on* Δ_F: for any scalar α, $\frac{dF}{d\lambda}(\alpha F) = \alpha \frac{dF}{d\lambda}(F)$,
- *additive* in the sense that for any pair of set functions $F, G : \Sigma \to X$, $\frac{dF}{d\lambda}(F + G)(\omega) = \frac{dF}{d\lambda}(F)(\omega) + \frac{dF}{d\lambda}(G)(\omega)$ for every $\omega \in \Delta_F \cap \Delta_G$

In this setting, to establish a FTC is equivalent to finding necessary and sufficient conditions under which the λ-derivative $\frac{dF}{d\lambda}$ of an additive set function F is an element of the class of λ-density functions $d_\lambda F$. More formally, we say that

Definition 20. The FTC holds for an additive set function $F : \Sigma \subset 2^\Omega \to X$ if $\frac{dF}{d\lambda} \in d_\lambda F$.

We are now ready to state and prove our version of FTC for set functions taking values in complete locally bounded topological vector spaces.

Theorem 21. *Let Ω be a topological vector space, X a topologically complete locally bounded topological vector spaces and $\lambda : 2^\Omega \to [0, \infty)$ a regular translation invariant size-function. Then the following statements are equivalent for an additive λ-differentiable set function $F : \Sigma \to X$.*

(1) *For every $N_0 \in \mathcal{B}_0(X)$, there exists $P_0 \in \Pi(\Omega, \lambda)$ consisting of elements of*

$$A_0 = \left\{(I, \omega) \in \Pi(\Omega, \lambda) : F(I) - \lambda(I)\frac{dF}{d\lambda}(\omega) \in N_0\right\}$$

such that whenever $P \succ P_0$, $\sum_{(I,\omega)\in P\setminus A_0} F(I) \in N_0$, and $\sum_{(I,\omega)\in P\setminus A_0} \lambda(I)\frac{dF}{d\lambda}(\omega) \in N_0$.

(2) $\frac{dF}{d\lambda} \in d_\lambda F$.

Proof. 1. $\Rightarrow$ 2. Since $\lambda(\Omega) < \infty$, we may assume all the subpartitions $P \in \Pi(\Omega)$ are partitions by simply adjoining if necessary the complement of $\bigsqcup P$. Fix $N_0 \in \mathcal{B}_0(X)$. Let $P_0 \in \Pi(\Omega)$ as in 1. Then by the additivity of F we have for $P \succ P_0$,

$$\begin{aligned} F(\Omega) - \left(\frac{dF}{d\lambda}\right)_\lambda (P) &= \sum_{(I,\omega)\in P} F(I) - \sum_{(I,\omega)\in P} \lambda(I)\frac{dF}{d\lambda}(\omega) \\ &= \sum_{(I,\omega)\in P\cap A_0} \left(F(I) - \lambda(I)\frac{dF}{d\lambda}(\omega)\right) \\ &\quad + \sum_{(I,\omega)\in P\setminus A_0} F(I) - \sum_{(I,\omega)\in P\setminus A_0} \lambda(I)\frac{dF}{d\lambda}(\omega) \\ &\in \sum_{(I,\omega)\in P\cap A_0} \lambda(I_i)N_0 + N_0 + N_0 \subset (\lambda(\Omega) + 2)N_0 \end{aligned}$$

Since $N_0 \in \mathcal{B}_0(X)$ is arbitrary, this shows that $\frac{dF}{d\lambda} \in \mathcal{I}(\Omega, X, \lambda)$ and $\int_\Omega \frac{dF}{d\lambda} d\lambda = F(\Omega)$. This proves 1. $\Rightarrow$ 2. .

2. $\Rightarrow$ 1. Assume that $\frac{dF}{d\lambda} \in d_\lambda F$. Then in particular $\frac{dF}{d\lambda} \in (\mathcal{I}(A, X, (\Sigma, \lambda)), \Theta)$. Fix $N_0 \in \mathcal{B}_0(X)$ and let $N_{00} \in \mathcal{B}_0(X)$ such that $\overline{N}_{00} + \overline{N}_{00} \subset 2N_0$. Let $E_k = \left\{\omega \in \Omega : \frac{dF}{d\lambda}(\omega) \in kN_{00} \setminus (k-1)N_{00}\right\}$. There exists $P_{k,0} \in \Pi(\Omega)$ such that for every $P \succ P_{k,0}$,

$$\sum_{(I,\omega)\in P} (F(I) - \lambda(I)f(\omega)) \in \frac{1}{k2^{k+1}} N_{00}.$$

For each $n \in \mathbb{N}$, let

$$f_n(\omega) = \begin{cases} \frac{dF}{d\lambda}(\omega) & \text{if } \omega \in \bigcup_{k=1}^{n} E_k \\ 0 & \text{otherwise.} \end{cases}$$

Then for $P \succ \bigvee_{k=1}^{n} P_{k,0}$, we have

$$\sum_{(I,\omega)\in P\setminus A_0} \lambda(I) f_n(\omega) \quad \in \quad \sum_{(I,\omega)\in P\setminus A_0} \lambda(I) k N_{00} \subset \sum_{(I,\omega)\in P\setminus A_0} k\left(F(I) - \lambda(I) f(\omega)\right) \subset \frac{1}{2}\overline{N}_{00}.$$

On taking the limit as $n \to \infty$, we infer that

$$\sum_{(I,\omega)\in P\setminus A_0} \lambda(I) f_n(\omega) \in \frac{1}{2}\overline{N}_{00} \subset N_0.$$

It then follows that

$$\sum_{(I,\omega)\in P\setminus A_0} F(I) \quad \subset \quad \sum_{(I,\omega)\in P\setminus A_0} (F(I) - \lambda(I) f(\omega)) + \sum_{(I,\omega)\in P\setminus A_0} \lambda(I) f(\omega) \subset \frac{1}{2}\overline{N}_{00} + \frac{1}{2}\overline{N}_{00} \subset N_0.$$

The proof is complete. □

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