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A Characterization for non-DCC Lattices

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Abstract. Join-irreducible elements in a lattice have an important role. They act like blocks of a lattice. In DCC lattices each element of the lattice has a unique finite representation as a join of join-irreducible elements. In this paper, we seek lattices which contains elements that can be represented as an infinite supremum of join-irreducible elements. One of these lattices is the lattice of sequences. Finally, we give a new characterization for such lattices.

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1. Introduction

The optimization problem over a bounded chain has been studied(See [9,10]). The extension of this concept to a bounded distributive lattice has studied in [6,7,8]. In [6] an algorithm was given which compute the minimum value of an objective function with respect to a linear system of inequalities. This algorithm totally depends on the set $J(a)$; the set of all join-irreducible elements which are less than or equal a.
It is well known that every element of a lattice can be represented as a finite supremum of join-irreducible elements if and only if it satisfies the DCC (see [1,5]). Moreover, one can find some lattices which do not satisfy the DCC (see Example 3.9); we call them non-DCC lattices.

Hence, there exist elements which are not a finite supremum of join-irreducible elements. In some cases there may exist completely join-irreducible elements, and we can show each element of lattice as a join of completely join-irreducible elements.
In [3] Crawley and Dilworth proved that, L contains elements which are a join of completely join-irreducible elements(i.e. L is non-DCC lattice), if and only if $L \cong D(P)$ for some partially ordered set P (where $D(P)$ is the ordered set of all order ideals of P, with inclusion).
Now, we will introduce the lattice of infinite sequences over a lattice. Clearly, the lattice of infinite sequences is non-DCC. Hence, it contains elements which are not a finite supremum of join-irreducible elements. In fact, the lattice of infinite sequences over a bounded below lattice which has a completely join-irreducible element, gives us a class of non-DCC lattices. After that, we show that non-DCC lattices are embedded to lattice of sequences. Hence, they are isomorphic to a sublattice of lattice of sequences. Therefore, we give a new characterization for non-DCC lattices.

2. Preliminaries

In this section we give some preliminaries which we need in next section. In the next exposition the terminology for lattice theory and algebra is according to [1,3,5].

Definition 2.1. *A partially ordered set P satisfies the descending chain condition (DCC) when every non-void subset of P has a minimal element.*
Clearly, if a partially ordered set P satisfies the DCC, then so do all its subsets(under the same partial ordering).

Example 2.2. (i) Any finite partially ordered set (and therefore any finite lattice) P satisfies the DCC.
(ii) Let N be the set of all natural numbers. Then, it satisfies the DCC, with respect to both usual order on N and divisibility relation.
(iii) Let $L \simeq N \times N \times ... \times N$ (n-times) and $\leqslant$ is the order induced by

usual order or divisibility relation on N. Then, L satisfies the DCC.

Definition 2.3. *(i) A lattice L is distributive if the two following equivalent conditions shold,*

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \quad \forall\, x, y, z \in L \tag{1}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z), \quad \forall\, x, y, z \in L \tag{2}$$

(ii) A lattice $(L, \leqslant)$ is called infinitely distributive if

$$a \wedge (\bigvee b_i \mid i \in I) = \bigvee (a \wedge b_i \mid i \in I) \tag{3}$$

$$a \vee (\bigwedge b_i \mid i \in I) = \bigwedge (a \vee b_i \mid i \in I) \tag{4}$$

Equality (3) is called join infinite distributive or JID. In the same way, (4) is called meet infinite distributive or MID.
Note that, JID and MID may not hold in every complete distributive lattice. Also they may not imply each other.

Example 2.4.([1]). Let $(L, \leqslant)$ be the complete lattice of all closed subset of the plane. Let c denote the disc $x^2 + y^2 = 1$ and d_k denote the disc $x^2 + y^2 = 1 - k^{-2}$, then $c \wedge (\bigvee_{i=1}^{\infty} d_k) = c$ and $\bigvee_{k=1}^{\infty} (c \wedge d_k)$ is the empty set. Therefore, JID does not hold. On the other hand, MID holds, since in this case $\vee$ and $\wedge$ coincide with the set theoretic operations $\cup$ and $\cap$, respectively.

Definition 2.5. *(i) In a bounded lattice L, an element a is called complement of b if $a \wedge b = 0$ and $a \vee b = 1$. In this case b is shown by $a^{'}$.*
(ii) A complemented lattice is a bounded lattice in which every element has a complement.
(iii) A Boolean lattice L is a complemented distributive lattice.
(iv) A Boolean algebra is an algebra $(B; \wedge, \vee, 0, 1)$ with two binary operations, one unary operation (called complemention) which satisfies:
1) $(B; \wedge, \vee)$ is distributive lattice.
2) $\forall x \in B$, $x \wedge 0 = 0$, $x \vee 1 = 1$,
3) $\forall x \in B \;\; \exists x^{'} \in B$, $x \wedge x' = 0$, $x \vee x^{'} = 1$.
The following theorem gives a well-known characterization for DCC lattices.

Theorem 2.6. **([1])** *Let $(L, \leqslant)$ be a lattice. Then every element of L is a finite supremum of join-irreducible elements if and only if L satisfies the DCC.* □
Of course, there are some lattices which are non-DCC. (See Example 3.9.)

Definition 2.7. *(i) Let $(L, \leqslant)$ be a lattice. Then, every non-zero minimal element of L is called an atom.*
(ii) An atomic lattice L is a lattice in which every element is a join of atoms, and hence of the atoms which it contains.
(iii) An element $a \in L$ is called join-irreducible if $a = b \vee c$ implies $a = b$ or $a = c$. We denote the set of all join-irreducible elements of L, by $J(L)$.
(iv) An element $a \in L$ is completely join-irreducible if $a = \vee K$ implies $a \in K$, for each non-empty $K \subseteq L$. Denote the set of all completely join-irreducible elements of L by $CJ(L)$.
(v) For each $a \in L$, let $J(a)$ and $CJ(a)$, be the set of all join-irreducible elements and completely join-irreducible elements less than or equal a, respectively. In the other words, $J(a) = \{j \in J(L) | j \leqslant a\}$ and $CJ(a) = \{j \in CJ(L) | j \leqslant a\}$.

Example 2.8. (i) Let X be an arbitrary set. Then $P(X)$, the set of all subsets of X, is a Boolean atomic lattice.
(ii) Let $L = [0, 1]$ be the bounded chain of real numbers between 0 and 1, Then L is an atom-less lattice while every element $a \in L$ is join-irreducible.
(iii) Let $L = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 27, 36, 54, 72, 108, 216\}$ be the set of all divisors of 216 and $x \leqslant y$ if x divides y. The set of join-irreducible elements of L is $J(L) = \{1, 2, 3, 4, 8, 9, 27\}$ while L has only two atoms; 2 and 3. The relationship between elements of L is shown in Figure 1.
(iv) [4] Let $O_{reg}(R)$ be the set of all regular open subsets of real numbers, that is, those sets equal to interior of their closure. The *sup* is not the union of the regular open sets but the interior of the closure of the union. The inf is the interior of the intersection. Remarkably, $O_{reg}(R)$ is a complete Boolean lattice where the lattice complement of $U \in O_{reg}(R)$ is the interior of $R \backslash U$. Note that $O_{reg}(R)$ has neither atom nor join-

irreducible element.

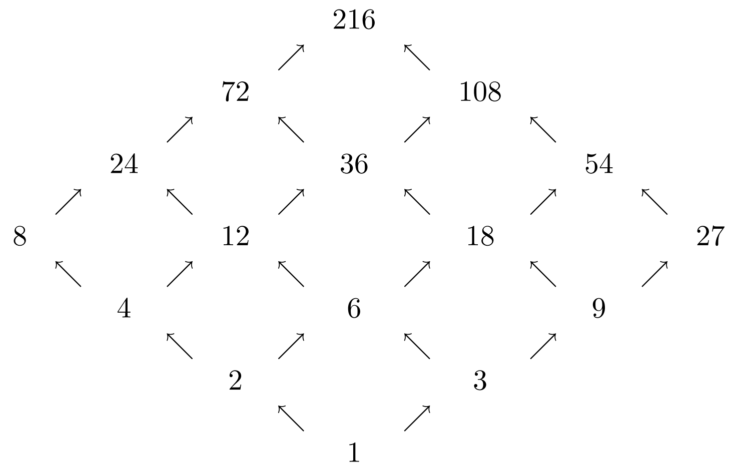

Figure 1

Remark 2.9. *Clearly, all completely join-irreducible are join-irreducible elements, but not vice versa.*

Example 2.10. Let $L = [0, 1]$ be the bounded chain of the real number between 0 and 1. As in Example 2.8(ii), every element of L is join-irreducible but not completely join-irreducible. Because of completeness of R, each real number can be the supremum of all numbers less than it. So, L has not any completely join-irreducible element.
One can find a lattice such that every join-irreducible is completely join-irreducible. (see Example 3.6.)

Definition 2.11. (**[3]**) *(i) Let L be a complete lattice and $a \in L$. Then, a is called compact if $a \leqslant \vee X$ for some $X \subseteq L$ implies that $a \subseteq X_1$ for some finite subset X_1 of X.*
A complete lattice is called algebraic iff every element is the join of compact element.

Remark 2.12. *The name algebraic lattice, is due to G. Birkhoff, however Birkhoff does not assume completeness. In this paper, algebraic lattices are also called compactly generated lattices, due to Grätzer(See [5]).*

Example 2.13. (**[2]**) (i) Let $L = P(X)$ for some arbitrary set X. Then, every finite subset of X is a compact element of L.
(ii) The subset $[0, 1]$ of the real line is a complete lattice, but is not compactly generated.
In the following theorem one can see a characterization of lattices in which each element is a join of completely join- irreducible elements.

Theorem 2.14. (**[3]**) *For a lattice L, the following conditions are equivalent:*
(1) L is distributive and both L, and its dual are compactly generated.
(2) L is a compactly generated distributive lattice, in which each element is a join of completely join-irreducible elements.
(3) $L \cong D(P)$ for some partially ordered set P (where $D(P)$ is the set of all order ideals of P, with inclusion).
(4) L is a complete sublattice of some complete and atomic Boolean algebra. □

3. Main Results

In this section we looking for lattices that every elements of them can be represented by a join of completely join-irreducible elements. First, we construct a lattice of sequences of a certain lattice. Then, we find the join-irreducible and completely join-irreducible elements of lattice of sequences. In Theorem 3.7 we show that the lattice of sequences has elements which can be represented by an infinite join of completely join-irreducible elements. Finally, in Theorem 3.13 and its corollary we find a new characterization for non-DCC lattices.

Definition 3.1. *Let $(L, \leqslant)$ be a lattice. Then, the lattice of infinite sequences over L is denoted by $IS(L)$ and defined as follows*:

$$IS(L) := \{\{a_i\}_{i\in N} | a_i \in L, \forall i \in N\}.$$

Remark 3.2. *Note that the order relation on L, induced a component-wise order on $IS(L)$ as*

$$\{a_i\} \leqslant \{b_i\} \Longleftrightarrow a_i \leqslant b_i, \quad \forall i \in N.$$

Hence, $IS(L)$ is a lattice.
In the following theorem one can find the relationship between join irreducible elements of L and $IS(L)$.

Theorem 3.3. *Let $(L, \leqslant)$ be a bounded below lattice, $CJ(L)$ and $CJ(IS(L))$ be the sets of completely join-irreducible elements of L and $IS(L)$, respectively. An element $\{a_i\} \in IS(L)$ belongs to $CJ(IS(L))$ if and only if there exists a unique $k \in N$ such that $a_k \in CJ(L)$ and $a_i = 0, \forall i \neq k$, where 0 is the least element of L.*

Proof. Let $\{a_i\} \in IS(L)$ and there exists a unique $k \in N$ such that $a_k \in CJ(L)$ and $a_i = 0, \quad \forall i \neq k$. We show that $\{a_i\}$ is completely join-irreducible. Suppose $\{a_i\}_{i=1}^{\infty} = \bigvee_{j=1}^{\infty}\{p_{i_j}\}$. Then, $a_i = \bigvee_{j=1}^{\infty} p_{i_j}$ for all i.
If $a_i = 0$ then, $\forall j \;\; p_{i_j} = 0$. For $a_k = \bigvee p_{k_j}$, we have $a_k = p_{k_{j_0}}$ for some j_0, since $a_k \in CJ(L)$. Now, let $\{a_i\} \in CJ(IS(L))$. Suppose, $\{a_i\}$ has at least two non zero components say, a_s and a_t. In this case we have, $\{a_i\} = \{b_i\} \vee \{c_i\}$ where,

$$b_i = \begin{cases} a_s & i = s \\ 0 & i = t \\ a_i & i \neq s, t \end{cases} \quad \text{and } c_i = \begin{cases} 0 & i = s \\ a_t & i = t \\ a_i & i \neq s, t \end{cases}$$

Clearly, $\{a_i\} \neq \{b_i\}$ and $\{a_i\} \neq \{c_i\}$ which contradicts the fact that $\{a_i\} \in CJ(IS(L))$. Hence $\{a_i\}$ has only one nonzero component, say a_k. Obviously, $a_k \in CJ(L)$. □

Example 3.4. Let L be the lattice of non negative integers. Then every element of L is completely join-irreducible. Hence

$$CJ(IS(L)) = \{\{a_i\} | \exists! k, a_k \neq 0 \text{ and } a_i = 0, \forall i \neq k\}.$$

For simplicity, we give the following definition.

Definition 3.5. ([3]) *A lattice L is call CJ-lattice, if each element of L is join of $CJ(L)$.*

Example 3.6. (i) Let L be the lattice of non negative integers, then every element is join-irreducible and every join-irreducible is completely

join-irreducible, so L is CJ-lattice.
(ii) let L be the chain of $s_t's$ where $s_t := [t, \infty)$ and $t \in R$ which ordered by inclusion. Note that, an infinite supremum of elements of L, is a closed half- interval in R. Hence, every s_t is join-irreducible and every join-irreducible is completely join-irreducible, so L is CJ-lattice.

Theorem 3.7. *Let $(L, \leqslant)$ be a bounded below lattice and $CJ(L) \neq \emptyset$. Then, $IS(L)$ contains elements such that they can represented by infinite supremum of completely join-irreducible elements. Moreover, $IS(L)$ is CJ-lattice.*

Proof. It is clear by construction of $IS(L)$. □

Remark 3.8. *Note that there exist some lattices which are not lattice isomorphic to $IS(L)$ for any L. But they contain an element such that it can be represented as an infinite supremum of completely join-irreducible elements.*

Example 3.9. Let X be an infinite set. Consider $(P(X), \subseteq)$ be the Boolean lattice of all subsets of X. We know that completely join-irreducible elements of $P(X)$ are singletons. If Y is an infinite subset of X, then it can be represented as infinite supremum of completely join-irreducible elements.
How ever, in special cases we find an isomorphism between $P(X)$ (or any non-DCC lattice) and lattice of sequence. To do this, we need the following definition.

Definition 3.10. **([3])** *The representation $a = \bigvee_{i \in I} p_i$ is irredundant, if $a \neq \bigvee_{i \in I - \{J\}}$ for all $J \subseteq I$. In the other words, we can not omit any element p_i in the representation of a.*

Proposition 3.11. *Let L be a lattice satisfy JID. If a has a representation $\bigvee_{i \in I} p_i$ of completely join irreducible elements of L, Then this representation is unique.*

Proof. Suppose
$$\bigvee_{i \in I} p_i = \bigvee_{j \in J} q_j \tag{5}$$

are two irredundant representations of a, and there exists an index m such that $p_m \neq q_i$, $\forall i$. Then, $p_m \leqslant \bigvee_{i\in I} p_i = \bigvee_{j\in J} q_j$. Hence, $p_m = p_m \wedge \bigvee_{j\in J} q_j$. Since L satisfies JID then, $p_m = \bigvee_{j\in J}(p_m \wedge q_i)$. As p_m is completely join-irreducible, it must be equal $p_m \wedge q_j$ for some j. Hence $p_m \leqslant q_j$. Again replacing p_m with q_j, $q_j \leqslant p_n$, which contradicts the fact that the $\bigvee_{i\in I} p_i$ is an irredundant representation of a. Therefore, $p_m = q_j$. Replacing p_m with any join-irreducible appearing in (5) and repeating this argument, we find that the two representations must be identical. □

Definition 3.12. *Let L be a lattice which satisfies JID. Define $ICJ(x)$ to be the set of all completely join-irreducible less than or equal of x, which appear in irredundant representation of x.*

Theorem 3.13. *Let $(L, \leqslant)$ be a bounded below lattice, that satisfies JID and L be CJ-lattice. If $CJ(L)$ indexed by N, the set of all natural numbers, then L is embedded into $IS(CJ(L))$.*

Proof. As $CJ(L)$ is indexed by N, $CJ(L) = \{j_i\}_{i=1}^{\infty}$. Define

$$\varphi : L \to IS(CJ(L))$$

by $\varphi(x) = \{a_i\}_{i=1}^{\infty}$ where,

$$\begin{cases} a_i = j_i & j_i \in ICJ(x). \\ a_i = 0 & \textit{otherewise}. \end{cases}$$

By Proposition 3.11, φ is well defined. φ is also order preserving. Suppose $x \leqslant y$. Then,

$$CJ(x) \subseteq CJ(y)$$

and by definition $\varphi(x) \leqslant \varphi(y)$, so φ is order preserving. Now, suppose $\varphi(x) = \varphi(y)$. Then, $\{a_i\} = \{b_i\}$, for $a_i, b_i \in CJ(L)$. Then, $a_i = b_i$ for all i. On the other hand $x = \bigvee_{j=1}^{\infty} p_{i_j}$ and $y = \bigvee_{j=1}^{\infty} q_{i_j}$. So, $x = y$. □

Corollary 3.14. *Let L be as in Theorem 3.13. Then, L is isomorphic to a sublattice of $IS(CJ(L))$.* □

4. Conclusion

In this research, we introduced a lattice of sequence and in Theorem 3.7. we proved that every element of this lattice can be represented by infinite supremum of completely join irreducible elements. Then, we gave an example which is not isomorphic with this lattices. Finally, we found a characterization for a lattice L which every element of L is a join of completely join-irreducible elements.

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