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Arrow-Hurwitcz-Uzawa Constraint Qualification for Nonsmooth Semi-Infinite Optimization with Mixed Constraints

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Abstract. This paper is devoted to the study of semi-infinite optimization with nonsmooth data. We introduce the Arrow-Hurwitcz-Uzawa constraint qualification which is based on the Clarke subdifferential. Then, we derive a suitable Karush-Kuhn-Tucker type necessary optimality condition.

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1. Introduction

In this paper we study the following semi-infinite programming problem (SIP, in brief)

$$\begin{array}{lll} (SIP) & \inf f(x) & \\ s.t. & g_i(x) \leqslant 0 & i \in \mathfrak{I}, \\ & h_j(x) = 0 & j \in \mathfrak{A}, \\ & x \in \mathbb{R}^n, & \end{array}$$

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where f and g_i, $i \in \mathfrak{I}$, and h_j, $j \in \mathfrak{A}$ are locally Lipschitz functions from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$, and the index sets $\mathfrak{I}$ and $\mathfrak{A}$ are arbitrary sets with $\mathfrak{I} \cup \mathfrak{A} \neq \emptyset$, not necessarily finite. In the review papers [4, 12], as well as in [3], we will find many applications of SIP in different fields such as Chebyshev approximation, robotics, mathematical physics, engineering design, optimal control, transportation problems, fuzzy sets, robust optimization, etc.

Some constraint qualifications for nonconvex and nondifferentiable SIPs (with $\mathfrak{A} = \emptyset$) are introduced in [6, 7, 8, 9, 10]; for instance Abadie, Basic, Zangwill, Mangasarian-Fromovitz, Slater, and Guignard constraint qualifications. There presented Fritz-John and Karush-Kuhn-Tucker type necessary and sufficient optimality conditions for these problem.

The aim of this paper is to introduce the Arrow-Hurwicz-Uzawa constraint qualification and to provide the Karush-Kuhn-Tucker type condition for optimal solution of nonsmooth SIP.

We organize the paper as follows. In Section 2, basic notations and results of nonsmooth analysis are reviewed. In Section 3, we present our main results.

2. Notations and Preliminaries

In this section we briefly overview some notions of onvex analysis and nonsmooth analysis from [2, 5].

Given a nonempty set $M \subseteq \mathbb{R}^n$, we denote by $cl(M)$, $conv(M)$, and $cone(M)$, the closure of M, convex hull and convex cone (containing the origin) generated by M, respectively. The polar cone and strict polar cone of M are defined respectively by:

$$M^0 := \{d \in \mathbb{R}^n \mid \langle x, d\rangle \leqslant 0, \quad \forall x \in M\},$$

$$M^- := \{d \in \mathbb{R}^n \mid \langle x, d\rangle < 0, \qquad \forall x \in M\},$$

where $\langle ., .\rangle$ exhibits the standard inner product in $\mathbb{R}^n$. Notice that M^0 is always closed convex cone. It is easy to show that if $M^- \neq \emptyset$, then $cl(M^-) = M^0$.

Definition 2.1. *Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function and $\hat{x} \in dom(f)$.*

I: *The generalized Clarke directional derivative of φ at $\hat{x}$ in the direction $d \in \mathbb{R}^n$ is defined by*

$$\varphi^0(\hat{x}; d) := \limsup_{y \to \hat{x}, t \downarrow 0} \frac{\varphi(y + td) - \varphi(y)}{t}.$$

II: *The Clarke subdifferential of φ at $\hat{x}$ is defined by*

$$\partial_c \varphi(\hat{x}) := \{\xi \in \mathbb{R}^n \mid \varphi^0(\hat{x}; d) \geqslant \langle \xi, d \rangle, \quad \forall d \in \mathbb{R}^n\}.$$

Observe that the Clarke subdifferential of a locally Lipschitz function at an interior point of its domain is always nonempty, compact, and convex cone. The Clarke subdifferential reduce to the classical gradient for continuously differentiable functions and to the subdifferential of convex analysis for convex ones.

Let us recall the following theorems which will be used in the sequel.

Theorem 2.2. ([5]) *Let $\{M_\alpha | \alpha \in \Lambda\}$ be an arbitrary collection of nonempty convex sets in $\mathbb{R}^n$. Then, every non-zero vector of conv$\big(\bigcup_{\alpha \in \Lambda} M_\alpha\big)$ can be expressed as a non-negative linear combination of n or fewer linearly independent vectors, each belonging to a different M_α.*

Theorem 2.3. ([2]) *Let φ and ψ are locally Lipschitz from $\mathbb{R}^n$ to $\mathbb{R}$, and $\hat{x} \in dom(\varphi) \cap dom(\psi)$. Then, the following properties hold:*

a: $\varphi^0(\hat{x}; d) = \max\{\langle \xi, d \rangle \mid \xi \in \partial_c \varphi(\hat{x})\}, \qquad \forall d \in \mathbb{R}^n.$

b: $d \to \varphi^0(\hat{x}; d)$*is a convex function, and*

$$\partial_c \varphi(x) = \partial \varphi^0(x; .)(0),$$

where $\partial \varphi(\hat{x})$ denotes the subdifferential of convex function φ at $\hat{x}$.

c: $x \mapsto \varphi(x)$*is an upper semicontinuous set-valued function.*

d: $\partial_c(\varphi+\psi)(\bar{x}) \subseteq \partial_c\varphi(\bar{x}) + \partial_c\psi(\bar{x})$.

Furthermore, if φ and ψ are convex, then equality holds in above virtue.

e: *If $\hat{x}$ is a minimum point of φ over $\mathbb{R}^n$, then $0 \in \partial_c\varphi(\hat{x})$.*

Definition 2.4. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. φ is said to be ∂_c-pseudoconcave at $\hat{x}$ if for all $x \in \mathbb{R}^n$,*

$$\varphi^0(\hat{x}; x-\hat{x}) \leqslant 0 \Longrightarrow \varphi(x) \leqslant \varphi(\hat{x}).$$

If $(-\varphi)$ is ∂_c-pseudococave at $\hat{x}$,then φ is said to be ∂_c-pseudoconvex at $\hat{x}$. φ is said to be ∂_c-pseudoaffine at $\hat{x}$ if it is both ∂_c-pseudoconcave and ∂_c-pseudoconvex at $\hat{x}$.

3. Main Results

At starting point of this section, let P denotes the feasible solutions of SIP

$$P := \{x \in \mathbb{R}^n \mid g_i(x) \leqslant 0,\ h_j(x) = 0, \qquad \forall\ (i,j) \in \mathfrak{I} \times \mathfrak{A}\}.$$

For a given $\hat{x} \in P$, let $\mathfrak{I}^{\hat{x}}$ denotes the index set of all active constraints at $\hat{x}$; that is

$$\mathfrak{I}^{\hat{x}} := \{i \in \mathfrak{I} \mid g_i(\hat{x}) = 0\}.$$

Set

$$\begin{aligned} &\mathfrak{I}_1 := i \in \mathfrak{I} \mid g_i \text{ is } \partial_c-\text{pseudoconcave at } \hat{x}, \\ &\mathfrak{I}_2 := \mathfrak{I} \backslash \mathfrak{I}_1, \\ &G(x) := \sup_{i \in \mathfrak{I}_2} g_i(x), \qquad \forall x \in P. \end{aligned}$$

One reason for difficulty of extending the results from a finite inequality problem to SIP is that in the finite case $G(.)$ is locally Lipschitz and we have (see [2, Propisition 2.3.12])

$$\partial_c G(\hat{x}) \subseteq conv\Big(\bigcup_{i \in \mathfrak{I}_2 \cap \mathfrak{I}^{\hat{x}}} \partial_c g_i(\hat{x})\Big), \qquad \forall x \in P, \tag{1}$$

but in general, (1) does not hold if $\mathfrak{I}$ is infinite (see [2, Theorem 2.8.2]). Let

$$\begin{aligned}
\mathcal{G}_1(\hat{x}) &:= \bigcup_{i\in\mathfrak{I}_1\cap\mathfrak{I}^{\hat{x}}} \partial_c g_i(\hat{x}),\\
\mathcal{G}_2(\hat{x}) &:= \bigcup_{i\in\mathfrak{I}_2\cap\mathfrak{I}^{\hat{x}}} \partial_c g_i(\hat{x}),\\
\mathcal{G}(\hat{x}) &:= \mathcal{G}_1(\hat{x})\cup\mathcal{G}_2(\hat{x}) = \bigcup_{i\in\mathfrak{I}^{\hat{x}}} \partial_c g_i(\hat{x}),\\
\mathcal{H}(\hat{x}) &:= \left(\bigcup_{j\in\mathfrak{A}} \partial_c h_j(\hat{x})\right)\cup\left(\bigcup_{j\in\mathfrak{A}} \big(-\partial_c h_j(\hat{x})\big)\right),\\
\mathcal{H}_*(\hat{x}) &:= \left(\bigcup_{j\in\mathfrak{A}} \partial_c h_j(\hat{x})\right).
\end{aligned}$$

We should observe that $\mathcal{H}^0(\hat{x}) = (\mathcal{H}_*(\hat{x}))^\perp$, where $(\mathcal{H}_*(\hat{x}))^\perp$ denotes the orthogonal space of $\mathcal{H}_*(\hat{x})$, i.e.,

$$(\mathcal{H}_*(\hat{x}))^\perp := \{d\in\mathbb{R}^n \mid \langle d,h\rangle = 0,\quad \forall h\in\mathcal{H}_*(\hat{x})\}.$$

We now extend the *Arrow-Hurwicz-Uzawa constraint qualification* (AHUCQ, in brief) for SIP.

Definition 3.1. *Let $\hat{x}$ be a feasible solution of SIP. We say that the AHUCQ is satisfied at $\hat{x}$ if $h_j(.)$ is ∂_c- pseudoaffine at $\hat{x}$ for each $j\in\mathfrak{A}$, and $G(.)$ is Lipschitz around $\hat{x}$, and*

(i):

$$\partial_c G(\hat{x}) \subseteq conv\big(\mathcal{G}_2(\hat{x})\big). \tag{2}$$

(ii):

$$\mathcal{G}_1^0(\hat{x})\cap\mathcal{G}_2^-(\hat{x})\cap\mathcal{H}^0(\hat{x}) \neq \emptyset. \tag{3}$$

Remarks 3.2.

1. *The definition 3.1 reduces to the classical AHUCQ -which is considered in [1]- for finite differentiable problems with* $\mathfrak{A} = \emptyset$.

2. *It is proved in [8] that if for all* $i \in \mathfrak{I}$, g_i *is convex function;* $\mathfrak{I}$ *is a compact set in some metric space; for each fixed* $\tilde{x} \in P$ *the function* $i \to g_i(\tilde{x})$ *is upper semicontinuous on* $\mathfrak{I}$, *and* $\mathfrak{A} = \emptyset$, *then (2) verifies at every* $\hat{x} \in P$.

3. *It is shown in [9] that there is no any relation of implication between the inclusions (2) and (3).*

Now, the optimality condition of KKT-type for SIP is stated as follows.

Theorem 3.3. *Suppose that* $\hat{x}$ *is an optimal solution of SIP. Assume that the AHUCQ is satisfied at* $\hat{x}$.

(a): *One has*

$$0 \in \partial_c f(\hat{x}) + cl\big(cone\big(\mathcal{G}(\hat{x})\big)\big) + span\big(\mathcal{H}_*(\hat{x})\big). \tag{4}$$

(b): *If, in addition,* $cone\big(\mathcal{G}(\hat{x})\big) + span\big(\mathcal{H}_*(\hat{x})\big)$ *is closed, then there exist scalers* $\lambda_i \geqslant 0$, $i \in \mathfrak{I}^{\hat{x}}$ *and* $\mu_j \in \mathbb{R}$, $j \in \mathfrak{J}$, *which finite numbers of them are nonzero, such that*

$$0 \in \partial_c f(\hat{x}) + \sum_{i \in \mathfrak{I}^{\hat{x}}} \lambda_i \partial_c g_i(\hat{x}) + \sum_{j \in \mathfrak{J}} \mu_j \partial_c h_j(\hat{x}). \tag{5}$$

Proof. We can choice a vector $d \in \mathcal{G}_1^0(\hat{x}) \cap \mathcal{G}_2^-(\hat{x}) \cap \mathcal{H}^0(\hat{x})$ by (2). Thus

$$\langle \xi, d < 0, \qquad \forall \xi \in \mathcal{G}_2(\hat{x}), \tag{6}$$

$$\langle \eta, d \leqslant 0, \qquad \forall \eta \in \mathcal{G}_1(\hat{x}), \tag{7}$$

$$\langle \eta, d = 0, \qquad \forall \eta \in \mathcal{H}(\hat{x}). \tag{8}$$

By (8) and the ∂_c-affinity of h_j for $j \in \mathfrak{A}$, we have (for each $\beta > 0$)

$$h_j^0\big(\hat{x}; (\hat{x}+\beta d)-\hat{x}\big) = \beta h_j^0(\hat{x}; d) = 0 \;\Rightarrow\; h_j(\hat{x}+\beta d) = \beta h_j(\hat{x}) = 0, \quad \forall j \in \mathfrak{A}. \tag{9}$$

On the other hand, with regard to (7), we have

$$g_i^0(\hat{x}; d) \leqslant 0, \qquad \forall\, i \in \mathfrak{I}_1.$$

Thus, for all $\hat{\beta} \in (0, 1]$ we obtain

$$g_i^0(\hat{x}; \frac{1}{\hat{\beta}}[(\hat{x} + \hat{\beta}d) - \hat{x}]) = g_i^0(\hat{x}; d) \leqslant 0, \quad \forall\, i \in \mathfrak{I}_1.$$

Using the pseudoconcavity of g_i for $i \in \mathfrak{I}_1$, we get

$$g_i(\hat{x} + \hat{\beta}d) \leqslant g_i(\hat{x}) \leqslant 0, \quad \forall\, \hat{\beta} \in (0, 1], \quad \forall\, i \in \mathfrak{I}_1. \tag{10}$$

Now, suppose that $\widehat{\xi} \in conv\big(\mathcal{G}_2(\hat{x})\big)$. Then, there exist scalers $\gamma_1, ..., \gamma_s \geqslant 0$, and vectors $\xi_1, ..., \xi_s \in \mathcal{G}_2(\hat{x})$, such that

$$\sum_{v=1}^{s} \gamma_v = 1, \qquad \widehat{\xi} = \sum_{v=1}^{s} \gamma_v \xi_v.$$

Using the virtue of (6) we have

$$\big\langle \widehat{\xi}, d \big\rangle = \sum_{v=1}^{s} \gamma_v \big\langle \xi_v, d \big\rangle < 0,$$

and hence –in view of (2)– we conclude

$$d \in \Big(conv\big(\mathcal{G}_2(\hat{x})\big)\Big)^{-} \subseteq \big(\partial_c G(\hat{x})\big)^{-}.$$

Thus $G^0(\hat{x}; d) < 0$, and consequently, there exists a scaler $\delta_1 > 0$, such that

$$g_i(\hat{x} + \underline{\mathrm{d}}) \leqslant G(\hat{x} + \underline{\mathrm{d}}) < G(\hat{x}) \leqslant 0, \quad \forall\, 0 \leqslant \beta \leqslant \delta, \quad \forall\, i \in \mathfrak{I}_2. \tag{11}$$

Therefore, in view of (9)-(11), we have

$$\hat{x} + td \in P, \qquad \forall\, 0 \leqslant t \leqslant \min 1, \delta_1,$$

and by minimality of $\hat{x}$, we conclude that

$$\frac{1}{\hat{\beta}}(f(\hat{x} + td) - f(\hat{x})) \geqslant 0, \qquad \forall\, 0 \leqslant t \leqslant \min 1, \delta_1.$$

Summarizing, –since d is an arbitrary element of $\mathcal{G}_1^0(\hat{x}) \cap \mathcal{G}_2^-(\hat{x}) \cap \mathcal{H}^0(\hat{x})$– we have

$$f^0(\hat{x}; d) \geqslant 0, \qquad \forall\, d \in \mathcal{G}_1^0(\hat{x}) \cap \mathcal{G}_2^-(\hat{x}) \cap \mathcal{H}^0(\hat{x}).$$

Since $f(\hat{x}; .)$ is a continuous function, the above relation implies that the inequality $f^0(\hat{x}; d) \geqslant 0$ holds for all d satisfying

$$\begin{aligned} & d \in cl\left(\mathcal{G}_1^0(\hat{x}) \cap \mathcal{G}_2^-(\hat{x}) \cap \mathcal{H}^0(\hat{x})\right) \\ & = \mathcal{G}_1^0(\hat{x}) \cap \mathcal{G}_2^0(\hat{x}) \cap \mathcal{H}^0(\hat{x}) = \left(\mathcal{G}_1(\hat{x}) \cup \mathcal{G}_2(\hat{x}) \cup \mathcal{H}(\hat{x})\right)^0 \\ & = \left(\mathcal{G}(\hat{x}) \cup \mathcal{H}(\hat{x})\right)^0 = \left(cl\left(cone\left(\mathcal{G}(\hat{x}) \cup \mathcal{H}(\hat{x})\right)\right)\right)^0 =: \mathcal{X}. \end{aligned}$$

Thus, the following convex function attains its minimum at $\hat{d} = 0$:

$$\Psi(.) := \Phi_{\mathcal{X}}(.) + f^0(\hat{x}; .),$$

where $\Phi_{\mathcal{X}}(.)$ denotes the indicator function of $\mathcal{X}$, it is defined as

$$\Phi_{\mathcal{X}}(y) := \begin{cases} 0 & if \quad y \in \mathcal{X}, \\ +\infty & if \quad y \notin \mathcal{X}. \end{cases}$$

Hence – in view of Theorem 2.3 – we get

$$0 \in \partial\Psi(0) = \partial\Phi_{\mathcal{X}}(0) + \partial f^0(\hat{x}; .)(0) = cl\left(cone\left(\mathcal{G}(\hat{x}) \cup \mathcal{H}(\hat{x})\right)\right) + \partial_c f(\hat{x}).$$

The above inclusion and the fact that

$$cl\left(cone\left(\mathcal{G}(\hat{x}) \cup \mathcal{H}(\hat{x})\right)\right) = cl\big(cone\big(\mathcal{G}(\hat{x})\big)\big) + span\big(\mathcal{H}_*(\hat{x})\big),$$

justify the result.

(b): It follows from of inclusion (4) and Theorem 2.2. □

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