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A Fixed Point Theorem with Application to a Class of Integral Equations

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Abstract. The notion of dualistic Geraghty Contraction is introduced. A new fixed point theorem is proved in the settings of complete dualistic partial metric spaces. The counterpart theorem provided in partial metric spaces is retrieved as a particular case of our new results. We give example to prove that the contractive conditions in the statement of our new fixed point theorem can not be replaced by those contractive conditions in the statement of the partial metric counterpart fixed point theorem. Moreover, we give an application of our fixed point theorem to show the existence of solution of integral equations.

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1. Introduction

In [7], Matthews introduced the concept of partial metric space as a suitable mathematical tool for program verification and proved an analogue

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of Banach fixed point theorem in complete partial metric spaces. O'Neill [8] introduced the concept of dualistic partial metric, which is more general than partial metric and established a robust relationship between dualistic partial metric and quasi metric. In [9], Oltra and Valero presented a Banach fixed point theorem on complete dualistic partial metric spaces. They also showed that the contractive condition in Banach fixed point theorem in complete dualistic partial metric spaces cannot be replaced by the contractive condition of Banach fixed point theorem for complete partial metric spaces. Later, Valero [10] generalized the main theorem of [9] using nonlinear contractive condition instead of Banach contractive condition.
For the sake of completeness, we recall Geraghty's Theorem. For this purpose, we first remind the class S of all functions $\beta : [0,\infty) \to [0,1)$ which satisfies the condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0.$$

Theorem 1.1. [5] *Let (M,d) be a complete metric space and $T : M \to M$ be a mapping. Assume that there exists $\beta \in S$ such that, for all $k,l \in M$,*

$$d(T(j),T(k)) \leqslant \beta(d(j,k))d(j,k).$$

Then T has a unique fixed point $v \in M$ and, for any choice of the initial point $j_0 \in M$, the sequence $\{j_n\}$ defined by $j_n = T(j_{n-1})$ for each $n \geqslant 1$ converges to the point v.

In [6], Rosa and Vetro have extended the notion of Geraghty contraction to the context of partial metric spaces. Besides, they have yielded partial metric versions of Theorem 1.1 under α-admissible mappings.

In this paper, we shall prove Theorem 1.1 in dualistic partial metric space and then apply it to show the existence of solution of particular class of integral equations.

$$j(w) = g(w) + \int_0^1 G_n(w,s,j(s))\,ds \ \ \forall\ w \in [0,1],\ n \in \mathbb{N}.$$

We need some mathematical basics of dualistic partial metric space and results to make this paper self sufficient.

Throughout this paper, the letters $\mathbb{R}_0^+$, $\mathbb{R}$ and $\mathbb{N}$ will represent set of nonnegative real numbers, set of real numbers and set of natural numbers respectively.
According to O'Neill, a dualistic partial metric can be defined as follows:

Definition 1.2. [8] *Let M be a nonempty set. If a function $D : M \times M \to \mathbb{R}$ satisfies, for all $j, k, l \in M$, the following properties:*

(D_1) $j = k \Leftrightarrow D(j, j) = D(k, k) = D(j, k)$.

(D_2) $D(j, j) \leqslant D(j, k)$.

(D_3) $D(j, k) = D(k, j)$.

(D_4) $D(j, l) + D(k, k) \leqslant D(j, k) + D(k, l)$.

Then D is called dualistic partial metric and the pair (M, D) is known as dualistic partial metric space.

Remark 1.3. *It is obvious that every partial metric is dualistic partial metric but converse is not true. To support this comment, define $D_\vee : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by*

$$D_\vee(j, k) = j \vee k = \sup\{j, k\} \ \forall \ j, k \in \mathbb{R}.$$

It is clear that $D_\vee$ is a dualistic partial metric. Note that $D_\vee$ is not a partial metric, because $D_\vee(-1, -2) = -1 \notin \mathbb{R}^+$. However, the restriction of $D_\vee$ to $\mathbb{R}^+$, $D_\vee|_{\mathbb{R}^+}$, is a partial metric.

Example 1.4. If (M, d) is a metric space and $c \in \mathbb{R}$ is an arbitrary constant, then

$$D(j, k) = d(j, k) + c.$$

defines a dualistic partial metric on M.

Following [8], each dualistic partial metric D on M generates a T_0 topology $\tau(D)$ on M which has, as a base, the family of D-open balls $\{B_D(j, \epsilon) : j \in M, \epsilon > 0\}$ and $B_D(j, \epsilon) = \{k \in M : D(j, k) < \epsilon + D(j, j)\}$.

If (M, D) is a dualistic partial metric space, then $d_D : M \times M \to \mathbb{R}_0^+$ defined by

$$d_D(j,k) = D(j,k) - D(j,j).$$

is called quasi metric on M such that $\tau(D) = \tau(d_D)$ for all $j, k \in M$. Moreover, if d_D is a quasi metric on M, then

$$d_D^s(j,k) = \max\{d_D(j,k), d_D(k,j)\}$$

defines a metric on M.
A sequence $\{j_n\}_{n\in\mathbb{N}}$ in (M, D) converges to a point $j \in M$ if and only if $D(j,j) = \lim_{n\to\infty} D(j, j_n)$.

Definition 1.5. [8] *Let (M, D) be a dualistic partial metric space, then*

(1) *A sequence $\{j_n\}_{n\in\mathbb{N}}$ in (M, D) is called a Cauchy sequence if $\lim_{n,m\to\infty} D(j_n, j_m)$ exists and is finite.*

(2) *A dualistic partial metric space (M, D) is said to be complete if every Cauchy sequence $\{j_n\}_{n\in\mathbb{N}}$ in M converges, with respect to $\tau(D)$, to a point $j \in M$ such that $D(j,j) = \lim_{n,m\to\infty} D(j_n, j_m)$.*

Following lemma will be helpful in the sequel.

Lemma 1.6. [8, 9]

(1) *A dualistic partial metric (M, D) is complete if and only if the metric space (M, d_D^s) is complete.*

(2) *A sequence $\{j_n\}_{n\in\mathbb{N}}$ in M converges to a point $j \in M$, with respect to $\tau(d_D^s)$ if and only if $\lim_{n\to\infty} D(j, j_n) = D(j,j) = \lim_{n\to\infty} D(j_n, j_m)$.*

(3) *If $\lim_{n\to\infty} j_n = \upsilon$ such that $D(\upsilon, \upsilon) = 0$ then $\lim_{n\to\infty} D(j_n, k) = D(\upsilon, k)$ for every $k \in M$.*

Oltra and Valero [9] established a Banach fixed point theorem for dualistic partial metric spaces in such a way that the Matthews fixed point theorem is obtained as a particular case. The aforesaid result can be stated as follows:

Theorem 1.7. [9] *Let* (M, D) *be a complete dualistic partial metric space and let* $T : M \to M$ *be a mapping such that there exists* $\alpha \in [0, 1[$ *satisfying*

$$|D(T(j), T(k))| \leqslant \alpha |D(j, k)|,$$

for all $j, k \in M$. *Then* T *has a unique fixed point* $\upsilon \in M$. *Moreover,* $D(\upsilon, \upsilon) = 0$ *and the Picard iterative sequence* $\{T^n(j_0)\}_{n\in\mathbb{N}}$ *converges to* υ *with respect to* $\tau(d_D^s)$, *for any* $j_0 \in M$.

2. The Results

In this section, we shall prove dualistic partial version of Theorem 1.1. To this end, we begin with following definition.

Definition 2.1. *Let* (M, D) *be a dualistic partial metric space. A mapping* $T : M \to M$ *is called dualistic Geraghty contraction provided, there exists* $\beta \in S$ *such that,*

$$|D(T(j), T(k))| \leqslant \beta(|D(j, k)|)|D(j, k)|, \tag{1}$$

for all $j, k \in M$.

Observe that if $D(j, k) \in \mathbb{R}^+ \ \forall \ j, k \in M$, then dualistic Geraghty contraction mapping is same as the Geraghty contraction mapping defined in [5].

Theorem 2.2. *Let* (M, D) *be a complete dualistic partial metric space and let* $T : M \to M$ *be a dualistic Geraghty contraction mapping. Then* T *has a fixed point* $\upsilon \in M$ *and the Picard iterative sequence* $\{T^n(j)\}_{n\in\mathbb{N}}$ *converges to* υ *with respect to* $\tau(d_D^s)$, *for every* $j \in M$. *Moreover,* $D(\upsilon, \upsilon) = 0$.

Proof. Let j_0 be an initial point of the iterative algorithm in M and let us define Picard iterative sequence $\{j_n\}$ by

$$j_n = T(j_{n-1}) \text{ for all } n \in \mathbb{N}.$$

If there exists a positive integer r such that $j_r = j_{r+1}$, then $j_r = j_{r+1} = T(j_r)$, so j_r is a fixed point of T. Now we suppose that $j_n \neq j_{n+1}$ for all

$n \in \mathbb{N}$, from contractive condition (1), we have for all $j_{n+1}, j_{n+2} \in M$ and $n \geqslant 1$

$$\begin{aligned} |D(j_{n+1}, j_{n+2})| &= |D(T(j_n), T(j_{n+1}))|, \\ &\leqslant \beta(|D(j_n, j_{n+1})|)|D(j_n, j_{n+1})|, \\ |D(j_{n+1}, j_{n+2})| &\leqslant |D(j_n, j_{n+1})|. \end{aligned}$$

This implies that $\{|D(j_n, j_{n+1})|\}_{n=1}^{\infty}$ is a monotone and bounded sequence, so it converges to a point α and

$$\lim_{n\to\infty} |D(j_n, j_{n+1})| = \alpha \geqslant 0.$$

If $\alpha = 0$ then we are done but if $\alpha > 0$ then again from (1) we have

$$|D(j_{n+1}, j_{n+2})| \leqslant \beta(|D(j_n, j_{n+1})|)|D(j_n, j_{n+1})|.$$

And

$$\left|\frac{D(j_{n+1}, j_{n+2})}{D(j_n, j_{n+1})}\right| \leqslant \beta(|D(j_n, j_{n+1})|).$$

Whence we deduce that

$$\lim_{n\to\infty} \beta(|D(j_n, j_{n+1})|) = 1.$$

Since $\beta \in S$, $\lim_{n\to\infty} |D(j_n, j_{n+1})| = 0 \Rightarrow \alpha = 0$.
Hence

$$\lim_{n\to\infty} D(j_n, j_{n+1}) = 0. \tag{2}$$

Similarly we can prove that

$$\lim_{n\to\infty} D(j_n, j_n) = 0.$$

Since

$$d_D(j_n, j_{n+1}) = D(j_n, j_{n+1}) - D(j_n, j_n)$$

we deduce that

$$\lim_{n\to\infty} d_D(j_n, j_{n+1}) = 0 \ \forall \ n \geqslant 1. \tag{3}$$

Now, we show that $\{j_n\}$ is a Cauchy sequence in (M, d_D^s). Suppose, on contrary, that $\{j_n\}$ is not a Cauchy sequence. Then given $\epsilon > 0$, we can construct a pair of subsequences $\{j_{m_r}\}$ and $\{j_{n_r}\}$ violating the following condition for the least integer n_r such that $m_r > n_r > r$ where $r \in \mathbb{N}$

$$d_D(j_{m_r}, j_{n_r}) \geqslant \epsilon. \tag{4}$$

In addition, upon choosing the smallest possible m_r, we may assume that

$$d_D(j_{m_r}, j_{n_{r-1}}) < \epsilon. \tag{5}$$

By (D_4), we have

$$\begin{aligned} \epsilon &\leqslant d_D(j_{m_r}, j_{n_r}); \\ &\leqslant d_D(j_{m_r}, j_{n_{r-1}}) + d_D(j_{n_{r-1}}, j_{n_r}); \\ &< \epsilon + d_D(j_{n_{r-1}}, j_{n_r}). \end{aligned}$$

That is,

$$\epsilon < \epsilon + d_D(j_{n_{r-1}}, j_{n_r}), \tag{6}$$

for all $r \in \mathbb{N}$. In the view of (6), (3), we have

$$\lim_{r\to\infty} d_D(j_{m_r}, j_{n_r}) = \epsilon. \tag{7}$$

Again using (D_4), we have

$$d_D(j_{m_r}, j_{n_r}) \leqslant d_D(j_{m_r}, j_{m_{r+1}}) + d_D(j_{m_{r+1}}, j_{n_{r+1}}) + d_D(j_{n_{r+1}}, j_{n_r}).$$

Moreover, we can have

$$d_D(j_{m_{r+1}}, j_{n_{r+1}}) \leqslant d_D(j_{m_{r+1}}, j_{m_r}) + d_D(j_{m_r}, j_{n_r}) + d_D(j_{n_r}, j_{n_{r+1}}).$$

Taking limit as $r \to +\infty$ and using (3) and (7), we obtain

$$\lim_{r\to+\infty} d_D(j_{m_{r+1}}, j_{n_{r+1}}) = \epsilon. \tag{8}$$

Now from contractive condition (1), we have

$$\begin{aligned} |D(j_{n_{r+1}}, j_{m_{r+2}})| &= |D(T(j_{n_r}), T(j_{m_{r+1}}))|, \\ &\leqslant \beta(|D(j_{n_r}, j_{m_{r+1}})|)|D(j_{n_r}, j_{m_{r+1}})|. \end{aligned}$$

We conclude that

$$\left|\frac{D(j_{n_{r+1}}, j_{m_{r+2}})}{D(j_{n_r}, j_{m_{r+1}})}\right| \leqslant \beta(|D(j_{n_r}, j_{m_{r+1}})|).$$

By using (3), taking limit as $r \to +\infty$ in the above inequality, we obtain

$$\lim_{r\to\infty} \beta(|D(j_{n_r}, j_{m_{r+1}})|) = 1. \tag{9}$$

Since $\beta \in S$, $\lim_{r\to\infty} |D(j_{n_r}, j_{m_{r+1}})| = 0$ and hence $\lim_{r\to\infty} d_D(j_{n_r}, j_{m_{r+1}}) = 0 < \epsilon$ which is a contradiction of our supposition (4). Hence $\{j_n\}$ is a Cauchy sequence in (M, d_D^s). Since (M, d_D^s) is a complete metric space, there exists $\upsilon \in M$ such that

$$\lim_{n\to\infty} d_D^s(j_n, \upsilon) = 0 \iff D(\upsilon, \upsilon) = \lim_{n\to\infty} D(j_n, \upsilon) = \lim_{n,m\to\infty} D(j_n, j_m) = 0.$$

We are left to prove that υ is a fixed point of T. Since T is continuous,

$$\upsilon = \lim_{n\to\infty} j_n = \lim_{n\to\infty} T^n(j_0) = \lim_{n\to\infty} T^{n+1}(j_0) = T(\lim_{n\to\infty} T^n(j_0)) = T(\upsilon).$$

Hence $\upsilon = T(\upsilon)$ that is υ is fixed point of T. $\square$

Note that :

Theorem 1.1. can be viewed as a corollary of Theorem 2.2.
A natural question that can be raised, is, whether the contractive condition in the statement of Theorem 2.2 can be replaced by the contractive condition in Theorem 1.1, the following easy example provides a negative answer to such a question.

Example 2.3. Consider the complete dualistic partial metric space $(\mathbb{R}, D_\vee)$ and the mapping $T_1 : \mathbb{R} \to \mathbb{R}$ defined by,

$$T_1(j) = \begin{cases} 0 & \text{if } j \neq 0 \\ -1 & \text{if } j = 0 \end{cases}.$$

It is easy to check that contractive condition in the statement of Theorem 1.1 holds true, that is

$$D_\vee(T_1(j), T_1(k)) \leqslant \beta(D_\vee(j,k))D_\vee(j,k),$$

for all $j, k \in \mathbb{R}^+$. However, T_1 has no fixed point. Observe that T_1 does not satisfy the contractive condition in the statement of Theorem 2.2. Indeed,

$$1 = |D_\vee(-1,-1)| = |D_\vee(T_1(0), T_1(0))| > \beta(|D_\vee(0,0)|)|D_\vee(0,0)| = 0.$$

3. Application to Integral Equations

In this section, we shall show, how Theorem 2.2 can be applied to prove the existence of solution of integral equation (10).
Let Ω represent the class of functions $\phi : [0,\infty) \to [0,\infty)$ with properties;

(1) ϕ is increasing.

(2) For each $j > 0$, $\phi(j) < j$

(3) $\int_0^1 \phi(j)\, dj \leqslant \phi(\int_0^1 j\, dj)$.

(4) $\beta(j) = \frac{\phi(j)}{j} \in S$.

For example, $\phi(j) = \frac{1}{5}j$, $\phi(j) = tan(j)$ are elements of Ω .

Let us consider the following class of integral equations:

$$j(w) = g(w) + \int_0^1 G_n(w, s, j(s))\, ds \ \ \forall\ w \in [0,1],\ n \in \mathbb{N}. \tag{10}$$

To show the existence of solution of integral equation (10), we need following lemma.

Lemma 3.1. *Let* $\mathbb{B} = \bar{B}(0,\rho) = \{j : j \in L^2([0,1],\mathbb{R})\ ;\ \|j\| \leqslant \rho\}$. *Assume following hypotheses are satisfied:*

(1) $g \in L^2([0,1],\mathbb{R})$

(2) $G : [0,1] \times [0,1] \times L^2([0,1],\mathbb{R}) \to \mathbb{R}$.

(4) $|G_n(w,s,j)| \leqslant f(w,s) + \upsilon|j|$ where $f \in L^2([0,1]\times[0,1])$ and $\upsilon < \frac{1}{2}$.

Then operator T defined by

$$(Tk)(w) = g(w) + \int_0^1 \tilde{G}(w)(k)(s)\, ds$$

satisfies $T(\mathbb{B}) \subset \mathbb{B}$.

Proof. We begin by defining the operator $\tilde{G}(w)(k)(s) = G_n(w, s, k(s))$.

$$\begin{aligned}
\|Tj\|^2_{L^2([0,1],\mathbb{R})} &= \int_0^1 |Tj(w)|^2\, dw. \\
&= \int_0^1 (|g(w) + \int_0^1 \tilde{G}(w)(j)(s)\, ds|^2)\, dw. \\
&\leqslant 2\int_0^1 |g(w)|^2\, dw + 2\int_0^1 \int_0^1 |\tilde{G}(w)(j)(s)|^2\, dsdw. \\
&\leqslant 2\int_0^1 |g(w)|^2\, dw + 2\int_0^1 \int_0^1 |f(w,s) + \upsilon|j(s)||^2\, dsdw \\
&\leqslant 2\int_0^1 |g(w)|^2\, dw + 4\int_0^1 \int_0^1 |f(w,s)|^2\, dsdw \\
&+ 4\upsilon^2 \|j\|^2_{L^2([0,1],\mathbb{R})}. \\
&\leqslant 2\int_0^1 |g(w)|^2\, dw + 4\int_0^1 \int_0^1 f^2(w,s)\, dsdw + 4\upsilon^2\rho^2.
\end{aligned}$$

Since $\upsilon < \frac{1}{2}$, choose ρ such that

$$\frac{2}{1-4\upsilon^2}\int_0^1 |g(w)|^2\, dw + \frac{4}{1-4\upsilon^2}\int_0^1 \int_0^1 f^2(w,s)\, dsdw \leqslant \rho^2$$

This implies that $T(j) \in \mathbb{B}$, hence $T(\mathbb{B}) \subset \mathbb{B}$. □

Now we are in position to state our result regarding application.

Theorem 3.2. *Assume that the following hypotheses are satisfied:*

(1) *The conditions supposed in Lemma 3.1 hold.*

(2) $G_n(w, s, j) - G_n(w, s, k) \leqslant \phi(j - k)$, $\forall\ j, k \in M$ *and for large* n,

Then integral equation (10) has a solution.

Proof. Let $M = L^2([0,1],\mathbb{R})$ and $D(j,k) = d(j,k)+c_n \; \forall \, j,k \in M$ where $d(j,k) = \|j-k\|_M$ and $\{c_n\}$ is a sequence of real numbers satisfying, $|c_n| \to 0$ for large n. Suppose that $T : M \to M$ be a mapping defined by

$$(Tk)(w) = g(w) + \int_0^1 \tilde{G}(w)(k)(s)\, ds.$$

Then (M,D) is a complete ordered dualistic partial metric space. Notice that T is well-defined and (10) has a solution if and only if the operator T has a fixed point. Precisely, we have to show that our Theorem 2.2 is applicable to the operator T. Then, for all $j,k \in M$, we write

$$\begin{aligned}
|D(T(j),T(k))|^2 &= |d(T(j),T(k)) + c_n|^2 \\
&\leqslant |d(T(j),T(k))|^2 + |c_n|^2 + 2|d(T(j),T(k))||c_n|. \\
&\leqslant \|T(j) - T(k)\|^2 + |c_n|^2 + 2|d(T(j),T(k))||c_n|. \\
&\leqslant \int_0^1 (\int_0^1 \tilde{G}(w)(j)(s) - \tilde{G}(w)(k)(s)\, ds)^2 dw \\
&+ |c_n|^2 + 2|d(Tj,Tk)||c_n|. \\
&\leqslant \int_0^1 (\int_0^1 G_n(w,s,j) - G_n(w,s,k)\, ds)^2 dw + |c_n|^2 \\
&+ 2|d(T(j),T(k))||c_n|. \\
&\leqslant \int_0^1 (\int_0^1 \phi(j(s) - k(s))\, ds)^2 dw \;\text{ for large n} \\
&\leqslant \phi^2(\int_0^1 (j(s) - k(s))^2\, ds).
\end{aligned}$$

It follows $|D(T(j),T(k))|^2 \leqslant [\phi(|D(j,k)|)]^2.$

$$|D(T(j),T(k))| \leqslant [\phi(|D(j,k)|)] = \frac{\phi(|D(j,k)|)}{|D(j,k)|}|D(j,k)|.$$

This implies

$$|D(T(j),T(k))| \leqslant \beta(|D(j,k)|)|D(j,k)|$$

and hence T satisfies all the conditions of Theorem 2.2, so it has a fixed point and hence (10) has a solution. □

Remark 3.3. *Significance of the above results lies in the fact that these results are true for all real numbers whereas such results proved in partial metric spaces are only true for positive real numbers.*

Conflict of Interests
The authors declare that they have no competing interests.

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