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Conditional Entropy of Infinite Partitions on Quantum Logic

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Abstract. In this work the notions of the entropy and conditional entropy on quantum logic with infinite partitions by a new formula are introduced. Some ergodic properties concerning this measures are proved. The entropy and conditional entropy under the relations of s-refinement and s-independent are studied.

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1. Introduction

The quantum logic approach was introduced by Birkhoff and Von Neumann [1]. Entropy is a tool to measure the amount of uncertainty in random events. Entropy has been applied in a variety of problem areas including physics, computer science, general systems theory, information theory, statistics, biology, chemistry, sociology and many other fields. Riecan and Dvurecenskij proposed a new model for quantum mechanics [8]. Some investigations concerning the notion of information function was carried in [4]. Entropy serves as a measure of information of the considered experiment and entropy of partitions in quantum logic, has an important application in dynamical systems and is a useful tool in studying the isomorphism of dynamical systems and also, the notion is very useful in studying quantum information theory ([3], [5], [6], [10], [14]).
In some previous papers ([3], [9], [11], [12], [13]), the notions of Shannon entropy and logical entropy of finite partitions on quantum logic, have been defined

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and studied. In [13], a construction of conditional entropy of finite partitions on quantum logic was given and in [11], for finite partitions on quantum logic, the Renyi and Tsallis conditional entropies of finite partitions were introduced. In my previous paper [2] the entropy with infinte partitions on quantum logic was defined and studied. The main idea of the present research is replacing the entropy of finite partitions with the entropy of infinite partitions by a new formulla. In fact, in this paper, we would like to define and study entropy and conditional entropy with countable partitions by an other formulla. Some researchers by using this formula have been defined and studied the concept entropy ([5], [7]). We will prove some of their ergodic properties and we will study the entropy and conditional entropy under the relations of s-refinement and s-independent.

2. Basic Definitions

In this section we present some basic definitions that will be useful in further considerations.

Definition 2.1. ([12]) *A quantum logic QL is a σ-orthomodular lattice, i.e., a lattice L $(L, \leqslant, \vee, \wedge, 0, 1)$ with the smallest element* 0 *and the greatest element* 1*, an operation $' : L \to L$ such that the following properties are hold for all $a, b \in L$:*
i)$a'' = a$, $a \leqslant b \Rightarrow b' \leqslant a'$, $a \vee a' = 1$, $a \wedge a' = 0$;
ii) Given any countable sequence $(a_i)_{i \in I}$, $a_i \leqslant a_j'$, $i \neq j$, the join $\vee_{i \in \mathbb{N}} a_i$ exists in L;
iii) L is orthomodular: $a \leqslant b \Rightarrow b = a \vee (b \wedge a')$. Two elements $a, b \in QL$ are called orthogonal if $a \leqslant b'$ and denoted by $a \perp b$. A sequence $(a_i)_{i \in I}$ is said orthogonal if $a_i \perp a_j$, $\forall i \neq j$.

Definition 2.2. ([12]) *Let L be a QL. A map $s : L \to [0, 1]$ is a state iff $s(1) = 1$ and for any orthogonal sequence $(a_i)_{i \in I}$, $s(\vee_{i \in I} a_i) = \sum_{i \in I} s(a_i)$.*

Definition 2.3. ([2]) *Let $P = \{a_i : i \in \mathbb{N}\}$ be a countable system of elements of the QL, L. P is called to be a $\vee$-orthogonal system iff $\vee_{i=1}^{k} a_i \perp a_{k+1}$, $\forall k$.*

Definition 2.4. ([2]) *A system $P = \{a_i : i \in \mathbb{N}\} \subset L$ is said the partition of L corresponding to a state s iff:*
i)P is a $\vee$-orthogonal system;
ii) $s(\vee_{i \in \mathbb{N}} a_i) = 1$.

Definition 2.5. ([2]) *Let $P = \{a_i : i \in \mathbb{N}\}$ and $Q = \{b_j : j \in \mathbb{N}\}$ be two countable partitions of L. We say Q is a refinement of P, denoted by $P \leqslant_s Q$, if for each $b_j \in Q$ there exists $a_i \in P$ with $s(b_j \wedge a_i) = s(b_j)$.*

3. Entropy and Conditional Entropy

In this section we introduce the notions of entropy and conditional entropy of countable partitions on a quantum logic L corresponding to a state s. Then we study some ergodic properties concerning this measures.

Definition 3.1. *Let $P = \{a_i : i \in \mathbb{N}\} \subset L$ be a partition of a quantum logic L corresponding to a state s. The entropy of P with respect to state s is defined by:*

$$H_s(P) := -\log \sup_{i\in\mathbb{N}} s(a_i).$$

It is obvious that $H_s(P) \geqslant 0$.

Example 3.2. $QL = ([0,1], \leqslant, \vee, \wedge, 0, 1)$ with an operation $' : [0,1] \to [0,1]$ is a quantum logic such that for all $a, b \in L$:
$a \vee b := \min\{a+b, 1\}$, $a \wedge b := \max\{0, a+b-1\}$, and $a^{'} := 1-a$. Consider $s : [0,1] \longrightarrow [0,1]$ as $s(t) = t$, then the sequence $P = \{\frac{1}{2^n} : n \in \mathbb{N}\}$ is a partition of QL corresponding to the state s. Then

$$H_s(P) = -\log \sup_{i\in\mathbb{N}} s(\frac{1}{2^i}) = -\log \frac{1}{2} = \log 2.$$

Definition 3.3. *Let $P = \{a_i : i \in \mathbb{N}\}$ and $Q = \{b_j : j \in \mathbb{N}\}$ be two countable partitions of L. The conditional entropy of P given Q is defined as following:*

$$H_s(P|Q) = -\log \frac{\sup_{i,j\in\mathbb{N}} s(a_i \wedge b_j)}{\sup_{j\in\mathbb{N}} s(b_j)}.$$

Note that $H_s(P|Q) \geqslant 0$.
Let $P = \{a_i : i \in \mathbb{N}\}$ and $Q = \{b_j : j \in \mathbb{N}\}$ be two countable partitions of L corresponding to a state s and $s(\vee_{i\in\mathbb{N}}(a_i \wedge b)) = s(b), \forall b \in L$. Then by Definition 2., we obtain $\sum_{i=1}^{\infty} s(a_i \wedge b) = s(b)$. In the remaining of this paper, s has this property. Then the common refinement of these partitions is the partition $P \vee Q = \{a_i \wedge b_j : a_i \in P, b_j \in R, i, j \in \mathbb{N}\}$.

Proposition 3.4. *Let P and Q be countable partitions of L. Then*

$$H_s(P \vee Q|R) = H_s(P|R) + H_s(Q|P \vee R).$$

Proof. Since $\sup_{i,k\in\mathbb{N}} s(a_i \wedge c_k) \neq 0$ we have

$$\frac{\sup_{i,j,k\in\mathbb{N}} s(a_i \wedge b_j \wedge c_k)}{\sup_{k\in\mathbb{N}} s(c_k)} = \frac{\sup_{i,k\in\mathbb{N}} s(a_i \wedge c_k)}{\sup_{k\in\mathbb{N}} s(c_k)} \times \frac{\sup_{i,j,k\in\mathbb{N}} s(b_j \wedge a_i \wedge c_k)}{\sup_{i,k\in\mathbb{N}} s(a_i \wedge c_k)}.$$

Thus we get

$$H_s(P \vee Q|R) = -\log \frac{\sup_{i,j,k\in\mathbb{N}} s(a_i \wedge b_j \wedge c_k)}{\sup_{k\in\mathbb{N}} s(c_k)}$$

$$= -\log \frac{\sup_{i,k\in\mathbb{N}} s(a_i \wedge c_k)}{\sup_{k\in\mathbb{N}} s(c_k)} - \log \frac{\sup_{i,j,k\in\mathbb{N}} s(b_j \wedge a_i \wedge c_k)}{\sup_{i,k\in\mathbb{N}} s(a_i \wedge c_k)}$$

$$= H_s(P|R) + H_s(Q|P \vee R). \quad \square$$

In the next proposition, it is proved subadditivity of entropy of countable partitions on a quantum logic.

Proposition 3.5. *Let P and Q be countable partitions of L. Then*
i) $H_s(P \vee Q) = H_s(Q) + H_s(P|Q)$
ii) $\max\{H_s(P), H_s(Q)\} \leqslant H_s(P \vee Q)$,
iii) $H_s(P|Q) \leqslant H_s(P \vee Q)$
iv) $H_s(P|Q) \leqslant H_s(P)$
v) $H_s(P \vee Q) \leqslant H_s(P) + H_s(Q)$

Proof. Let $P = \{a_i : i \in \mathbb{N}\}$, $Q = \{b_j : j \in \mathbb{N}\}$ and $R = \{c_k : k \in \mathbb{N}\}$.
i) Since $\sup_{j\in\mathbb{N}} s(b_j) \neq 0$ we may write

$$\sup_{i,j\in\mathbb{N}} s(a_i \wedge b_j) = \sup_{j\in\mathbb{N}} s(b_j) \times \frac{\sup_{i,j\in\mathbb{N}} s(a_i \wedge b_j)}{\sup_{j\in\mathbb{N}} s(b_j)}.$$

So the result holds.
ii) Since the conditional entropy is nonnegative, from the first part of this proposition we have $H_s(Q) \leqslant H_s(P \vee Q)$. But $P \vee Q = Q \vee P$ and this implies $H_s(P) \leqslant H_s(P \vee Q)$.
iii) Follows from the first part of this proposition and from the fact that $H_s(Q)$ is nonnegative.
iv) For each $i, j \in \mathbb{N}$, we have $s(a_i) \leqslant s(a_i \wedge b_j)$. Since $\sup_{j\in\mathbb{N}} s(b_j) \leqslant 1$, we get

$$\sup_{i\in\mathbb{N}} s(a_i) \leqslant \sup_{i,j\in\mathbb{N}} s(a_i \wedge b_j) \leqslant \frac{\sup_{i,j\in\mathbb{N}} s(a_i \wedge b_j)}{\sup_{j\in\mathbb{N}} s(b_j)}.$$

v) It follows immediately from the parts *i)* and *iv)* of this proposition. $\square$

Now the entropy of infinite partitions under the relation s-refinement will be studied.

Proposition 3.6. *Let P and Q be countable partitions of L and $P \leqslant_s Q$. Then $H_s(P) \leqslant H_s(Q)$.*

Proof. Let $P = \{a_i : i \in \mathbb{N}\}$ and $Q = \{b_j : j \in \mathbb{N}\}$. For each $b_j \in Q$ there exists $a_i \in P$ such that $s(b_j) = s(b_j \wedge a_i)$. So $s(b_j) \leqslant \sum_{j=1}^{\infty} s(a_i \wedge b_j) = s(a_i)$. Thus $\sup_{j\in\mathbb{N}} s(b_j) \leqslant \sup_{i\in\mathbb{N}} s(a_i)$ and this means $H_s(P) \leqslant H_s(Q)$. □

In the following proposition, the conditional entropy under the relation of s-refinement will be studied.

Proposition 3.7. *Let P and Q be countable partitions of L and $P \leqslant_s Q$. Then*

$$H_s(P|R) \leqslant H_s(Q|R).$$

Proof. Let $P = \{a_i : i \in \mathbb{N}\}$, $Q = \{b_j : j \in \mathbb{N}\}$ and $R = \{c_k : k \in \mathbb{N}\}$. Let $b_{j_0} \wedge c_{k_0}$ be an arbitrary element of $Q \vee R$. Since $P \leqslant_s Q$, there exists $a_{i_0} \in P$ such that $s(a_{i_0} \wedge b_{j_0}) = s(b_{j_0})$. We have

$$s((a_{i_0} \wedge c_{k_0}) \wedge (b_{j_0} \wedge c_{k_0})) = s(a_{i_0} \wedge b_{j_0} \wedge (c_{k_0} \wedge c_{k_0})) = s(a_{i_0} \wedge b_{j_0} \wedge c_{k_0}).$$

We show that $s(a_{i_0} \wedge b_{j_0} \wedge c_{k_0}) = s(b_{j_0} \wedge c_{k_0})$. Since $P \leqslant_s Q$, we get

$$s(a_{i_0} \wedge b_{j_0}) = s(b_{j_0}) = \sum_{i=1}^{\infty} s(a_i \wedge b_{j_0}).$$

So for each $i \neq i_0$, $s(a_i \wedge b_{j_0}) = 0$, therefore for each $i \neq i_0$, $s(a_i \wedge b_{j_0} \wedge c_{k_0}) = 0$ and this implies that

$$s(a_{i_0} \wedge b_{j_0} \wedge c_{k_0}) = \sum_{i=1}^{\infty} s(a_i \wedge b_{j_0} \wedge c_{k_0}) = s(b_{j_0} \wedge c_{k_0}).$$

Thus $P \vee R \leqslant_s Q \vee R$. Now by Proposition 3.6, we obtain

$$\begin{aligned} H_s(P|R) &= H_s(P \vee R) - H_s(R) \\ &\leqslant H_s(Q \vee R) - H_s(R) = H_s(Q|R) \quad \square \end{aligned}$$

Proposition 3.8. *Let P and Q be countable partitions of L. Then*

$$P \leqslant_s Q \Longleftrightarrow H_s(P \vee Q) = H_s(Q).$$

Proof. Let $P = \{a_i : i \in \mathbb{N}\}$, $Q = \{b_j : j \in \mathbb{N}\}$ and $P \leqslant_s Q$, then for each $b_j \in Q$ there exists $a_{i_j} \in P$ such that $s(b_j) = s(a_{i_j} \wedge b_j)$. So

$$\sup_{j\in\mathbb{N}} s(b_j) \leqslant \sup_{j\in\mathbb{N}} s(a_{i_j} \wedge b_j) \leqslant \sup_{i,j\in\mathbb{N}} s(a_i \wedge b_j).$$

On the other hand, since $s(b_j) = \sum_{i=1}^{\infty} s(a_i \wedge b_j)$, we obtain

$$\sup_{i,j\in\mathbb{N}} s(a_i \wedge b_j) \leqslant \sup_{j\in\mathbb{N}} s(b_j).$$

Thus from the above relations we obtain $\sup_{i,j\in\mathbb{N}} s(a_i \wedge b_j) = \sup_{j\in\mathbb{N}} s(b_j)$, and this means $H_s(P \vee Q) = H_s(Q)$. Conversely, if $H_s(P \vee Q) = H_s(Q)$, then by Proposition 3. i), $H_s(P|Q) = 0$. So for each i, j, $s(b_j) = s(a_i \wedge b_j)$. Thus $P \leqslant_s Q$. □

Let s be a state. Two countable partitions P and Q of a QL are called s-independent if $s(a \wedge b) = s(a)s(b)$ for all $a \in P$, and $b \in Q$.
In the next proposition, we study the entropy and conditional entropy under the relation of s-independent.

Proposition 3.9. *Let s be a state and let P, Q and R be s-independent countable partitions of a QL. Then*
i) $H_s(P \vee Q) = H_s(P) + H_s(Q)$;
ii) $H_s(P|Q) = H_s(P)$;
iii) if P and $Q \vee R$ are s-independent partitions of QL, then

$$H_s(P \vee Q|R) = H_s(P) + H_s(Q|R).$$

Proof. Let $P = \{a_i : i \in \mathbb{N}\}$, $Q = \{b_j : j \in \mathbb{N}\}$ and $R = \{c_k : k \in \mathbb{N}\}$.
i) Since P and Q are s-independent, we may write

$$\begin{aligned} H_s(P \vee Q) &= -\log \sup_{i,j\in\mathbb{N}} s(a_i \wedge b_j) \\ &= -\log \sup_{i,j\in\mathbb{N}} s(a_i)s(b_j) \\ &= -\log \sup_{i\in\mathbb{N}} s(a_i) - \log \sup_{j\in\mathbb{N}} s(b_j) \\ &= H_s(P) + H_s(Q). \end{aligned}$$

ii) Follows from the part *i*) of this proposition and Proposition 3..
iii) Since P and $Q \vee R$ are s-independent we have

$$\begin{aligned} H_s(P \vee Q|R) &= -\log \frac{\sup_{i,j,k\in\mathbb{N}} s(a_i \wedge b_j \wedge c_k)}{\sup_{k\in\mathbb{N}} s(c_k)} \\ &= -\log \frac{\sup_{i,j,k\in\mathbb{N}} s(a_i)s(b_j \wedge c_k)}{\sup_{k\in\mathbb{N}} s(c_k)} \\ &= -\log \sup_{i\in\mathbb{N}} s(a_i) - \log \frac{\sup_{j,k\in\mathbb{N}} s(b_j \wedge c_k)}{\sup_{k\in\mathbb{N}} s(c_k)} \\ &= H_s(P) + H_s(Q|R). \quad \square \end{aligned}$$

4. Conclusion

In this paper entropy and conditional entropy of countable partitions on a quantum logic were defined. The entropy and conditional entropy under the relations of s-refinement and s-independent were studied and some ergodic properties of the mesures were investigated.

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