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Homotopy Perturbation Method for a Modified

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Abstract. In this article, the Homotopy Perturbation Method (HPM) is employed to approximate solutions of a modified Lotka - Volterra equation. HPM has been introduced by He to solve approximately linear or nonlinear differential equations. Approximate polynomials have also been constructed to find approximate solutions of a modified Lotka - Volterra system. Numerical comparisons are made between HPM and maple numerical results.

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1. Introduction

Consider the nonlinear system of ODE's given by

$$\begin{cases} u' = (a - bv)u - pu^2, \\ v' = (cu - d)v - qv^2. \end{cases} \tag{1}$$

These are the modified Lotka-Volterra equations representing predator-prey (u, v) populations. We can think of the u^2 and v^2 terms as the effects of overcrowding or competition within a species leading to a decline in population (assuming $p, q > 0$).

Recently, a very effective method that based on perturbation techniques and homotopy topology, well known as Homotopy Perturbation

Method (HPM) is proposed by the Chinese researcher He [6-12]. A homotopy parameter θ which takes values from zero to one relates initial solutions of a problem to its final solutions. This method can be successfully applied to various kinds of linear and nonlinear ordinary differential equations ([1,4,5,12-15,17]), partial differential equations ([8]), integro - differential equations ([16]), difference differential equations, fractional differential equations, systems of differential equations ([2,3]) , etc.

In this paper we will employ HPM to approximate solutions of the Lotka - Volterra system with some initial conditions ([1]). The obtained results are compared with numerical results which we can find by using of Runge - Kutta method ([4,5]).

2. Homotopy Perturbation Method (HPM)

To illustrate the homotopy perturbation method, we consider a general equation as follows

$$A(u) - f(r) = 0, \qquad r \in \Omega \tag{2}$$

with the boundary conditions

$$B(u, \frac{\partial u}{\partial n}) = 0, \qquad r \in \Gamma \tag{3}$$

where A is a general differential operator, B is a boundary operator, Γ is the boundary of the domain Ω and $f(r)$ is a known analytical function. Generally speaking, the operator A can be divided into a linear part L and a nonlinear part N. So equation (2) can be rewritten as:

$$L(u) + N(u) - f(r) = 0. \tag{4}$$

By the homotopy perturbation method, we construct a homotopy as $v(r, \theta) : \Omega \times [0, 1] \to R$ which satisfies

$$H(v, \theta) = (1 - \theta) \; [\; L(v) - L(u_0) \;] + \theta \; [\; A(v) - f(r) \;] = 0, \tag{5}$$

where $\theta \in [0, 1]$ is an embedding parameter and u_0 is an initial approximation of equation (2) which satisfies the boundary conditions.

Considering equation (4) we will have

$$H(v,0) = L(v) - L(u_0) = 0, \tag{6}$$

$$H(v,1) = A(v) - f(r) = 0. \tag{7}$$

The changing process of θ from zero to one is just that of $v(r,\theta)$ from $u_0(r)$ to $u(r)$. In topology $H(v,\theta)$ is called deformation where $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotop. According to the homotopy perturbation theory, we can first use the embedding parameter θ as a small parameter and assume that the solution of equation (4) can be written as a power series in θ :

$$v = v_0 + \theta v_1 + \theta^2 v_2 + \cdots \tag{8}$$

Setting $\theta = 1$ one can have the approximation solution of equation (2) as the following

$$u = \lim_{\theta \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{9}$$

The series (9) is convergent for most cases. However, the convergent rate depends on the nonlinear operator $A(v)$.

3. HPM for Lotka - Volterra Equation

Consider the Lotka - Volterra equation with initial values of the form

$$\begin{cases} u' = (a - bv)u - pu^2, \\ v' = (cu - d)v - qv^2, \\ u(0) = \alpha, \quad v(0) = \beta, \end{cases} \tag{10}$$

where α and β are constants. The general homotopy can be defined as follows

$$\begin{cases} H(u,\theta) = (1-\theta)(u' - u_0') + \theta \left[u' - (a - bv)u + pu^2\right] = 0, \\ H(v,\theta) = (1-\theta)(v' - v_0') + \theta \left[v' - (cu - d)v + qv^2\right] = 0, \end{cases} \tag{11}$$

where $\theta \in [0,1]$ is the embedding parameter. Choosing the leading terms u_0 and v_0 to be constants, one can rewrite the system (11) in the form

$$\begin{cases} u' - \theta \left[(a - bv)u - pu^2\right] = 0, \\ v' - \theta \left[(cu - d)v - qv^2\right] = 0. \end{cases} \tag{12}$$

Let

$$\begin{cases} u = u_0 + \theta u_1 + \theta^2 u_2 + \cdots \\ v = v_0 + \theta v_1 + \theta^2 v_2 + \cdots \end{cases} \tag{13}$$

be the solutions of (10). Substituting (13) in (12), we obtain

$$\begin{cases} \sum_{i=0}^{\infty} \theta^i u_i' - \theta \left[(a - b \sum_{i=0}^{\infty} \theta^i v_i)(\sum_{i=0}^{\infty} \theta^i u_i) - p \ (\sum_{i=0}^{\infty} \theta^i u_i)^2 \right] = 0, \\ \sum_{i=0}^{\infty} \theta^i v_i' - \theta \left[(c \sum_{i=0}^{\infty} \theta^i u_i - d \ )(\sum_{i=0}^{\infty} \theta^i v_i) - q \ (\sum_{i=0}^{\infty} \theta^i v_i)^2 \right] = 0, \end{cases} \tag{14}$$

Equating the coefficients of the same powers of θ in (14) yields

$$\theta^0 : \begin{cases} u_0' = 0, \\ v_0' = 0, \\ u_0(0) = \alpha, \quad v_0(0) = \beta. \end{cases}$$

$$\theta^1 : \begin{cases} u_1' - (a - bv_0 \ )u_0 + p \ u_0^2 = 0, \\ v_1' - (cu_0 - d \ )v_0 + q \ v_0^2 = 0, \\ u_1(0) = 0, \ v_1(0) = 0. \end{cases}$$

$$\theta^2 : \begin{cases} u_2' - (a - bv_0 \ )u_1 + bu_0v_1 + 2pu_0u_1 = 0, \\ v_2' - (cu_0 - d \ )v_1 - cu_1v_0 + 2q \ v_0v_1 = 0, \\ u_2(0) = 0, \ v_2(0) = 0. \end{cases}$$

$$\theta^3 : \begin{cases} u_3' - (a - bv_0 \ )u_2 + bu_1v_1 + bu_0v_2 + p(u_1^2 + 2u_0u_2) = 0, \\ v_3' - (cu_0 - d \ )v_2 - cu_1v_1 - cu_2v_0 + q(v_1^2 + 2v_0v_2) = 0, \\ u_3(0) = 0, \ v_3(0) = 0. \end{cases}$$

$$\vdots$$

4. Applications

In this section we will apply the general homotopy perturbation method to some cases of the Lotka - Volterra differential equations. In the following examples this equation with setting $a = b = c = d = 1$ and $p = 0$ or 1 with also $q = 0$ or 1 will be considered.

Example 1. We consider $p = q = 0$. The equations (1) turns in the following form

$$\begin{cases} u' = (1-v)u, \\ v' = (u-1)v. \end{cases} \tag{15}$$

Choosing $u(0) = 1, v(0) = 0$ yields

$$\alpha = 1, \quad \beta = 0,$$

and so we have the following system of equations

$$\theta^0 : \begin{cases} u_0' = 0, \\ v_0' = 0, \\ u_0(0) = 1, \quad v_0(0) = 0. \end{cases}$$

$$\theta^1 : \begin{cases} u_1' - (1 - v_0\ )u_0 = 0, \\ v_1' - (u_0 - 1\ )v_0 = 0, \\ u_1(0) = 0, \ v_1(0) = 0. \end{cases}$$

$$\theta^2 : \begin{cases} u_2' - (1 - v_0\ )u_1 + u_0v_1 = 0, \\ v_2' - (u_0 - 1\ )v_1 - u_1v_0 = 0, \\ u_2(0) = 0, \ v_2(0) = 0. \end{cases}$$

$$\theta^3 : \begin{cases} u_3' - (1 - v_0\ )u_2 + u_1v_1 + u_0v_2 = 0, \\ v_3' - (u_0 - 1\ )v_2 - u_1v_1 - u_2v_0 = 0, \\ u_3(0) = 0, \ v_3(0) = 0. \end{cases}$$

$$\vdots$$

Solutions of these problems are as follows

$u_0(t) = 1, \qquad v_0(t) = 0$

$u_1(t) = t\ , \qquad v_1(t) = 0$

$u_2(t) = \frac{1}{2}t^2, \qquad v_2(t) = 0$

$u_3(t) = \frac{1}{6}t^3, \qquad v_3(t) = 0$

$u_4(t) = \frac{1}{24}t^4, \qquad v_4(t) = 0$

$u_5(t) = \frac{1}{120}t^5, \qquad v_5(t) = 0$

Therefore, by applying

$$\begin{cases} u = u_0 + u_1 + u_2 + \cdots \\ v = v_0 + v_1 + v_2 + \cdots \end{cases} \tag{16}$$

the solutions will be obtained from the following series

$$\begin{cases} u(t) = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + \cdots \\ v(t) = 0. \end{cases}$$

This yields $u(t) = e^t$ and $v(t) = 0$.

Example 2. Considering (15) with choosing $u(0) = v(0) = \frac{1}{2}$ yields

$$\alpha = \frac{1}{2}, \quad \beta = \frac{1}{2},$$

which give $u_0(0) = \frac{1}{2}$, $v_0(0) = \frac{1}{2}$. Doing the same procedure as in Example 1. the following solutions will be obtained

$u_0(t) = \frac{1}{2}$, $v_0(t) = \frac{1}{2}$

$u_1(t) = \frac{1}{4}t$, $v_1(t) = -\frac{1}{4}t$

$u_2(t) = \frac{1}{8}t^2$, $v_2(t) = \frac{1}{8}t^2$

$u_3(t) = \frac{1}{48}t^3$, $v_3(t) = -\frac{1}{48}t^3$

$u_4(t) = \frac{1}{192}t^4$, $v_4(t) = \frac{1}{192}t^4$

$u_5(t) = -\frac{1}{960}t^5$, $v_5(t) = -\frac{1}{960}t^5$

$\vdots$

Applying (16), the solutions of the system can be approximated by the following truncated series

$$\begin{cases} u(t) \approx \frac{1}{2} + \frac{1}{4}t + \frac{1}{8}t^2 + \frac{1}{48}t^3 + \frac{1}{192}t^4 - \frac{1}{960}t^5 - \frac{1}{5760}t^6 - \frac{1}{5040}t^7, \\ v(t) \approx \frac{1}{2} - \frac{1}{4}t + \frac{1}{8}t^2 - \frac{1}{48}t^3 + \frac{1}{192}t^4 - \frac{1}{960}t^5 - \frac{1}{5760}t^6 + \frac{1}{5040}t^7. \end{cases}$$

Table 1. and Figure 1. a show comparisons between the HPM and Maple procedure results which uses the Fehlberg fourth-fifth order Runge-Kutta method.

Table 1. Comparisons between HPM and Maple procedure results.

t	$u(t)_{HPM}$	$v(t)_{HPM}$	$u(t)_{Maple\ proc}$	$v(t)_{Maple\ proc}$
-0.5	0.4040027	0.6591429	0.4040026	0.6591427
-0.4	0.4188103	0.6214549	0.4188104	0.6214549
-0.3	0.4357326	0.5868520	0.4357326	0.5868516
-0.2	0.4548420	0.5551746	0.4548422	0.5551744
-0.1	0.4762297	0.5262713	0.4762297	0.5262713
0.0	0.5000000	0.5000000	0.5000000	0.5000000
0.1	0.5262713	0.4762297	0.5262713	0.4762297
0.2	0.5551746	0.4548420	0.5551744	0.4548422
0.3	0.5868520	0.4357326	0.5868516	0.4357326
0.4	0.6214549	0.4188103	0.6214549	0.4188104
0.5	0.6591429	0.4040027	0.6591427	0.4040026

Example 3. Considering $p = 0, q = 1$, the equations (1) will turn out to the following form

$$\begin{cases} u' = (1-v)u, \\ v' = (u-1)v - v^2. \end{cases} \tag{17}$$

With the initial conditions $u(0) = 0.001, v(0) = 0.01$ one obtaines

$$\alpha = 0.001, \quad \beta = 0.01.$$

So the solutions would be as follows

$$\begin{aligned}
&u_0(t) = \tfrac{1}{1000}, && v_0(t) = \tfrac{1}{100} \\
&u_1(t) = \tfrac{99}{10^5}t, && v_1(t) = -\tfrac{1009}{10^5}t \\
&u_2(t) = \tfrac{99019}{2\times10^8}t^2, && v_2(t) = \tfrac{1029161}{2\times10^8}t^2 \\
&u_3(t) = \tfrac{98997469}{6\times10^{11}}t^3, && v_3(t) = -\tfrac{1070084309}{6\times10^{11}}t^3
\end{aligned}$$

$\vdots$

Therefore, the series solutions will be demonstrated as

$$\begin{cases} u(t) = \frac{1}{1000} + \frac{99}{10^5}t + \frac{99019}{2\times10^8}t^2 + \frac{98997469}{6\times10^{11}}t^3 + \cdots \\ v(t) = \frac{1}{100} - \frac{1009}{10^5}t + \frac{1029161}{2\times10^8}t^2 - \frac{1070084309}{6\times10^{11}}t^3 + \cdots \end{cases}$$

The approximate polynomial solutions can be obtained by the truncated series $u(t) \approx \sum_{i=0}^{N} u_i$ and $v(t) \approx \sum_{i=0}^{N} v_i$ where they are calculated up to $N = 8$ and the results are tabulated in Table (2). Figure 1-b illustrates a graphical comparison.

Table 2. Comparisons between HPM and Maple procedure results.

t	$u(t)_{HPM}$	$v(t)_{HPM}$	$u(t)_{Maple\ proc}$	$v(t)_{Maple\ proc}$
-0.5	0.000610490	0.016588295	0.000610489	0.016588297
-0.4	0.000673632	0.014987015	0.000673632	0.014987008
-0.3	0.000743419	0.013542457	0.000743418	0.013542451
-0.2	0.000820547	0.012238916	0.000820546	0.012238913
-0.1	0.000905790	0.011062291	0.000905789	0.011062293
0.0	0.001000000	0.010000000	0.001000000	0.010000000
0.1	0.001104120	0.009040721	0.001104120	0.009040719
0.2	0.001219192	0.008174300	0.001219192	0.008174285
0.3	0.001346369	0.007391610	0.001346368	0.007391600
0.4	0.001486921	0.006684445	0.001486921	0.006684442
0.5	0.001642258	0.006045406	0.001642256	0.006045424

Example 4. Considering $p = 1, q = 1$ one can obtain

$$\begin{cases} u' = (1 - v)u - u^2, \\ v' = (u - 1)v - v^2, \end{cases} \tag{18}$$

with the initial conditions $u(0) = 0.1, v(0) = 0.1$, one obtains

$$\alpha = 0.1, \quad \beta = 0.1$$

Therefore, the following solutions will be found

$$\begin{aligned} &u_0(t) = 0.1, && v_0(t) = 0.1 \\ &u_1(t) = \tfrac{2}{25}t, && v_1(t) = -\tfrac{1}{10}t \\ &u_2(t) = \tfrac{33}{1000}t^2, && v_2(t) = \tfrac{59}{1000}t^2 \\ &u_3(t) = \tfrac{47}{7500}t^3, && v_3(t) = -\tfrac{199}{7500}t^3 \\ &\vdots \end{aligned}$$

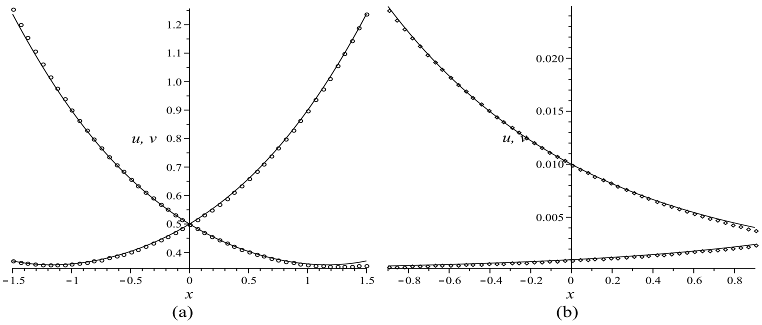

Figure 1: Comparison between HPM and Maple results. Continuous lines: Solutions by Runge - Kutta (4,5). Circle points : Solutions by HPM. (a): comparisons for example 2. (b): comparisons for example 3.

So the series solutions are

$$\begin{cases} u(t) = \frac{1}{10} + \frac{2}{25}t + \frac{33}{1000}t^2 + \frac{47}{7500}t^3 + \cdots \\ v(t) = \frac{1}{10} - \frac{1}{10}t + \frac{59}{1000}t^2 - \frac{199}{7500}t^3 + \cdots \end{cases}$$

The approximate polynomial solutions can be obtained by the truncated series $u(t) \approx \sum_{i=0}^{N} u_i$ and $v(t) \approx \sum_{i=0}^{N} v_i$ where they are calculated up to $N = 8$ and the results are tabulated in Table 3.

Table 3. Comparisons between HPM and Maple procedure results.

t	$u(t)_{HPM}$	$v(t)_{HPM}$	$u(t)_{Maple\ proc}$	$v(t)_{Maple\ proc}$
-0.5	0.06748481	0.16890619	0.06748480	0.16890643
-0.4	0.07288541	0.15146575	0.07288539	0.15146585
-0.3	0.07880253	0.13612537	0.07880253	0.13612536
-0.2	0.08527014	0.12259097	0.08527014	0.12259106
-0.1	0.09232375	0.11061765	0.09232375	0.11061765
0.0	0.10000000	0.10000000	0.10000000	0.10000000
0.1	0.10833627	0.09056450	0.10833627	0.08216364
0.2	0.11737011	0.08216367	0.11737012	0.08216364
0.3	0.12713868	0.07467145	0.12713868	0.07467145
0.4	0.13767804	0.06797950	0.13767804	0.06797944
0.5	0.14902239	0.06199438	0.14902239	0.06199420

5. Conclusion

In this paper the homotopy perturbation method is applied to find the approximate solutions of a modified Lotka - Volterra system with the given initial conditions. The numerical solutions are compared with those from Maple which uses Runge - Kutta in Table 1., Table 2., Table 3., and Figure 1. The results showed that the homotopy perturbation method is a powerful technique through which we can achieve a desired approximation to the solution by simple calculations.

MAPLE 12.0 software has been employed for the computations in this work.

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References

[1] A. Ghorbani and J. S. Nadjfi, He's homotopy perturbation method for calculating Adomian's polynomials. *Int. J. Nonlin. Sci. Num. Simul*, 8 (2) (2007), 229-332.

[2] D. D. Ganji and H. Mirgolbabaei, Me. Miansari, Mo. Miansari, Application of homotopy perturbation method to solve linear and non-Linear systems of ordinary differential equations and differential equation of order three, *J. Appl. Scie*, 8 (7) (2008), 1256-1261.

[3] J. Biazar, E. Babolian, and R. Islam, Solution of the system of ordinary differential equations by Adomian decomposition method, *Applied Math. Comput*, 147 (3) (2004), 713-719.

[4] E. Hesameddini and A. Peyrovi, The use of variational iteration method and homotopy perturbation method for painleve equation I, *Applied Mathematical Sciences*, 178 (3) (2009), 1861-1871.

[5] E. Hesameddini and A. Peyrovi, Homotopy perturbation method for second painleve equation and comparisons with analytic continuation extension and chebishev series method, *International Mathematical Forum*, (2010) (in print).

[6] J. H. He, Homotopy perturbation technique, *Comput. Methods. Appl. Mech. Eng*, 178 (3-4) (1999), 257-262.

[7] J. H. He, A coupling method of a homotopy technique and a perturbation technique for nonlinear problems, *Int. J. Non-Linear. Mech*, 351 (1) (2000), 37-43.

[8] J. H. He, Homotopy pertubation method for solving boundary value problems, *Phys. Lett*, A 350 (1-2) (2006), 87-88.

[9] J. H. He, Variational iteration method some recent result and new interpretations, *Jornal of Computational and Applied Mathematics*, 207 (2007), pp, 3-17.

[10] J. H. He, Recent developments of the homotopy perturbation method, *Top. Meth. Nonlin. Anzl*, 31 (2008), 205-209.

[11] J. H. He, Homotopy perturbation technique, *Comput. Math.Appl, Mech. Engy*, (1999) pp, 178-257.

[12] J. H. He, Some asymptotic methods for strongly nonlinear equation, *Int. J. N. Phy*, 20(20), 10 (2006), 1144-1199.

[13] M. A. Noor and S. T. Mohyud-Din, Homotopy perturbation method for nonlinear higher-order boundary value problems, *Int. J. Nonlin. Sci. Num. Simul*, 9 (4) (2008), 395-408.

[14] M. A. Noor and S. T. Mohyud-Din, Homotopy perturbation method for solving sixth-order boundary value problems, *Comput. Math. Appl*, 55 (12) (2008), 2953-2972.

[15] M. Gorji, D. D. Ganji, and S. Soleimani, New application of He's homotopy perturbation method, *Int. J. Nonl. Sci. Num. Sim*, 8 (3) (2007), 319-325.

[16] S. Abbasbandy, Numerical solutions of the integral equations, *App. Math. Comput*, 173 (2006), 493-500.

[17] S. Abbasbandy, Homotopy perturbation method for quadratic Riccati differential equations and comparsion with Adomian's decomposition method, *Applied Math.Comput*, 172 (1) (2006), 485-490.

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