Journal of Mathematical Extension
Vol. 3, No. 2 (2009), 69-75

Effect of Polynomial Identity $\mathbf{x[x,y] = (x[x,y])^n}$ in the Commutativity of Rings

Z. Tabatabaei
Islamic Azad University-Marvdasht Branch

Abstract. In this paper we study some sufficient conditions for commutativity of a ring according to Jacobsons'idea. Jacobson proved that if R is a ring satisfying $x^n = x$ $(n > 1)$ for each $x \in R$, then R is commutative. In this paper, we show that R is commutative if for every $x, y \in R$ there exists a positive integer $n = n(x, y)$ such that $(x[x, y])^n = x[x, y]$.

AMS Subject Classification: 13PXX, 14A05.
Keywords and Phrases: Commutator, left(right) s-unital, left semisimple ring, Jacobson Radical, left Primitive ring, division ring, faithful simple left R-module.

1. Introduction

Throughout this paper R denotes an associative ring with center C and Jacobson radical $J(R)$.
In 1950, Jacobson proved that if for each x in R there exists a positive integer $n > 1$ such that $x^n = x$, then R is commutative . After that in [1], Searcoid and MacHale proved that if for each x, y in R, there exists a positive integer $n = n(x, y) > 1$ such that $(xy)^n = xy$, then R is commutative. In [3], Hirano and Yaqub studied the rings satisfying $(x - x^n)(y - y^n) = 0$ for every $x, y \in R$ and recently, Bell, yaqub and Abu-khuzam in [5], [6], [7] have considered some conditions and periodicity conditions for rings to be commutative.
We will fix commutator $[x, y] = xy - yx$ in place of y and obtain commutative results for rings.

A ring R is called left (resp. right) s-unital ([2]) if for each $x \in R$ we have $x \in Rx$ (resp. $x \in xR$). A ring R is called s-unital if for each x in R, $x \in xR \cap Rx$.

If R is a s-unital ring, then for any finite subset K of R, there exists an element e in R such that $xe = ex = x$ for all $x \in K$ (see [2]). Such an element e will be called a pueudo-identity of K.

A ring R is called to be semisimple if its Jocobson radical $J(R)$ is zero.

A ring R is called left primitive if there exists a simple faithful left R-module.

A ring R is said to be a subdirect product of the family of rings $\{R_i | i \in I\}$ if R is a subring of the direct product $\prod_{i \in I} R_i$ such that $\pi_k(R) = R_k$ for every $k \in I$, where $\pi_k : \prod_{i \in I} R_i \longrightarrow R_k$ is the canonical epimorphism.

2. Preliminaries

First of all, we recall some concepts and prove some results that are used in the sequel.

Theorem 2.1. *A nonzero ring R is semisimple if and only if R is isomorphic to a subdirect product of primitive rings.*

Proof. See [8. pro. 3.2]. □

Theorem 2.2. *If R is a ring, then we have*

(i) $J(R)$ *is the intersection of all the left annihilators of simple left R-modules.*

(ii) $J(R)$ *is the intersection of all the regular maximal left ideals of* R.

(iii) $J(R)$ *is the intersection of all the left primitive ideals of* R.

(iv) $J(R)$ *is a left quasi-regular ideal which contains every left quasi-regular left ideal of* R.

Proof. See [8. Theorem 2.3]. □

We say R satisfies $(*)$ if, for each x, y in R, there exists a positive integer $n = n(x, y) > 1$ such that $(x[x, y])^n = x[x, y]$.

Lemma 2.3. *Let R and S be rings such that R satisfies $(*)$ and $\varphi : R \longrightarrow S$ is a ring epimorphism. Then S satisfies $(*)$.*

Proof. Let $x, y \in S$. Since φ is onto, there exist $s, t \in R$ such that $\varphi(t) = x$, $\varphi(s) = y$. But R satisfies $(*)$. Therefore there exists a positive integer $n = n(t,s) > 1$ such that $(t[t,s])^n = t[t,s]$. On the other hand, we have

$$\varphi([t,s]) = \varphi(ts - st),$$

and since φ is a ring homomorphism, we have

$$\begin{aligned}\varphi([t,s]) &= \varphi(t)\varphi(s) - \varphi(s)\varphi(t),\\ &= [\varphi(t), \varphi(s)],\\ &= [x,y].\end{aligned}$$

Thus

$$\begin{aligned}x[x,y] &= \varphi(t)\left[\varphi(t), \varphi(s)\right],\\ &= \varphi\left(t[t,s]\right),\\ &= \varphi\left((t[t,s])^n\right),\\ &= \left(\varphi(t)[\varphi(t), \varphi(s)]\right)^n,\\ &= \left(x[x,y]\right)^n.\end{aligned}$$

The result follows. □

Lemma 2.4. *If R is a ring, then the quotient ring $\frac{R}{J(R)}$ is semisimple.*

Proof. See [8. Theorem 2.14.] □

Lemma 2.5. *Let $b \in R$ and $a \in J(R)$ such that $ab = b$. Then $b = 0$.*

Proof. Let $a \in J(R)$. By theorem 2.2, there exists $r \in R$ such that $r + a - ra = 0$ and therefore

$$0 = 0b = (r + a - ra)b = rb + ab - rab = b. \ \square$$

Theorem 2.6. *Let K be a division ring. If for any $x, y \in K$ there exists a positive integer $n = n(x, y) > 1$ such that $[x, y]^n = [x, y]$, then K is commutative.*

Proof. see [9. Theorem 12.10]. □

Theorem 2.7. *Let K be a division ring which satisfies $(*)$. Then K is commutative.*

Proof. Let x, y be arbitrary elements in K. If $x = 0$, then $[x, y] = 0$ and $xy = yx$. If $x \neq 0$, we put $z = x^{-1}y$. Hence by $(*)$, there exists a positive integer $n = n(x, z) > 1$ such that

$$(x[x, z])^n = x[x, z] \quad ,$$

$$([x, xz])^n = [x, xz] \quad ,$$

$$[x, y]^n = [x, y].$$

Hence for every $x, y \in K$, there exists a positive integer $n = n(x, y) > 1$ such that $[x, y]^n = [x, y]$. Thus, by theorem 2.6, K is commutative. □

Theorem 2.8. *Let R be a left primitive ring which satisfies $(*)$. Then R is commutative.*

Proof. Let R be a left primitive ring. By Structure Theorem for Left primitive ring ([9]), there exists a division ring $K = End_R(V)$ (V is a faitful simple left R-module) such that we have one of the following statements:

i) There exists a positive integer m such that $R \cong M_m(K)$.

ii) For any integer $m > 1$, there exists a subring R_m of R which admits a ring homomorphism onto $M_m(K)$.

But for any $m \geqslant 2$, the division ring $M_m(K)$ doesn't satisfy $(*)$. For example, if $x = E_{11}$, $y = E_{12}$, then $x[x, y] = y$ and therefore $(x[x, y])^n = y^n = 0$. Thus $m = 1$ and $R \cong K$. By theorem 2.6, R is commutative. □

Theorem 2.9. *Let R be a semisimple ring which satifies $(*)$. Then R is commutative.*

Proof. By theorem 2.1. R is isomorphic to a subdirect product of primitive rings. By Lemma 2.3., every R_i satisfies $(*)$. Therefore, by theorem 2.8., each R_i is commutative and so R is commutative. □

3. Main Results

Theorem 3.1. *Let R be a ring which satisfies $(*)$. Then $x[x,y] = 0$, for every $x, y \in R$.*

Proof. The semisimple ring $\overline{R} = \frac{R}{J(R)}$ satisfies $(*)$ and so by theorem 2.9, $\overline{R}$ is commutative. Therefore, for each x, y in R, $[x,y] \in J(R)$ and there exist a positive integer $n = n(x,y) > 1$ such that

$$([x,xy])^n = (x[x,y])^n = x[x,y] = [x,xy],$$

and so

$$[x,xy]^{n-1}[x,xy] = [x,xy]$$

by lemma 2.5., $x[x,y] = 0$. □

Corollary 1. *Let R be a left s-unital ring satisfies $(*)$. Then R is commutative.*

Proof. Since R is left s-unital, so for every x in R, there exists $e \in R$ that $x = ex$. Now we show that R is right s-unital.

If $x \neq xe$, then by $(*)$ there exists a positive integer $n > 1$ such that

$$e[e, xe - x] = (e[e, xe - x])^n.$$

But we have

$$(e[e, xe - x])^2 = (xe - x - xe^2 + xe)^2 = 0.$$

Therefore,

$$e[e, xe - x] = 0$$

and so

$$xe - x - xe^2 + xe = 0.$$

and hence, $x = x(2e - e^2)$. If $é = 2e - e^2$, for every $x \in R$, there exists $é \in R$ such that $x = xé$ and thus R is s-unital.
For every x, y in R, there exists $e \in R$ such that $xe = ex = x$ and $ye = ey = y$ $(see[2])$ and, by theorem 3.1, we have

$$\begin{aligned} 0 &= [x + e, \quad (x + e)y], \\ &= [x + e, \quad xy + y], \\ &= [x, \ xy + y] + [e, \ xy + y], \end{aligned}$$

since $[e, xy + y] = 0$, so

$$\begin{aligned} 0 &= [x, \quad xy + y], \\ &= [x, \quad xy] + [x, y]. \end{aligned}$$

By theorem 3.1., $[x, xy] = 0$, thus $[x, y] = 0$. Therefore, R is commutative. $\square$

Corollary 2. *In corollary 1, being s-unital is necessary, for example the following noncommutative ring isn't s-unital:*

$$A = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : \ a, b, c \text{ are any real numbers} \right\}.$$

But for every x, y, z in A, $xyz = 0$ and so A satisfies $(*)$, and A is noncommutative ring.

References

[1] M. O. Searcoid and D. Machale, Two elementary generalizations of boolean rings, *Amer. Math. Monthly*, 93 (1986), 121-122.

[2] A. H. Yamini, Some commutativity results for rings with certain Polynomial identities, *Math. Okayama Univ*, 26 (1994), 133-136.

[3] Y. Hirano and A. Yaqub, Rings satisfying the identity $(x-x^n)(y-y^n) = 0$, *Math. Okayama Univ.*, 29 (1997).

[4] A. H. Yamini and S. Sahebi, Rings satisfying the generalized Polynomial identity $(x - x^n)([x,y]_k - [x,y]_k^m) = 0$, *Riv. Math. Univ. Parma*, 6 (2) (1999), 11-18.

[5] H. Abu-khuzam, H. E. Bell, and A. Yaqub, A weak periodicity condition for rings, *Math. Science*, 9 (2005), 1387-1391.

[6] H. E. Bell and A. Yaqub, Near-commutativity and Partial-Periodicity Conditions for rings, *Result. Math.*, 46 (2004), 24-30.

[7] H. E. Bell and A. Yaqub, On commutativity of semiperiodic rings, *Result. Math.*, 53 (2009), 19-26.

[8] T. W. Hungerford, *Algebra*, by Springer-Verlag New York Inc., 1974.

[9] T. Y. Lam, *A First course in Noncommutative rings*, Springer-Verlag New York Inc., 1991.

Zohre Tabatabaei
Department of Mathematics
Islamic Azad University-Marvdasht Branch
Shiraz, Iran.
E-mail: Parivash.tabatabaee@yahoo.com