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Effect of Polynomial Identity $\mathbf{x[x,y] = (x[x,y])^n}$ in the Commutativity of Rings

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Abstract. In this paper we study some sufficient conditions for commutativity of a ring according to Jacobsons'idea. Jacobson proved that if R is a ring satisfying $x^n = x$ $(n > 1)$ for each $x \in R$, then R is commutative. In this paper, we show that R is commutative if for every $x, y \in R$ there exists a positive integer $n = n(x, y)$ such that $(x[x, y])^n = x[x, y]$.

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1. Introduction

Throughout this paper R denotes an associative ring with center C and Jacobson radical $J(R)$.
In 1950, Jacobson proved that if for each x in R there exists a positive integer $n > 1$ such that $x^n = x$, then R is commutative . After that in [1], Searcoid and MacHale proved that if for each x, y in R, there exists a positive integer $n = n(x, y) > 1$ such that $(xy)^n = xy$, then R is commutative. In [3], Hirano and Yaqub studied the rings satisfying $(x - x^n)(y - y^n) = 0$ for every $x, y \in R$ and recently, Bell, yaqub and Abu-khuzam in [5], [6], [7] have considered some conditions and periodicity conditions for rings to be commutative.
We will fix commutator $[x, y] = xy - yx$ in place of y and obtain commutative results for rings.

A ring R is called left (resp. right) s-unital ([2]) if for each $x \in R$ we have $x \in Rx$ (resp. $x \in xR$). A ring R is called s-unital if for each x in R, $x \in xR \cap Rx$.
If R is a s-unital ring, then for any finite subset K of R, there exists an element e in R such that $xe = ex = x$ for all $x \in K$ (see [2]). Such an element e will be called a pueudo-identity of K.
A ring R is called to be semisimple if its Jocobson radical $J(R)$ is zero.
A ring R is called left primitive if there exists a simple faithful left R-module.
A ring R is said to be a subdirect product of the family of rings $\{R_i | i \in I\}$ if R is a subring of the direct product $\prod_{i\in I} R_i$ such that $\pi_k(R) = R_k$ for every $k \in I$, where $\pi_k : \prod_{i\in I} R_i \longrightarrow R_k$ is the canonical epimorphism.

2. Preliminaries

First of all, we recall some concepts and prove some results that are used in the sequel.
Theorem 2.1. *A nonzero ring R is semisimple if and only if R is isomorphic to a subdirect product of primitive rings.*

Proof. See [8. pro. 3.2]. □

Theorem 2.2. *If R is a ring, then we have*
(i) $J(R)$ *is the intersection of all the left annihilators of simple left R-modules.*
(ii) $J(R)$ *is the intersection of all the regular maximal left ideals of* R.
(iii) $J(R)$ *is the intersection of all the left primitive ideals of* R.
(iv) $J(R)$ *is a left quasi-regular ideal which contains every left quasi-regular left ideal of* R.

Proof. See [8. Theorem 2.3]. □
We say R satisfies $(*)$ if, for each x, y in R, there exists a positive integer $n = n(x, y) > 1$ such that $(x[x, y])^n = x[x, y]$.

Lemma 2.3. *Let R and S be rings such that R satisfies $(*)$ and $\varphi : R \longrightarrow S$ is a ring epimorphism. Then S satisfies $(*)$.*

Proof. Let $x, y \in S$. Since φ is onto, there exist $s, t \in R$ such that $\varphi(t) = x$, $\varphi(s) = y$. But R satisfies $(*)$. Therefore there exists a positive integer $n = n(t, s) > 1$ such that $(t[t, s])^n = t[t, s]$. On the other hand, we have

$$\varphi([t, s]) = \varphi(ts - st),$$

and since φ is a ring homomorphism, we have

$$\begin{aligned} \varphi([t, s]) &= \varphi(t)\varphi(s) - \varphi(s)\varphi(t), \\ &= [\varphi(t), \varphi(s)], \\ &= [x, y]. \end{aligned}$$

Thus

$$\begin{aligned} x[x, y] &= \varphi(t)\left[\varphi(t), \varphi(s)\right], \\ &= \varphi\left(t[t, s]\right), \\ &= \varphi\left((t[t, s])^n\right), \\ &= \left(\varphi(t)[\varphi(t), \varphi(s)]\right)^n, \\ &= \left(x[x, y]\right)^n. \end{aligned}$$

The result follows. □

Lemma 2.4. *If R is a ring, then the quotient ring $\frac{R}{J(R)}$ is semisimple.*

Proof. See [8. Theorem 2.14.] □

Lemma 2.5. *Let $b \in R$ and $a \in J(R)$ such that $ab = b$. Then $b = 0$.*

Proof. Let $a \in J(R)$. By theorem 2.2, there exists $r \in R$ such that $r + a - ra = 0$ and therefore

$$0 = 0b = (r + a - ra)b = rb + ab - rab = b. \ \square$$

Theorem 2.6. *Let K be a division ring. If for any $x, y \in K$ there exists a positive integer $n = n(x, y) > 1$ such that $[x, y]^n = [x, y]$, then K is commutative.*

Proof. see [9. Theorem 12.10]. $\square$

Theorem 2.7. *Let K be a division ring which satisfies $(*)$. Then K is commutative.*

Proof. Let x, y be arbitrary elements in K. If $x = 0$, then $[x, y] = 0$ and $xy = yx$. If $x \neq 0$, we put $z = x^{-1}y$. Hence by $(*)$, there exists a positive integer $n = n(x, z) > 1$ such that

$$(x[x, z])^n = x[x, z] \quad ,$$

$$([x, xz])^n = [x, xz] \quad ,$$

$$[x, y]^n = [x, y].$$

Hence for every $x, y \in K$, there exists a positive integer $n = n(x, y) > 1$ such that $[x, y]^n = [x, y]$. Thus, by theorem 2.6, K is commutative. $\square$

Theorem 2.8. *Let R be a left primitive ring which satisfies $(*)$. Then R is commutative.*

Proof. Let R be a left primitive ring. By Structure Theorem for Left primitive ring ([9]), there exists a division ring $K = End_R(V)$ (V is a faitful simple left R-module) such that we have one of the following statements:

i) There exists a positive integer m such that $R \cong M_m(K)$.

ii) For any integer $m > 1$, there exists a subring R_m of R which admits a ring homomorphism onto $M_m(K)$.

But for any $m \geqslant 2$, the division ring $M_m(K)$ doesn't satisfy $(*)$. For example, if $x = E_{11}$, $y = E_{12}$, then $x[x, y] = y$ and therefore $(x[x, y])^n = y^n = 0$. Thus $m = 1$ and $R \cong K$. By theorem 2.6, R is commutative. $\square$

Theorem 2.9. *Let R be a semisimple ring which satifies* $(*)$. *Then R is commutative.*

Proof. By theorem 2.1. R is isomorphic to a subdirect product of primitive rings. By Lemma 2.3., every R_i satisfies $(*)$. Therefore, by theorem 2.8., each R_i is commutative and so R is commutative. □

3. Main Results

Theorem 3.1. *Let R be a ring which satisfies* $(*)$. *Then $x[x,y]=0$, for every $x,y \in R$.*

Proof. The semisimple ring $\overline{R} = \frac{R}{J(R)}$ satisfies $(*)$ and so by theorem 2.9, $\overline{R}$ is commutative. Therefore, for each x,y in R, $[x,y] \in J(R)$ and there exist a positive integer $n = n(x,y) > 1$ such that

$$([x,xy])^n = (x[x,y])^n = x[x,y] = [x,xy],$$

and so

$$[x,xy]^{n-1}[x,xy] = [x,xy]$$

by lemma 2.5., $x[x,y]=0$. □

Corollary 1. *Let R be a left s-unital ring satisfies* $(*)$. *Then R is commutative.*

Proof. Since R is left s-unital, so for every x in R, there exists $e \in R$ that $x = ex$. Now we show that R is right s-unital.

If $x \neq xe$, then by $(*)$ there exists a positive integer $n > 1$ such that

$$e[e, xe-x] = (e[e, xe-x])^n.$$

But we have

$$(e[e, xe-x])^2 = (xe - x - xe^2 + xe)^2 = 0.$$

Therefore,

$$e[e, xe-x] = 0$$

and so

$$xe - x - xe^2 + xe = 0.$$

and hence, $x = x(2e - e^2)$. If $é = 2e - e^2$, for every $x \in R$, there exists $é \in R$ such that $x = xé$ and thus R is s-unital.
For every x, y in R, there exists $e \in R$ such that $xe = ex = x$ and $ye = ey = y$ $(see[2])$ and, by theorem 3.1, we have

$$\begin{aligned} 0 &= [x + e, \quad (x + e)y], \\ &= [x + e, \quad xy + y], \\ &= [x, \ xy + y] + [e, \ xy + y], \end{aligned}$$

since $[e, xy + y] = 0$, so

$$\begin{aligned} 0 &= [x, \quad xy + y], \\ &= [x, \quad xy] + [x, y]. \end{aligned}$$

By theorem 3.1., $[x, xy] = 0$, thus $[x, y] = 0$. Therefore, R is commutative. $\square$

Corollary 2. *In corollary 1, being s-unital is necessary, for example the following noncommutative ring isn't s-unital:*

$$A = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : \ a, b, c \text{ are any real numbers} \right\}.$$

But for every x, y, z in A, $xyz = 0$ and so A satisfies $(*)$, and A is noncommutative ring.

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