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Application of He's Variational Iteration Method for solving the Equation Governing the Unsteady Flow of a Polytropic Gas

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Abstract. In this paper, He's variational iteration method (VIM) is used to obtain the exact solution of the Equation Governing the Unsteady Flow of a Polytropic Gas. This method is based on Lagrange multiplier for identification of optimal value of parameter in a functional. Using this method creates a sequence which tends to the exact solution of problem. The method is capable of reducing the size of calculation and easily overcomes the difficulty of the Adomian polynomials. The results reveal that He's variational iteration method is very effective for these types of equations.

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1. Introduction

Although there are few phenomena in different fields of science occurring linearly, most of them occur nonlinearly. We know that except a limited number of these problems, most of them do not have precise analytical solutions; therefore they have to be solved using other approximate methods such as He's variational iteration method . The Equation Governing the Unsteady Flow of a Polytropic Gas in two dimensions is given by Feng [3], Billingham [2], Rogers and Ames [14]. In this paper we find the exact solution of the Equation Governing the Unsteady Flow of a Polytropic Gas by He's variational iteration method.

2. Basic Idea of VIM

To clarify the Basic ideas of the variational iteration method, we Consider the following differential equation:

$$L[u(t) + N[u(t)] = g(t),$$

where L is a linear operator , N is a non-linear operator and $g(t)$ is an inhomogeneous term. According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau))d\tau,$$

where λ is a general Lagrange multiplier [1, 4, 5, 6, 7, 8, 9] which can be identified optimally via the variational theory [10, 11, 12, 13, 15]. The subscript n indicates the nth approximation and $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$. The variational iteration method proposed by Ji Huan He has been shown to solve effectively, easily and accurately a large class of non-linear problems with approximations converging rapidly to accurate solutions.

3. Implementation of the Method

Consider the Equation Governing the Unsteady Flow of a Polytropic Gas:

$$u_t + uu_x + vu_y + \frac{p_x}{\rho} = 0,$$

$$v_t + uv_x + vv_y + \frac{p_y}{\rho} = 0,$$

$$\rho_t + u\rho_x + v\rho_y + \rho(u_x + v_y) = 0,$$

$$p_t + up_x + vp_y + \gamma p(u_x + v_y) = 0,$$

where ρ is the density, p the pressure, u and v the velocity components in the x and y directions, respectively, and the adiabatic index γ is the ratio of the specific heats.

With the initial conditions
$u(x, y, 0) = e^{x+y}, v(x, y, 0) = -1 - e^{x+y}, \rho(x, y, 0) = e^{x+y}$ and
$p(x, y, 0) = c$.

The exact solutions of the equations are
$u(x, y, t) = e^{x+y+t}, v(x, y, t) = -1 - e^{x+y+t}, \rho(x, y, t) = e^{x+y+t}$ and
$p(x, y, t) = c$.

Their correction functionals can be written down as follows

$$\begin{aligned} u_{n+1}(x,y,t) &= u_n(x,y,t) + \int_0^t \lambda_1(\tau)[u_{n\tau}(x,y,\tau) + u_n(x,y,\tau)u_{nx}(x,y,\tau) \\ &+ v_n(x,y,\tau)u_{ny}(x,y,\tau) + \frac{p_{nx}(x,y,\tau)}{\rho_n(x,y,\tau)}]d\tau, \end{aligned}$$

$$\begin{aligned} v_{n+1}(x,y,t) &= v_n(x,y,t) + \int_0^t \lambda_2(\tau)[v_{n\tau}(x,y,\tau) + u_n(x,y,\tau)v_{nx}(x,y,\tau) \\ &+ v_n(x,y,\tau)v_{ny}(x,y,\tau) + \frac{p_{ny}(x,y,\tau)}{\rho_n(x,y,\tau)}]d\tau, \end{aligned}$$

$$\begin{aligned} \rho_{n+1}(x,y,t) &= \rho_n(x,y,t) + \int_0^t \lambda_3(\tau)[\rho_{n\tau}(x,y,\tau) + u_n(x,y,\tau)\rho_{nx}(x,y,\tau) \\ &+ v_n(x,y,\tau)\rho_{ny}(x,y,\tau) + \rho_n(u_{nx} + v_{ny})]d\tau, \end{aligned}$$

$$\begin{aligned} p_{n+1}(x,y,t) &= p_n(x,y,t) + \int_0^t \lambda_4(\tau)[p_{n\tau}(x,y,\tau) + u_n(x,y,\tau)p_{nx}(x,y,\tau) \\ &+ v_n(x,y,\tau)p_{ny}(x,y,\tau) + \gamma p_n(u_{nx} + v_{ny})]d\tau, \end{aligned}$$

Their stationary conditions are obtained as follows:

$$\begin{cases} 1 + \lambda_i(\tau)_{\tau=t} = 0, \\ \acute{\lambda_i}(\tau) = 0. \qquad\qquad fori = 1, 2, 3, 4 \end{cases}$$

Lagrange multipliers can be obtained as $\lambda_{1,2,3,4}(\tau) = -1$, thus we have:

$$\begin{aligned} u_{n+1}(x,y,t) &= u_n(x,y,t) - \int_0^t [u_{n\tau}(x,y,\tau) + u_n(x,y,\tau)u_{nx}(x,y,\tau) \\ &+ v_n(x,y,\tau)u_{ny}(x,y,\tau) + \frac{p_{nx}(x,y,\tau)}{\rho_n(x,y,\tau)}]d\tau, \end{aligned}$$

$$\begin{aligned} v_{n+1}(x,y,t) &= v_n(x,y,t) - \int_0^t [v_{n\tau}(x,y,\tau) + u_n(x,y,\tau)v_{nx}(x,y,\tau) \\ &+ v_n(x,y,\tau)v_{ny}(x,y,\tau) + \frac{p_{ny}(x,y,\tau)}{\rho_n(x,y,\tau)}]d\tau, \end{aligned}$$

$$\begin{aligned} \rho_{n+1}(x,y,t) &= \rho_n(x,y,t) - \int_0^t [\rho_{n\tau}(x,y,\tau) + u_n(x,y,\tau)\rho_{nx}(x,y,\tau) \\ &+ v_n(x,y,\tau)\rho_{ny}(x,y,\tau) + \rho_n(u_{nx} + v_{ny})]d\tau, \end{aligned}$$

$$\begin{aligned} p_{n+1}(x,y,t) &= p_n(x,y,t) - \int_0^t [p_{n\tau}(x,y,\tau) + u_n(x,y,\tau)p_{nx}(x,y,\tau) \\ &+ v_n(x,y,\tau)p_{ny}(x,y,\tau) + \gamma p_n(u_{nx} + v_{ny})]d\tau. \end{aligned}$$

Now by using initial approximations and the above iteration formulas, we can obtain the following iterations :

$$u_1(x,y,t) = e^{x+y}(1+t),$$

$$u_2(x,y,t) = \frac{1}{2}e^{x+y}(2+2t+t^2),$$

$$u_3(x,y,t) = \frac{1}{6}e^{x+y}(6+6t+3t^2+t^3),$$

$$u_4(x,y,t) = \frac{1}{24}e^{x+y}(24+24t+12t^2+4t^3+t^4),$$

and etc.. Also we have

$$v_1(x,y,t) = -1 - e^{x+y} - e^{x+y}t,$$

$$v_2(x,y,t) = -1 - e^{x+y} - e^{x+y}t - \frac{1}{2}e^{x+y}t^2,$$
$$v_3(x,y,t) = -1 - e^{x+y} - e^{x+y}t - \frac{1}{2}e^{x+y}t^2 - \frac{1}{6}e^{x+y}t^3,$$
$$v_4(x,y,t) = -1 - e^{x+y} - e^{x+y}t - \frac{1}{2}e^{x+y}t^2 - \frac{1}{6}e^{x+y}t^3 - \frac{1}{24}e^{x+y}t^4,$$
$$\vdots$$

and

$$\rho_1(x,y,t) = e^{x+y}(1+t),$$
$$\rho_2(x,y,t) = \frac{1}{2}e^{x+y}(2+2t+t^2),$$
$$\rho_3(x,y,t) = \frac{1}{6}e^{x+y}(6+6t+3t^2+t^3),$$
$$\rho_4(x,y,t) = \frac{1}{24}e^{x+y}(24+24t+12t^2+4t^3+t^4),$$
$$\vdots$$

and $p_i(x,y,t) = c$, for i=1,2, ...

By continuing this manner, the solutions of $u(x,y,t)$, $v(x,y,t)$, $\rho(x,y,t)$ and $p(x,y,t)$ are obtained as:

$u(x,y,t) = e^{x+y+t}$, $v(x,y,t) = -1 - e^{x+y+t}$, $\rho(x,y,t) = e^{x+y+t}$ and $p(x,y,t) = c$,

which are exactly the same as the exact solutions.

4. Conclusions

In this paper, we present the application of He's variational iteration method for obtaining the exact solution of the Equation Governing the Unsteady Flow of a Polytropic Gas. Some of the advantages of VIM are that the initial solution can be freely chosen with some unknown parameters that we can easily achieve the unknown parameters in the initial solution. An interesting point about VIM is that with the fewest number of iterations or even in some cases, once, it can converge to exact results. therefore this method give a powerful mathematical tool for nonlinear problems. In final, we use the Matlab Package to calculate the functions obtained from the He's variational iteration method.

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