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Reflexivity on Some Function Spaces*

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"Dedicated to Mola Ali"

Abstract. We give sufficient conditions on a domain Ω so that the associated canonical model is reflexive. Also, we discuss a class of shifts that are reflexive, and the operator M_z of multiplication by z on a Banach space of functions analytic on a domain is shown to be reflexive whenever M_z is polynomially bounded.

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1. Introduction

In this section we include some preparatory material which will be needed later. By a *domain* we understand a connected open subset of the plane. If Ω is a bounded domain in the plane, then the *Carathéodory hull* (or $\mathbb{C}$-hull) of Ω is the complement of the closure of the unbounded component of the complement of the closure of Ω. The $\mathbb{C}$-hull of Ω is denoted by Ω^*. Intuitively, Ω^* can be described as the interior of the outer boundary of Ω, and in analytic terms it can be defined as the interior of the set of all points z_0 in the plane such that $|p(z_0)| \leqslant \sup\{|p(z)| : z \in \Omega\}$ for all polynomials p. The components of Ω^* are simply connected; in fact, one can easily see that each of these components has a connected complement. The component of Ω^* that contains Ω is denoted by Ω_1.

*Invited paper

Note that for all polynomials p, $\|p\|_\Omega = \|p\|_{\Omega_1}$. The domain Ω is called a *Carathéodory domain* if $\Omega^* = \Omega$. In this case the Farrell-Rubel-Shields Theorem holds: let f be a bounded analytic function on Ω. Then there is a sequence $\{p_n\}$ of polynomials such that $||p_n||_\Omega \leqslant c$ for a constant c and $p_n(z) \to f(z)$ for all $z \in \Omega$ ([9, theorem 5.1, p.151]).

Now let X be a reflexive Banach space. For the algebra $\mathcal{B}(X)$ of all bounded operators on the Banach space X, the weak operator topology is the one in which a net A_α converges to A if $A_\alpha x \to Ax$ weakly, $x \in X$. Recall that if $A \in \mathcal{B}(X)$, then $Lat(A)$ is by definition the lattice of all invariant subspaces of A, and $AlgLat(A)$ is the algebra of all operators B in $\mathcal{B}(X)$ such that $Lat(A) \subset Lat(B)$. An operator A in $\mathcal{B}(X)$ is said to be *reflexive* if $AlgLat(A) = W(A)$, where $W(A)$ is the smallest subalgebra of $\mathcal{B}(X)$ that contains A and the identity I and is closed in the weak operator topology.

The operator M_z has been the focus of attention for several decades and many of its properties have been studied ([2]). In [12] Sarason proved that normal operators are reflexive. It was shown by J. Deddens ([6]) that every isometry is reflexive. Also, R. Olin and J. Thomson ([10]) have shown that subnormal operators are reflexive. H. Bercovici, C. Foias, J. Langsam, and C. Pearcy ([1]) have shown that (BCP)-operators are reflexive. The reflexive operators on a finite dimensional space were characterized by J. Deddens and P. A. Fillmore ([7]). Reflexivity of certain bilateral weighted shift are also studied in [24, 27]. In [18] we investigated some sufficient conditions for the reflexivity of multiplication operators on Dirichlet spaces. Also, in [28] we study the reflexivity of canonical models associated with generalized Bergman kernel. For some sources of reflexivity see [11, 14, 15, 25, 31].

2. Conditions for the Reflexivity of Canonical Model

For a connected open subset Ω of the plane and n a positive integer, let $B_n(\Omega)$ be the Cowen-Douglas class of operators. In this article we show that if Ω is an arbitrary domain and $T \in B_n(\Omega)$, then under sufficient conditions T is reflexive.

For a connected open subset Ω of the plane and n a positive integer, let $B_n(\Omega)$ denote the operators T defined on the Hilbert space $\mathcal{H}$ which satisfy

(a) $\Omega \subseteq \sigma(T)$,

(b) $ran(T-\lambda) = \mathcal{H}$ for λ in Ω,

(c) $\bigvee_{\lambda\in\Omega} \ker(T-\lambda) = \mathcal{H}$, and

(d) $\dim\ker(T-\lambda) = n$ for λ in Ω.

The spaces $B_n(\Omega)$ has been introduced by Cowen and Douglas ([3]). It is shown in [5] that every operator in the class $B_n(\Omega)$ is unitarily equivalent to the adjoint of the canonical model associated with a generalized Bergman kernel (g.B.k. for brevity) K. Actually K is the reproducing kernel for a coanalytic functional Hilbert space $\mathcal{K}$ on which we can define the operator $T_{\bar{z}}$ of multiplication by $\bar{z}$. The operator $T = T_{\bar{z}}^*$ acting on $\mathcal{K}$ is called the canonical model associated with K. We know that for every λ in Ω, $T-\lambda$ is onto,

$$\ker(T-\lambda) = ranK(\lambda,\cdot) = \{K(\lambda,\cdot)\xi : \xi \in \mathbb{C}^n\},$$

and $\dim\ker(T-\lambda) = n$. For a detailed treatment of the subject of Cowen-Douglas classes see [3,5,29,33].

From now on $T = T_{\bar{z}}^* \in B_n(\Omega)\cap\mathcal{B}(\mathcal{K})$. Indeed T is a canonical model associated with a g.B.k. K for a coanalytic functional Hilbert space $\mathcal{K}$. In this section we give some sufficient conditions so that the associated canonical model is reflexive. This answers the question 5.6 in [13, p.98].

Lemma 2.1. *Let K be a g.B.k. on Ω. If $X \in AlgLat(T)$, then there exists a function $\psi \in H^\infty(\Omega)$ such that $XK(\lambda,\cdot) = \psi(\lambda)K(\lambda,\cdot)$ for all λ in Ω.*

Proof. Let $X \in AlgLat(T)$. Then every invariant subspace of T is invariant under X too. In particular, the one-dimensional span of $K(\lambda,\cdot)\xi$ is invariant under X. Hence

$$XK(\lambda,\cdot)\xi = \psi(\lambda,\xi)K(\lambda,\cdot)\xi, \qquad \psi(\lambda,\psi) \in \mathbb{C}. \tag{1}$$

We shall show that ψ does not depend on ξ and that it is analytic on Ω.

To show that ψ is independent of ξ, let u and v be two distinct nonzero vectors in $\mathbb{C}^n$. First, suppose u is not a multiple of v. In (1) replace ξ by u and v and add the two equations to get

$$XK(\lambda,\cdot)(u+v)=\psi(\lambda,u)K(\lambda,\cdot)u+\psi(\lambda,v)K(\lambda,\cdot)v. \tag{2}$$

Again, replacing ξ by $u+v$ in (1) gives us

$$XK(\lambda,\cdot)(u+v)=\psi(\lambda,u+v)K(\lambda,\cdot)(u+v). \tag{3}$$

Subtract (3) from (2) to get

$$\psi(\lambda,u)K(\lambda,\cdot)u+\psi(\lambda,v)K(\lambda,\cdot)v=\psi(\lambda,u+v)K(\lambda,\cdot)(u+v).$$

Since $K(\lambda,\lambda)$ is invertible we have $\psi(\lambda,u)+\psi(\lambda,v)v=\psi(\lambda,u+v)(u+v)$. Hence

$$(\psi(\lambda,u)-\psi(\lambda,u+v))u=(\psi(\lambda,u+v)-\psi(\lambda,v))v.$$

Since u is not a multiple of v we get $\psi(\lambda,u)=\psi(\lambda,v)=\psi(\lambda,u+v)$.

Now let $v=cu$ where c is a nonzero constant. In (1) replace ξ by u and then multiply the resulting equation by c to get

$$XK(\lambda,\cdot)v=\psi(\lambda,u)K(\lambda,\cdot)v.$$

Again, replace ξ by v in (1) to get

$$XK(\lambda,\cdot)v=\psi(\lambda,v)K(\lambda,\cdot)v.$$

Comparing the two equations and using the fact that $K(\lambda,\lambda)$ is invertible we arrive at $\psi(\lambda,u)=\psi(\lambda,v)$.

The preceding argument shows that $XK(\lambda,\cdot)=\psi(\lambda)K(\lambda,\cdot)$. Writing

$$XK(\lambda,\cdot)-XK(\lambda_0,\cdot)=(\psi(\lambda)-\psi(\lambda_0))K(\lambda,\cdot)+\psi(\lambda_0)(K(\lambda,\cdot)-K(\lambda_0,\cdot)),$$

then dividing by $\lambda-\lambda_0$ and taking the limit shows that ψ is analytic on Ω. We also have

$$|\psi(\lambda)|\|K(\lambda,\cdot)\xi\|=\|XK(\lambda,\cdot)\xi\|\leqslant\|X\|\|K(\lambda,\cdot)\xi\|.$$

So $|\psi(\lambda)| \leqslant \|X\|$ for all λ in Ω and therefore $\psi \in H^\infty(\Omega)$. This completes the proof. □

Corollary 2.2. *Let K be a g.B.k. on Ω. Then $Alglat(T) \subseteq \{T\}'$. Also if $W(T) = \{T\}'$, then T is reflexive.*

Corollary 2.3. *If $T \in B_1(\Omega)$ and T^* is an injective unilateral weighted shift, then T is reflexive.*

Note that a compact subset F of the plane is a spectral set for a bounded operator A if F contains $\sigma(A)$ and $\|f(A)\| \leqslant \sup_{z\in F} |f(z)|$ for all rational functions f with poles off F.

Theorem 2.4.([8]) *Let K be a g.B.k. on a domain Ω such that any sequence of polynomials in T that converges on a dense subset of $\mathcal{K}$ is bounded. Then T is reflexive.*
By $\sigma(T)$, we mean the spectrum of T.

Corollary 2.5. *Let K be a g.B.k. on a Caratheodory region Ω such that $\sigma(T) = \overline{\Omega}$ is a spectral set for T. Then T is reflexive.*

Proof. Let $X \in AlgLat(T)$. Then $XK(\lambda, .) = \psi(\lambda)K(\lambda, .)$ for some function $\psi \in H^\infty(\Omega)$. Since Ω is a Caratheodory region, there is a uniformly bounded sequence $\{p_n\}_n$ of polynomials converging pointwise to ψ. Also $\{p_n(T)g\}_n$ converges for all g in a dense subset of $\mathcal{K}$. Since $\sigma(T) = \overline{\Omega}$ is a spectral set, $\{p_n(T)\}_n$ is bounded and so by the Theorem the proof is complete. □

Corollary 2.6. *If T is a contraction in $B_n(U)$ where U is the open unit disc, then T is reflexive.*

3. Certain Weighted Shifts are Reflexive

In this section we give sufficient conditions for reflexivity of the multiplication operator by the independent variable z, M_z, acting on Banach spaces of formal series. This work presents sufficient conditions to a problem considered by A. L. Shields.

Let $\beta = \{\beta(n)\}_{n=-\infty}^{\infty}$ be a sequence of positive numbers satisfying

$\beta(0) = 1$. If $1 \leqslant p < \infty$, the space $L^p(\beta)$ consists of all formal Laurent series $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n$ such that the norm $\|f\|^p = \|f\|_\beta^p = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \beta(n)^p$ is finite. When n just runs over $\mathbb{N} \cup \{0\}$, the space $L^p(\beta)$ only contains formal power series $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ and it is usually denoted by $H^p(\beta)$. If $p = 2$, such spaces were introduced by Allen L. Shields [17] to study weighted shift operators. Let $\hat{f}_k(n) = \delta_k(n)$. So $f_k(z) = z^k$ and then $\{f_k\}_{k\in\mathbb{Z}}$ is a basis for $L^p(\beta)$ such that $\|f_k\| = \beta(k)$. Now consider M_z, the operator of multiplication by z on $L^p(\beta)$: $(M_z f)(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^{n+1}$ where $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n \in L^p(\beta)$. In other words $(M_z f)\hat{}(n) = \hat{f}(n-1)$ for all $n \in \mathbb{Z}$. Clearly M_z shifts the basis $\{f_k\}_k$. The operator M_z is bounded if and only if $\{\beta(k+1)/\beta(k)\}_k$ is bounded and in this case $\|M_z^n\| = \sup_k[\beta(k+n)/\beta(k)]$ for all $n \in \mathbb{N} \cup \{0\}$. Clearly M_z is invertible if and only if $\beta(k)/\beta(k+1)$ is bounded.

We denote the set of multipliers $\{\varphi \in L^p(\beta) : \varphi L^p(\beta) \subseteq L^p(\beta)\}$ by $L^p_\infty(\beta)$ and the linear operator of multiplication by φ on $L^p(\beta)$ by M_φ. Also the set of multipliers on $H^p(\beta)$ is denoted by $H^p_\infty(\beta)$.

We say that a complex number λ is a bounded point evaluation on $L^p(\beta)$ if the functional $e(\lambda) : L^p(\beta) \longrightarrow \mathbb{C}$ defined by $e(\lambda)(f) = f(\lambda)$ is bounded.

By the same method used in [19] we can see that $L^p(\beta)^* = L^q(\beta^{\frac{p}{q}})$, where $\frac{1}{p} + \frac{1}{q} = 1$. Also if $f(z) = \sum_n \hat{f}(n)z^n \in L^p(\beta)$ and $g(z) = \sum_n \hat{g}(n)z^n \in L^q(\beta^{\frac{p}{q}})$, then clearly $< f, g >= \sum_n \hat{f}(n)\overline{\hat{g}(n)}\beta(n)^p$. For a good source in formal power series, we refer the reader to [4, 17, 19–23, 26].

First we note that the multiplication operator M_z on $L^p(\beta)(H^p(\beta))$ is unitarily equivalent to an injective bilateral (unilateral) weighted shift and conversely, every injective bilateral (unilateral) weighted shift is unitarily equivalent to M_z acting on $L^p(\beta)(H^p(\beta))$ for a suitable choice of β (the proof is similar to the case p=2 in [17]).

This work presents sufficient conditions to a problem that has been posed by A. L. Shields.

Question 3.1. *Which shifts are reflexive?*
We use the following notations:

$$\begin{array}{lcl}
r_{01} = \overline{\lim}\beta(-n)^{\frac{-1}{n}} & , & \Omega_{01} = \{z \in \mathbb{C} : |z| > r_{01}\} \\
r_{11} = \underline{\lim}\beta(n)^{\frac{1}{n}} & , & \Omega_{11} = \{z \in \mathbb{C} : |z| < r_{11}\} \\
r_{12} = r(M_z^{-1})^{-1} & , & \Omega_{12} = \{z \in \mathbb{C} : |z| > r_{12}\} \\
r_{22} = r(M_z) & , & \Omega_{22} = \{z \in \mathbb{C} : |z| < r_{22}\} \\
r_{23} = \|M_z^{-1}\|^{-1} & , & \Omega_{23} = \{z \in \mathbb{C} : |z| > r_{23}\} \\
r_{33} = \|M_z\| & , & \Omega_{33} = \{z \in \mathbb{C} : |z| < r_{33}\} \\
\Omega_1 = \Omega_{01} \cap \Omega_{11} & = & \{z \in \mathbb{C} : r_{01} < |z| < r_{11}\} \\
\Omega_2 = \Omega_{12} \cap \Omega_{22} & = & \{z \in \mathbb{C} : r_{12} < |z| < r_{22}\} \\
\Omega_3 = \Omega_{23} \cap \Omega_{33} & = & \{z \in \mathbb{C} : r_{23} < |z| < r_{33}\}.
\end{array}$$

If $r_{01} < r_{11}$, the same method used for formal power series in [6] yields that each point of Ω_1 is a bounded point evaluation on $L^p(\beta)$.

Theorem 3.2. *Let M_z be invertible on $L^p(\beta)$ and $r_{01} < r_{11}$. If there exists $c > 0$ such that $\|M_s\| \leqslant c\|s\|_{\Omega_1}$ for all Laurent polynomials s, then M_z is reflexive.*

Proof. Let $A \in AlgLat(M_z)$. Then $Lat(M_z) \subset Lat(A)$. By the same method used in the proof of Theorem 1 in [21] we can see that each point of Ω_1 is a bounded point evaluation on $L^p(\beta)$. Since $M_z^* e(\lambda) = \lambda e(\lambda)$ for all λ in Ω_1, the one dimensional span of $e(\lambda)$ is invariant under M_z^*. Therefore it is invariant under A^* and we write $A^* e(\lambda) = \varphi(\lambda)e(\lambda)$, $\lambda \in \Omega_1$. So

$$< Af, e(\lambda) >=< f, A^* e(\lambda) >= \varphi(\lambda) f(\lambda)$$

for all $f \in L^p(\beta)$ and $\lambda \in \Omega_1$. This implies that $A = M_\varphi$ and $\varphi \in L^p_\infty(\beta)$. Now since $\varphi \in L^p_\infty(\beta) \subset H^\infty(\Omega_1)$, by the same Lemma in [17, p.81] we can write $\varphi(z) = \sum_{n=-\infty}^{\infty} \hat{\varphi}(n) z^n = \varphi_1(z) + \varphi_2(z)$ where

$$\begin{aligned}
\varphi_1(z) &= \sum_{n=0}^{\infty} \hat{\varphi}_1(n) z^n \in H^\infty(\Omega_{11}), \\
\varphi_2(z) &= \sum_{n=-\infty}^{-1} \hat{\varphi}_2(n) z^n \in H^\infty(\Omega_{01}).
\end{aligned}$$

Since Ω_{11} is a Caratheodory domain, by the Farrell-Rubel-Shields Theorem, there exists a uniformly bounded sequence $\{s_n\}_n$ of polynomials converging pointwise to φ_1. By the assumption we have $\|s_n(M_z)\| = \|M_{s_n}\| \leqslant c\|s_n\|_{\Omega_1}$ for all n. Passing to a subsequence, if necessary, we may assume that $s_n(M_z) \longrightarrow X$ in the weak operator topology for some operator X. Now since every point in Ω_1 is a bounded point evaluation, we conclude that

$$M^*_{s_n}e(\lambda) = s_n(\lambda)e(\lambda) \longrightarrow \varphi_1(\lambda)e(\lambda) = M^*_{\varphi_1}e(\lambda)$$

for every $\lambda \in \Omega_1$. On the other hand $M^*_{s_n}e(\lambda) \longrightarrow X^*e(\lambda)$ weakly. Therefore $X^*e(\lambda) = M^*_{\varphi_1}e(\lambda)$. Since the linear span of $\{e(\lambda) : \lambda \in \Omega_1\}$ is dense in $L^p(\beta)^*$ we conclude that $M_{\varphi_1} = X \in W(M_z)$. If we show that $\varphi_2 \equiv c$, a constant, then $A = M_\varphi = M_{\varphi_1} + cI \in W(M_z)$ and the proof is complete. To see this, note that $L^p(\beta) \in Lat(M_z)$, so $L^p(\beta) \in Lat(A)$ and also $L^p(\beta) \in Lat(M_{\varphi_1})$. Hence $\varphi_2 = \varphi - \varphi_1 = A1 - M_{\varphi_1}1 \in L^p(\beta)$. If $\varphi_2 \neq c$, then $\hat{\varphi}_2(k) \neq 0$ for some $k < 0$. Since $\varphi_2 \in H^\infty(\Omega_{01})$, there is a sequence of polynomials in $\frac{1}{z}$, $\{s_n(\frac{1}{z})\}_n$, uniformly bounded on Ω_{01} and converging pointwise to $\varphi_2(z)$. Therefore $s_n(M_z^{-1}) \longrightarrow M_{\varphi_2}$ in the weak operator topology. To see this note that

$$M^*_{s_n(\frac{1}{z})}e(\lambda) = s_n(\frac{1}{\lambda})e(\lambda) \longrightarrow \varphi_2(\lambda)e(\lambda) = M^*_{\varphi_2}e(\lambda).$$

Hence $M^*_{s_n(\frac{1}{z})}f \longrightarrow M^*_{\varphi_2}f$ for every f in the linear span of $\{e(\lambda) : \lambda \in \Omega_{01}\}$ that is dense in $L^q(\beta^{\frac{p}{q}})$. But $\varphi_2 \in L^p_\infty(\beta)$, thus by using the assumption we get

$$\|M_{s_n(\frac{1}{z})}\| \leqslant c\|s_n(\frac{1}{z})\|_{\Omega_{01}}.$$

Therefore $\{M_{s_n(\frac{1}{z})}\}_n$ is uniformly bounded and hence

$$M^*_{s_n(\frac{1}{z})}f \longrightarrow M^*_{\varphi_2}f$$

for every $f \in L^p(\beta)$. We have actually shown that

$$s_n(M_z^{-1}) \longrightarrow M_{\varphi_2}$$

in the strong operator topology. Therefore

$$< s_n(M_z^{-1})1, f_k > \longrightarrow < M_{\varphi_2}1, f_k >$$

where $f_k(z) = z^k$. That is $\widehat{s_n(\frac{1}{z})}(k) \longrightarrow \hat{\varphi}_2(k)$ as $n \longrightarrow \infty$. This is a contradiction, since the left hand side is zero and the right hand side is nonzero. Hence φ_2 is a constant and the proof is complete. □

Theorem 3.3. (**[30]**) *Let M_z be invertible on $L^p(\beta)$ and $r_{12} \leqslant r_{01} \leqslant r_{11} \leqslant r_{22}$. Then M_z is reflexive.*

Theorem 3.4. *Let M_z be invertible on $L^p(\beta)$ and $r_{12} < r_{22}$. If there exists $c > 0$ such that $\|M_s\| \leqslant c\|s\|_{\Omega_2}$ for all Laurent polynomials s, then M_z is reflexive.*

Proof. Let $A \in AlgLat(M_z)$. Then $A = M_\varphi$ for some φ in $L^p_\infty(\beta)$. By the same proof of Theorem 3 in [21] we can see that $L^p_\infty(\beta) \subset H^\infty(\Omega_2)$. Thus $\varphi \in H^\infty(\Omega_2)$ and so we can write $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1 \in H^\infty(\Omega_{22})$ and $\varphi_2 \in H^\infty(\Omega_{12})$. Now, the proof runs as that one of Theorem 3.2. □

In the special case when $p = 2$, we have the following Corollary.

Corollary 3.5. *If M_z is invertible on $L^2(\beta)$ and $r_{01} = r_{12} = r_{23} < r_{11} = r_{22} = r_{33}$, then M_z is reflexive.*

Proof. By using the von Neumann's inequality we have $\|M_s\| \leqslant c\|s\|_{\Omega_3}$ for all Laurent polynomials s, ([17, Prop.23, P.82]). This completes the proof. □

Note that in Theorem 3.4, the inequality $\|M_s\| \leqslant c\|s\|_{\Omega_2}$ was assumed to be held for Laurent polynomials. In the next theorem, we assume it for all polynomials.

Theorem 3.6. (**[27]**) *Let M_z be invertible on $L^p(\beta)$ and $r_{12} < r_{22}$. If $L^p(\beta) = L^p_\infty(\beta)$ and there exists a constant $c > 0$ such that $\|M_s\| \leqslant c\|s\|_{\Omega_2}$ for all polynomials s, then M_z is reflexive.*

Theorem 3.7. (**[24]**) *Suppose that M_z is not invertible on $L^p(\beta)$. If*

$r_{01} < r_{11}$, *then* M_z *is reflexive.*

Theorem 3.8. **([24])** *If* $\underline{\lim}\beta(n)^{\frac{1}{n}} > 0$, *then* M_z *is reflexive on* $H^p(\beta)$.

4. Reflexivity on Banach Spaces of Analytic Functions

Let X be a reflexive Banach space of functions analytic on a plane domain. We give sufficient conditions for the multiplication operator, M_z, to be reflexive on X.

By (X, Ω) we mean that X is a Banach space of analytic functions on a plane domain Ω. Throughout this section (X, Ω) is a reflexive Banach space such that:

(1) For each $\lambda \in \Omega$ the linear functional, $e(\lambda)$, of evaluation at λ is bounded on X.

(2) X contains the constant functions.

(3) multiplication by the independent variable z defines a bounded linear operator M_z on X.

A complex valued function φ on Ω for which $\varphi f \in X$ for every $f \in X$ is called a *multiplier* of X and the collection of all these multipliers is denoted by $\mathcal{M}(X)$. Each multiplier φ of X determines a multiplication operator M_φ on X by $M_\varphi f = \varphi f$, $f \in X$. It is well-known that each multiplier is a bounded analytic function ([16]). Indeed $|\varphi(\lambda)| \leqslant ||M_\varphi||$ for each λ in Ω. Also $M_\varphi 1 = \varphi \in X$. But $X \subset H(\Omega)$, thus φ is a bounded analytic function. We also point out that if φ is a multiplier and $\lambda \in \Omega$ then $M_\varphi^* e(\lambda) = \varphi(\lambda) e(\lambda)$, since for all f in X we have

$$< f, M_\varphi^* e(\lambda) >= \varphi(\lambda) f(\lambda) = \varphi(\lambda) < f, e(\lambda) >=< f, \varphi(\lambda) e(\lambda) >$$

(here for simplicity we used the notation $< x, x^* >$ instead of $x^*(x)$ for $x \in X$ and $x^* \in X^*$).

By definition M_z is called *polynomially bounded* on a Banach space (X, Ω) if for some $c > 0$, $||M_p|| \leqslant c||p||_\Omega$ for all polynomials p.

Consider the circular plane domain $\Omega = U \backslash K_1 \cup \cdots \cup K_N$ where

$$K_i = \overline{D_i} = \{z : |z - z_i| \leqslant r_i\}$$

are disjoint closed subdiscs of the open unit disc U. We can choose circles

$$\gamma_i = \{z : |z - z_i| \leqslant r_i + \epsilon_i\} \ (i = 1, ..., N)$$

and $\gamma_0 = \{z : |z| = 1 - \epsilon_0\}$ in Ω and concentric to the boundary circles of Ω so that they do'nt meet each other. Put $\Omega_i = (\mathbb{C} \cup\{\infty\})\backslash K_i$ for $i = 1, 2, \cdots, N$. Then by the Cauchy integral formula it is proved that

$$H^\infty(\Omega) = H^\infty(\Omega_0) + H_0^\infty(\Omega_1) + \cdots + H_0^\infty(\Omega_N)$$

where $\Omega_0 = U$, $H_0^\infty(\Omega_i) = H^\infty(\Omega_i) \cap H_0(\Omega_i)$, and $H_0(\Omega_i)$ is the space of all analytic functions on Ω_i that vanish at infinity.

Theorem 4.1. *Let Ω be a circular plane domain and suppose that M_z is polynomially bounded on (X, Ω). Then M_z is reflexive.*

Proof. Let $A \in AlgLat(M_z)$. Then $A = M_\varphi$ for some multiplier φ, indeed since $M_z^* e(\lambda) = \lambda e(\lambda)$, the one dimensional span of $e(\lambda)$ is invariant under M_z^*, so it is invariant under A^*. That is,

$$A^* e(\lambda) = \varphi(\lambda) e(\lambda), \lambda \in \Omega.$$

Using the Hahn-Banach Theorem we see that the linear span of $\{e(\lambda)\}_{\lambda\in\Omega}$ is weak star dense in X^*. Hence $\varphi \in \mathcal{M}(X)$ and $A = M_\varphi$, and so $\varphi \in H^\infty(\Omega)$. Set $X_0 = X \cap H(U)$. Since X contains the constants, $X_0 \neq \{0\}$. Clearly every function in X_0 is analytic in U. Now we show that X_0 is a closed subspace of X that is invariant under M_z. To see this let $\{g_n\}$ be a sequence in X_0 such that g_n converges to f in X. By applying the Cauchy Integral Theorem we can write $f = f_0 + f_1 + \cdots + f_N$ where $f_0 \in H(U)$ and $f_i \in H_o(\Omega_i)$ for $i = 1, \cdots, N$. Set $h = f_1 + \cdots + f_N$. Clearly $g_n - f_0$ converges uniformly to h on compact subsets of Ω and so $z^i(g_n - f_0)$ converges uniformly to $z^i h$ on compact subsets of Ω for $i = 0, 1, 2, \cdots$. Since $z^i(g_n - f_0) \in H(U)$ we have

$$\int_{\gamma_0} \xi^i (g_n(\xi) - f_0(\xi)) d\xi = 0, \ \ i = 0, 1, 2, \cdots ; \ n = 1, 2, \cdots,$$

where γ_0 is the circle defined as before. Choose the circle γ_0' sufficiently close to γ_0 with smaller radius so that γ_0 lies in ext(γ_0'). We can write

$$h(z) = \sum_{n=-\infty}^{-1} a_n z^n \quad , \quad z \in ext(\gamma_0')$$

$$a_n = \frac{1}{2\pi i} \int_{\gamma_0} h(\xi)/\xi^{n+1}\, d\xi \quad , \quad n < 0.$$

But

$$\int_{\gamma_0} \xi^k h(\xi) = 0, \quad k = 0, 1, 2, \cdots .$$

From this it follows that $h(z) = 0$, $z \in ext(\gamma_0')$. Hence $h \equiv 0$. Therefore $f = f_0$ is analytic in U and so X_0 is closed. Clearly X_0 is invariant under M_z, and contains the constants.

Since $AX_0 \subset X_0$ and $1 \in X_0$, we see that $A1 = \phi \in X_0$. But $X_0 \subset H(U)$ and $\varphi \in H^\infty(\Omega)$. Thus $\varphi \in H^\infty(U)$. Since U is a Carathéodory domain, there exists a sequence $\{p_n\}$ of polynomials such that $\sup_n ||p_n||_\Omega < \infty$ and $p_n(z) \to \varphi(z)$, $z \in U$. By hypothesis $||M_{p_n}|| \leqslant c||p_n||_\Omega$, where c is a constant. But X is a reflexive Banach space, so the unit ball of $\mathcal{B}(X)$ is compact in the weak operator topology. Thus, we may assume by passing to a subsequence if necessary, that $M_{p_n} \to S$, in the weak operator topology, for some operator S. Thus

$$M_{p_n}^* e(\lambda) = p_n(\lambda) e(\lambda) \to \varphi(\lambda) e(\lambda) = M_\varphi^* e(\lambda)$$

for every $\lambda \in \Omega$. On the other hand, $M_{p_n}^* e(\lambda) \to S^* e(\lambda)$ weak star. Therefore $S^* e(\lambda) = M_\varphi^* e(\lambda)$. So $S^* = M_\varphi^*$, hence $S = M_\varphi$ on X. Since $A = M_\varphi$, $A = S$ and so it follows that $A \in W(M_z)$ and M_z is reflexive. This completes the proof. □

In the proof of the Theorem 4.1 we used the polynomially bounded condition. In the following Corollary we substitute it by another conditions that each of which implies the desired result.

Corollary 4.2. *The above Theorem holds also if we substitute the condition of polynomially bounded by one of the following conditions:*

(i) The map $\varphi \longrightarrow M_\varphi$ *of* $\mathcal{M}(X) \longrightarrow \mathcal{B}(X)$ *is an isometry,*

(ii) $\overline{\Omega}$ *is a spectral set for* M_z*,*
(iii) $\|M_\varphi\| \leqslant c\|\varphi\|_\Omega$ *for every multiplier* φ*,*
(iv) $H^\infty(\Omega_1) \subset \mathcal{M}(X)$*.*

Proof. It is clear to see that all conditions (i), (ii) and (iii) imply the polynomially bounded condition. So it is sufficient to show that the condition (iv) implies the polynomially bounded condition. For this we show that $L : H^\infty(\Omega_1) \longrightarrow \mathcal{B}(X)$ given by $L(\varphi) = M_\varphi$ is continuous. Suppose that the sequence $\{\varphi_n\}_n$ converges to φ in $H^\infty(\Omega_1)$ and $L(\varphi_n) = M_{\varphi_n}$ converges to A in $\mathcal{B}(X)$. Then for each f in X, $Af = \lim_n M_{\varphi_n} f = \lim_n \varphi_n f$ and so $\{\varphi_n f\}_n$ is convergent in X. Note that by the continuity of point evaluations $\varphi_n f$ converges pointwise to φf. Thus Af is analytic on Ω and agree with φf on Ω. Hence $A = M_\varphi$ and so L is continuous. This implies that there is a constant $c > 0$ such that $\|M_\varphi\| \leqslant c\|\varphi\|_{\Omega_1}$ for all φ in $H^\infty(\Omega_1)$. But $\|p\|_\Omega = \|p\|_{\Omega_1}$ for all polynomials p, hence $\|M_p\| \leqslant c\|p\|_\Omega$ holds and so M_z is polynomially bounded. $\square$

Theorem 4.3. **([32])** *If* $\{e(\lambda) : \lambda \in \Omega\}$ *is norm bounded and* $H^\infty(\Omega_1) \subset \mathcal{M}(X)$*, then* M_z *is reflexive on* (X, Ω)*.*

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