Generalized Lindley Distribution

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Abstract. In this paper, we introduce a three–parameter generalization of the Lindley distribution. This includes as special cases the exponential and gamma distributions. The distribution exhibits decreasing, increasing and bathtub hazard rate depending on its parameters. We study various properties of the new distribution and provide numerical examples to show the flexibility of the model. We also derive a bivariate version of the proposed distribution.

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1. Introduction

The one parameter family of distributions with the density function

$$f(x;\theta) = \frac{\theta^2(1+x)e^{-\theta x}}{1+\theta}, \quad x,\theta > 0, \tag{1}$$

used by Lindley [8] to illustrate a difference between fiducial distribution and posterior distribution. Sankaran [9] used it as the mixing distribution of a Poisson parameter and the distribution he derived is known as the Poisson–Lindley distribution. Recently, Ghitany et al. [3] rediscovered and studied various properties of (1). Because of having only one parameter, the Lindley distribution does not provide enough flexibility for analysing different types of lifetime data. To increase the flexibility for modelling purposes it will be useful to consider further alternatives of this distribution. This paper offers a three-parameter family of distributions which generalizes the Lindley distribution and includes as special cases the ordinary exponential and gamma distributions. The procedure used here is based on certain mixtures of the gamma distributions. The study examines various properties of the new model. The paper is organized as follows: Section 2, introduces the generalized Lindley distribution and presents its basic properties including: the behavior of the density and hazard rate functions, the distribution of the sums and some results on stochastic orderings. We also proposed an algorithm for generating random data from the new distribution in this section. Section 3, discusses the estimation of parameters. Two data modelling examples are provided in this section, where the generalized Lindley distribution fits marginally better than the gamma, Weibull and lognormal distributions. Finally, a bivariate derivation of the proposed

model is discussed in Section 4.

2.   Definition and Some Properties

In this section, we introduce the generalized Lindley distribution and study its basic properties.

2.1.   Generalization

Let

$$f_g(x; \alpha, \theta) = \frac{\theta(\theta x)^{\alpha-1} e^{-\theta x}}{\Gamma(\alpha)}, \ \alpha, \theta, x > 0, \tag{2}$$

be the density function of the gamma distribution with the shape parameter α and the scale parameter θ, denoted by gamma(α, θ). Let V_1 and V_2 be two independent random variables distributed according to gamma(α, θ) and gamma$(\alpha + 1, \theta)$, respectively. For $\gamma \geqslant 0$, consider the random variable $X = V_1$ with probability $\frac{\theta}{\theta+\gamma}$, and $X = V_2$ with probability $\frac{\gamma}{\theta+\gamma}$. It is then easy to verify that the density function of X is given by

$$f(x; \alpha, \theta, \gamma) = \frac{\theta^2 (\theta x)^{\alpha-1} (\alpha + \gamma x) e^{-\theta x}}{(\gamma + \theta) \Gamma(\alpha + 1)}, \quad \alpha, \theta, \gamma, x > 0. \tag{3}$$

This distribution contains the Lindley distribution as a particular case $\alpha = \gamma = 1$. When $\gamma = 0$, (3) reduces to the density function of the gamma distribution with the parameters α and θ. The case $(\alpha, \gamma) = (1, 0)$, it coincides with the density function of the ordinary exponential

distribution. We say that the random variable X has a *generalized Lindley* (GL) distribution, if X has the density function defined by (3). We denote the generalized Lindley distribution with the parameters α, θ and γ as $\mathrm{GL}(\alpha, \theta, \gamma)$.

2.2. Shape

For the density function of the GL distribution, the first and the second derivatives of $\log f(x)$ are

$$\frac{d}{dx} \log f(x) = \frac{(\alpha - 1)(\alpha + \gamma x) + (\gamma - \theta(\alpha + \gamma x))x}{x(\alpha + \gamma x)},$$

and

$$\frac{d^2}{dx^2} \log f(x) = \frac{(1 - \alpha)(\alpha + \gamma x)^2 - (\gamma x)^2}{x^2(\alpha + \gamma x)^2}.$$

If $\alpha \geqslant 1$, then $\frac{d^2}{dx^2} \log f(x) \leqslant 0$; i.e., the density function $f(x)$, is log-concave. Note that $(\log f)'(0) = \infty$ and $(\log f)'(\infty) = -\theta < 0$. This implies that for $\alpha \geqslant 1$, $f(x)$ has a unique mode at x_0, where $x_0 = \frac{\alpha(\gamma - \theta) + \sqrt{(\alpha(\gamma + \theta))^2 - 4\alpha\theta\gamma}}{2\gamma\theta}$, is the solution of the equation $\frac{d}{dx} \log f(x) = 0$. For $\alpha < 1$ we have that $\frac{d}{dx} \log f(x) \leqslant 0$; i.e., $f(x)$ is decreasing in x.

Let $h(x) = \frac{f(x)}{1 - F(x)}$ be the hazard rate function of the random variable X. Because the survival function of this distribution can be given only in terms of the incomplete gamma function when α is not an integer, the hazard rate function could not be expressed in closed form. However, properties of this function can still be determined.

Proposition 1. *Let $h(t)$ be the hazard rate function of a random variable X distributed according to $GL(\alpha, \theta, \gamma)$. Then*

(i) $h(t)$ is increasing for $\alpha \geqslant 1$;

(ii) $h(t)$ is bathtub shaped for $\alpha < 1$ and $\gamma > 0$;

(iii) $h(t)$ is decreasing for $\alpha \leqslant 1$ and $\gamma = 0$.

Proof. For the density function (3) we have

$$\rho(t) = -\frac{f'(t)}{f(t)} = \frac{1-\alpha}{t} - \frac{\gamma}{\alpha + \gamma t} + \theta.$$

It follows that $\rho'(t) = \frac{\alpha-1}{t^2} + \frac{\gamma^2}{(\alpha+\gamma t)^2} \geqslant 0$, for $\alpha \geqslant 1$; $\rho'(t) \leqslant 0$ when $\alpha \leqslant 1$ and $\gamma = 0$. When $\alpha < 1$ and $\gamma > 0$, we have $\rho'(t) < 0$ for $t < (\sqrt{1-\alpha} + 1 - \alpha)/\gamma$; $\rho'((\sqrt{1-\alpha} + 1 - \alpha)/\gamma) = 0$, and $\rho'(t) > 0$ for $t > (\sqrt{1-\alpha} + 1 - \alpha)/\gamma$. Now, parts (i), (ii) and (iii) follow from Glaser [4]. $\square$

2.3. Distribution of the Sums

It is well known that the distribution of a sum of independent gamma random variables with the same scale parameter is again a gamma distribution. The following result shows that the distribution of a sum of independent random variables from the GL distribution, could be written as a mixture of the gamma distributions.

Proposition 2. *Let $X_1, ..., X_n$ denote independent random variables from GL distribution with the parameters $(\alpha_i, \theta, \gamma)$, for $i = 1, ..., n$. Then*

the density function of $S = \sum\limits_{i=1}^{n} X_i$, *is given by*

$$f_S(x) = \sum_{k=0}^{n} p_k f_g(x; \alpha^* + k, \theta), \tag{4}$$

where f_g *is the density function of the gamma distribution,* $\alpha^* = \sum\limits_{i=1}^{n} \alpha_i$ *and* $p_k = \frac{\binom{n}{k} \theta^{n-k} \gamma^k}{(\gamma + \theta)^n}$, *for* $k = 0, ..., n$; *with* $\sum\limits_{k=0}^{n} p_k = 1$,.

Proof. The proof can be established by comparing the mgf of S and the mgf corresponding to the density function defined by (4). If X distributed according to (3) then the corresponding moment generating function (mgf) of X defined by $M(t) = E(e^{tX})$, is given by

$$M(t) = \left(\frac{\theta}{\theta - t} \right)^{\alpha+1} \frac{\theta - t + \gamma}{\theta + \gamma}. \tag{5}$$

From (5), the mgf of $S = \sum\limits_{i=1}^{n} X_i$ could be obtained as

$$M_S(t) = \left(\frac{\theta}{\theta - t} \right)^{\alpha^* + n} \left(\frac{\theta - t + \gamma}{\gamma + \theta} \right)^n.$$

Now, let $M_g(t; \alpha^* + k, \theta) = (\frac{\theta}{\theta - t})^{\alpha^* + k}$, $k = 0, ..., n$, be the mgf of the gamma distribution with the shape parameter $\alpha^* + k$ and the scale parameter θ. Then the mgf corresponding to the density function $f_S(x)$,

denoted by $M_f(t)$, is given by

$$
\begin{aligned}
M_f(t) &= \sum_{k=0}^{n} p_k M_g(x; \alpha^* + k, \theta) \\
&= \frac{\theta^{n+\alpha^*}}{(\gamma + \theta)^n} \left(\frac{1}{\theta - t}\right)^{\alpha^*} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\gamma}{\theta - t}\right)^k \\
&= \frac{\theta^{n+\alpha^*}}{(\gamma + \theta)^n} \left(\frac{1}{\theta - t}\right)^{\alpha^*} \left(1 + \frac{\gamma}{\theta - t}\right)^n \\
&= \left(\frac{\theta}{\theta - t}\right)^{\alpha^* + n} \left(\frac{\theta - t + \gamma}{\gamma + \theta}\right)^n,
\end{aligned}
$$

which completes the proof. $\square$

2.4. Stochastic Orders

A random variable X is said to be stochastically smaller than Y (denoted by $X \prec_s Y$), if $F_X(t) \geqslant F_Y(t)$ for all t. Two stronger stochastic dominance are the hazard rate order (denoted by $X \prec_{hr} Y$) if $h_X(t) \geqslant h_Y(t)$, for all t, and the likelihood ratio order (denoted by $X \prec_{lr} Y$) if $f_X(t)/f_Y(t)$ is decreasing in t. It is well known that $X \prec_{lr} Y \Rightarrow X \prec_{hr} \Rightarrow X \prec_s Y$. For more detail see Shaked and Shanthikumar [10].

Let Y_i be a random variable distributed according to (3) with the parameters $(\alpha_i, \theta_i, \gamma_i)$, for $i = 1, 2$. Then

$$
\frac{d}{dy} \log\left(\frac{f_{Y_1}(y)}{f_{Y_2}(y)}\right) = \frac{\gamma_1}{\alpha_1 + \gamma_1 y} - \frac{\gamma_2}{\alpha_2 + \gamma_2 y} + \frac{\alpha_1 - \alpha_2}{y} + \theta_2 - \theta_1. \quad (6)
$$

Clearly, if $\alpha_1 = \alpha_2$, then (6) is negative when $\theta_1 \geqslant \theta_2$ and $\gamma_1 \leqslant \gamma_2$. When $\theta_1 = \theta_2$ and $\gamma_1 = \gamma_2$, it could be verified that the expression (6)

is negative for $\alpha_1 \leqslant \alpha_2$. In short, we have proved that:

Proposition 3. *Let Y_1 and Y_2 be two random variables having GL distribution with the parameters $(\alpha_i, \theta_i, \gamma_i)$, $i = 1, 2$. Then the followings hold*

(i) If $\theta_1 = \theta_2$, $\gamma_1 = \gamma_2$ and $\alpha_1 \leqslant \alpha_2$, then $Y_1 \prec_{lr} Y_2$, $Y_1 \prec_{hr} Y_2$ and $Y_1 \prec_s Y_2$;

(ii) If $\alpha_1 = \alpha_2$, $\theta_1 \geqslant \theta_2$ and $\gamma_1 \leqslant \gamma_2$, then $Y_1 \prec_{lr} Y_2$, $Y_1 \prec_{hr} Y_2$ and $Y_1 \prec_s Y_2$.

2.5.   Random Variate Generation

The density function of the GL distribution can be written in terms of the gamma density function as

$$f(x; \alpha, \theta, \gamma) = \frac{\theta}{\gamma + \theta} f_g(x; \alpha, \theta) + \frac{\gamma}{\gamma + \theta} f_g(x; \alpha + 1, \theta).$$

To generate random data X_i, $i = 1, ..., n$, from $\mathrm{GL}(\alpha, \theta, \gamma)$, one can use the following algorithm:

1. Generate U_i, $i = 1, ..., n$, from $U(0, 1)$ distribution.

2. Generate V_{1i}, $i = 1, ..., n$, from the gamma(α, θ).

3. Generate V_{2i}, $i = 1, ..., n$, from the gamma$(\alpha + 1, \theta)$.

4. If $U_i \leqslant \frac{\theta}{\gamma + \theta}$, then set $X_i = V_{1i}$; otherwise set $X_i = V_{2i}$, $i = 1, ..., n$.

In next section, we consider the maximum likelihood estimation of the parameter of the GL distribution.

3. Estimation

3.1. Maximum Likelihood Estimates

In this section we consider the maximum likelihood estimation (MLE) of the parameters. If $X_1, ..., X_n$, is a random sample from X distributed according to $\text{GL}(\alpha, \theta, \gamma)$, then the log-likelihood function, $l(\alpha, \theta, \gamma)$, is:

$$
\begin{aligned}
l(\alpha, \theta, \gamma) \quad = \quad & n(\alpha + 1) \log(\theta) - n \log(\gamma + \theta) - n \log \Gamma(\alpha + 1) \\
& + (\alpha - 1) \sum_{i=1}^{n} \log(x_i) + \sum_{i=1}^{n} \log(\alpha + \gamma x_i) - \theta \sum_{i=1}^{n} x_i.
\end{aligned}
$$

The derivatives of $l(\alpha, \theta, \gamma)$ with respect to α, θ, and γ are:

$$
\frac{\partial l}{\partial \alpha} = n \log(\theta) - n \Psi(\alpha + 1) + \sum_{i=1}^{n} \log(x_i) + \sum_{i=1}^{n} \frac{1}{\alpha + \gamma x_i}, \qquad (7)
$$

$$
\frac{\partial l}{\partial \theta} = \frac{n(\alpha + 1)}{\theta} - \frac{n}{\gamma + \theta} - \sum_{i=1}^{n} x_i, \qquad (8)
$$

and

$$
\frac{\partial l}{\partial \gamma} = \sum_{i=1}^{n} \frac{x_i}{\alpha + \gamma x_i} - \frac{n}{\gamma + \theta}, \qquad (9)
$$

where $\Psi(t) = \frac{\Gamma'(t)}{\Gamma(t)}$, denotes the digamma function. The equations (7)–(9) can be solved simultaneously to find the maximum likelihood estimators of α, θ and γ.

The GL distribution satisfies all the regularity conditions (see, Bain, [1, pp. 86-87] in a way similar to gamma distribution, and therefore applying the usual large sample approximation, the estimators $(\hat{\alpha}, \hat{\theta}, \hat{\gamma})$ treated as being approximately bivariate normal with the mean vector

(α, θ, γ) and variance-covariance matrix I^{-1}, where I is the Fisher information matrix, whose elements are given by

$$
\begin{aligned}
-E(\frac{\partial^2 l}{\partial \alpha^2}) &= n\Psi'(1+\alpha) + nJ_0(\alpha, \theta, \gamma), \\
-E(\frac{\partial^2 l}{\partial \alpha \partial \theta}) &= -\frac{n}{\theta},
\end{aligned}
$$

$$
\begin{aligned}
-E(\frac{\partial^2 l}{\partial \alpha \partial \gamma}) &= nJ_1(\alpha, \theta, \gamma), \\
-E(\frac{\partial^2 l}{\partial \theta^2}) &= \frac{n(\alpha+1)}{\theta^2} - \frac{n}{(\gamma+\theta)^2}, \\
-E(\frac{\partial^2 l}{\partial \theta \partial \gamma}) &= -\frac{n}{(\gamma+\theta)^2}, \\
-E(\frac{\partial^2 l}{\partial \gamma^2}) &= nJ_2(\alpha, \theta, \gamma) - \frac{n}{(\gamma+\theta)^2}.
\end{aligned}
$$

Here for $i = 0, 1, 2$,

$$
\begin{aligned}
J_i(\alpha, \theta, \gamma) &= E\left(\frac{X^i}{(\alpha+\gamma X)^2}\right) \\
&= \frac{\theta^{1-i}}{\alpha\Gamma(\alpha+1)(\gamma+\theta)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\gamma}{\alpha\theta}\right)^k \Gamma(\alpha+k+i),
\end{aligned}
$$

where X distributed as $GL(\alpha, \theta, \gamma)$.

In what follows we provide numerical examples to show the flexibility of GL distribution for data modeling.

3.2. Numerical Examples

Two sets of real data are considered from Lawless (2003), pp. 204 and 263). The first set of data represents the failure times (in minutes)

for a sample of 15 electronic components in an accelerated life test and they are 1.4, 5.1, 6.3, 10.8, 12.1, 18.5, 19.7, 22.2, 23, 30.6, 37.3, 46.3, 53.9, 59.8, 66.2. The second set of data, are the number of cycles to failure for 25 100-cm specimens of yarn, tested at a particular strain level and they are 15, 20, 38, 42, 61, 76, 86, 98, 121, 146, 149, 157, 175, 176, 180, 180, 198, 220, 224, 251, 264, 282, 321, 325, 653. In addition to the generalized Lindley distribution, we have considered the gamma, Weibull and lognormal distributions with respective densities $f_g(x; \alpha, \theta) = \theta^\alpha x^{\alpha-1} e^{-\theta x} (\Gamma(\alpha))^{-1}$, $f_W(x; \alpha, \theta) = \alpha \theta x^{\alpha-1} e^{-\theta x^\alpha}$ and $f_{LN}(x; \alpha, \theta) = \frac{1}{\sqrt{2\pi}\alpha x} e^{-\frac{1}{2}(\frac{\log x - \theta}{\alpha})^2}$. The estimates, the log-likelihood (LL) and the Kolmogrov-Smirnov (K-S) statistic presented in Table 1. It is observed that, the generalized Lindley distribution competes well with three popular alternatives, the gamma, Weibull and lognormal models.

Table 1

Estimates, log-likelihood and Kolmogrov–Smirnov statistic

Data set	Distribution	α	θ	γ	LL	K–S
	GL	1.203	0.064	0.083	-64.080	0.095
1($n = 15$)	Gamma	1.442	0.052	—	-64.186	0.100
	Weibull	1.306	0.034	—	-64.020	0.451
	Lognormal	1.061	2.931	—	-65.617	0.161
	GL	1.505	0.012	0.018	-152.369	0.137
2($n = 25$)	Gamma	1.794	0.010	—	-152.438	0.135
	Weibull	1.414	0.005	—	-152.443	0.697
	Lognormal	0.891	4.880	—	-154.086	0.155

In next section, we introduce a bivariate version of the GL distribution.

4.   The Bivariate Case

A large number of bivariate distributions have been proposed in litera-
ture. A very wide survey on bivariate distributions is given in Kotz et al.
[6]. In this section, we provide a family of bivariate distribution whose
univariate margins are generalized Lindley distributions. For this, let
(V_1, V_2) and (W_1, W_2) be two vectors of independent random variables
distributed according to the gamma(α, θ) and gamma$(\alpha + 1, \theta)$, respec-
tively. For $\gamma \geqslant 0$, consider the random pair $(X_1, X_2) = (V_1, V_2)$ with the
probability $\frac{\theta}{\theta+\gamma}$, and $(X_1, X_2) = (W_1, W_2)$ with the probability $\frac{\gamma}{\theta+\gamma}$. It
is then easy to verify that the joint density function of the pair (X_1, X_2)
is given by

$$f(x_1, x_2) = \frac{\theta^{\alpha+2}(\theta x_1 x_2)^{\alpha-1}(\alpha^2 + \gamma\theta x_1 x_2)e^{-\theta(x_1+x_2)}}{(\gamma+\theta)\Gamma^2(\alpha+1)}, \quad \alpha, \theta, \gamma, x_1, x_2 > 0. \tag{10}$$

Note that the joint density function (10), may be written in terms of
the gamma density functions as

$$f(x_1, x_2) = \frac{\theta(\alpha^2 + \gamma\theta x_1 x_2)}{\alpha^2(\gamma+\theta)} f_g(x_1; \alpha, \theta) f_g(x_2; \alpha, \theta). \tag{11}$$

It is easy to see that the univariate marginal density functions of (10)
are of the form (3). When $\gamma = 0$, the random variables X_1 and X_2
become independent and the bivariate density function (11), reduces to
the product of two gamma density functions with the same parameters

The following proposition gives the mixture representation of the conditional density functions of (10) in terms of the gamma density functions.

Proposition 4. *If X_1 and X_2 are jointly distributed according to (10), then the conditional density function of X_j given $X_i = x_i$, denoted by $f_{j|i}(x_j|x_i)$, $(i \neq j = 1, 2)$, is given by*

$$f_{j|i}(x_j|x_i) = \frac{\alpha}{\alpha + \gamma x_i} f_g(x_j; \alpha, \theta) + \frac{\gamma x_i}{\alpha + \gamma x_i} f_g(x_j; \alpha + 1, \theta). \qquad (12)$$

Proof. The proof follows readily upon substituting for the joint density function of (X_1, X_2) in (10) and the marginal density function of X_i, $i = 1, 2$, in (3), in the relation

$$f_{j|i}(x_j|x_i) = \frac{f(x_i, x_j)}{f_{X_i}(x_i)}. \quad \square$$

The following result gives the joint and the marginal density functions of the random variables $U = X_1 + X_2$ and $V = X_1/(X_1 + X_2)$, when the pair (X_1, X_2) is distributed according to (10).

Proposition 5. *Let (X_1, X_2) be a random vector distributed according to (10). Then the joint and marginal density functions of $U = X_1 + X_2$, and $V = X_1/(X_1 + X_2)$, are given by*

$$f_{U,V}(u, v) = \frac{\theta}{\gamma + \theta} f_g(u; 2\alpha, \theta) f_B(v; \alpha, \alpha) + \frac{\gamma}{\gamma + \theta} f_g(u; 2\alpha + 2, \theta) f_B(v; \alpha + 1, \alpha + 1),$$
$$(13)$$

$$f_U(u; \alpha, \theta) = \frac{\theta}{\gamma + \theta} f_g(u; 2\alpha, \theta) + \frac{\gamma}{\gamma + \theta} f_g(u; ; 2\alpha + 2, \theta), \qquad (14)$$

and

$$f_V(v; \alpha, \theta) = \frac{\theta}{\gamma + \theta} f_B(v; \alpha, \alpha) + \frac{\gamma}{\gamma + \theta} f_B(v; \alpha + 1, \alpha + 1), \qquad (15)$$

respectively, where $f_g(u; a, b)$ and $f_B(v; a, b)$, respectively, denote the density functions of the gamma and the beta distributions with the parameters a and b, for all $0 < v < 1$ and $u > 0$.

Simple calculations show that for each positive integer m and n, the following expression for the moments could be obtained

$$E(X_1^n X_2^m) = \frac{\Gamma(\alpha + n)\Gamma(\alpha + m)}{\theta^{n+m}(\gamma + \theta)\Gamma^2(\alpha + 1)} (\alpha^2(\theta + \gamma) + \gamma(\alpha + m)(\alpha + n)).$$

In particular the correlation coefficient of X_1 and X_2 is given by

$$Corr(X_1, X_2) = \frac{\gamma\theta}{\alpha(\theta + \gamma)^2 + \gamma(2\theta + \gamma)}.$$

Clearly, $Corr(X_1, X_2) = 0$, when $\gamma = 0$. When $\alpha \to 0$ and $\theta \to \infty$, we have that $Corr(X_1, X_2) \to \frac{1}{2}$; which is also the maximum value of the correlation for this family and a limitation of the proposed model.

The proposed bivariate distribution is a mixture of i.i.d random variables, form the general theory, the resulting pair is positively correlated by mixtures (see, Droute-Mori and Kotz [2]. In what follows we study the dependence properties of (10) in detail. A bivariate distribution is said

to be positively likelihood ratio dependent (PLRD) if the density function $f(x,y)$ satisfies

$$f(x_1,y_1)f(x_2,y_2) \geqslant f(x_1,y_2)f(x_2,y_1), \qquad (16)$$

for all $x_1 \geqslant x_2$ and $y_1 \geqslant y_2$. For the bivariate density function (10), the above inequality is equivalent to

$$(\alpha^2 + \theta\gamma x_1 y_1)(\alpha^2 + \theta\gamma x_2 y_2) \geqslant (\alpha^2 + \theta\gamma x_1 y_2)(\alpha^2 + \theta\gamma x_2 y_1),$$

or $(x_1 - x_2)(y_1 - y_2) \geqslant 0$, which holds. The PLRD has several implications; in particular, it implies $P(X_1 \leqslant x | X_2 = y)$ is non increasing in y for all x, and similarly $P(X_2 \leqslant y | X_1 = x)$ is non increasing in x for all y. This property is called positive regression dependent (PRD). Furthermore, property PRD implies $P(X_2 > y | X_1 > x)$ is non decreasing in x for all y, and $P(X_2 \leqslant y | X_1 \leqslant x)$ is non increasing in y for all x; each of which implies that $P(X_1 \leqslant x, X_2 \leqslant y) \geqslant P(X_1 \leqslant x)P(X_2 \leqslant y)$, for all x and y, namely, X_1 and X_2 are positive quadrant dependent. For more details see Droute-Mori and Kotz ([2]).

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